Exact transport and spectral dimensions for the almost Mathieu operator*

Seth Douglas Nidhal Mghirbi

Version 2.0.1, November 2026

Abstract

For the almost Mathieu operator with coupling $\lambda > 1$ and irrational frequency α, let $\beta(\alpha) = \limsup_n q_n^{-1} \log q_{n+1}$ and $\gamma = \max\{0, 1 - \log\lambda/\beta(\alpha)\}$, read as 0 if $\beta = 0$ and 1 if $\beta = \infty$. For almost every phase, the upper Abel-averaged transport exponent of every positive moment, normalized by its order, the fastest-tail exponent, and both packing dimensions of every nonzero spectral measure equal γ. For finite β the phase set is an explicit set G_α^*, invariant under $\theta \mapsto \theta + \alpha$, that allows subexponential decay of $|\sin 2\pi q_n\theta|$ along resonant scales; for $\beta = \infty$ the equalities hold at every phase. The lower bounds hold at every phase. Moreover, for fixed $c > h > \log\lambda$, along scales with $b_n = q_n^{-1} \log|q_n\alpha - p_n|^{-1} \geq c$, the Abel-averaged mass beyond distance Te^{-hq_n} from the launch site tends to one, uniformly in the phase, the launch site and $e^{hq_n} \leq T \leq |q_n\alpha - p_n|^{-1}$; for a packet launched at the origin and fixed $b > 0$, $s \in (0,1)$, the instantaneous mass within that distance exceeds s at only $o(T)$ times in $[0, bT]$. For phases in G_α^*, and at every phase if $\beta = \infty$, the upper generalized Rényi dimensions of orders $0 < r < 1$ of the single-site spectral measure equal γ; the lower ones vanish at every phase. For $1 < \lambda < e^\beta$ the packing formula replaces the factor two in the bound of Jitomirskaya, Liu and Tcheremchantsev by the sharp factor one. The proof combines an exactly conserved polynomial with elliptic transfer-matrix estimates; its confining barrier holds at every phase. We give explicit finite-time and finite-volume comparison bounds. At coupling 2 and phase $1/4$, computable frequencies with eventually odd convergent denominators realize exactly the exponents in $[0,1]$ that are right-c.e. relative to the halting problem. One computable frequency has exponent of Turing degree $\mathbf{0}''$ for every computable coupling in $(1, 2^{23/12})$, although finite-time predictions at nonzero error tolerance do not determine it.

1 Introduction

The almost Mathieu operator

$$(H_{\lambda,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + 2\lambda\cos\bigl(2\pi(\theta + n\alpha)\bigr)u(n) \tag{1}$$

on $\ell^2(\mathbb{Z})$ is the central model of quasi-periodic spectral theory. Physicists know it as the Aubry–André model [1], and it is realized with cold atoms in bichromatic optical lattices [22] and with light in quasi-periodic waveguide arrays [20]. Here $\lambda > 0$ is the coupling, α is an irrational frequency and $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a phase. For $\lambda > 1$ the Lyapunov exponent on the spectrum equals $\log\lambda > 0$ [6], but positivity alone does not imply localization.

*The authors developed this work with AI models from more than one developer and take responsibility for every claim. How the work was done is described at the end of the paper.

That picture depends on the arithmetic of α. Let p_n/q_n be the continued-fraction convergents of α and

$$\beta(\alpha) = \limsup_{n\to\infty} \frac{\log q_{n+1}}{q_n} \in [0, \infty]. \tag{2}$$

Avila, You and Zhou [2] proved that the supercritical regime splits sharply at $\lambda = e^{\beta(\alpha)}$. For $\lambda > e^{\beta}$ the operator has Anderson localization for almost every phase. For $1 < \lambda < e^{\beta}$ the spectrum is purely singular continuous for every phase. Strongly Liouville frequencies, those with $\beta(\alpha) > 0$, can therefore destroy localization even though the Lyapunov exponent stays positive.

What does a wave packet do in that window? For almost every phase, we determine every upper Abel-averaged transport moment exponent and both packing dimensions of every nonzero spectral measure; they equal the same number. The matching lower bounds hold at every phase. The word *upper* matters: the exponent describes the largest long-time bursts, not a power law valid at every time.

The main conclusions are easy to separate. First, the competition between $\log\lambda$ and $\beta(\alpha)$ gives an exact upper transport law on the phase set of Theorem A; when $\beta(\alpha) > \log\lambda$, almost all of the Abel-averaged packet reaches the radii and times specified in Theorem B, at every phase. Second, the same law determines packing dimensions under Theorem C's phase conditions and, for a packet started at one site, the upper generalized Rényi dimensions of orders $0 < r < 1$ under Theorem 10.2's phase conditions. Third, a computable frequency can produce a noncomputable exponent even though finite-time probabilities for computable data remain effectively approximable. This computability degree persists under computable changes of coupling in an interval; at coupling 2 and phase 1/4, however, any finite frequency prefix and any nonzero finite-time error tolerance permit every exponent in the class of Theorem D(2).

1.1 Transport exponents

Let $\psi_t = e^{-itH}\delta_0$ and average in time with the exponential weight $(2/T)e^{-2t/T}$:

$$a(j,T) = \frac{2}{T}\int_0^\infty e^{-2t/T}\,|\langle\delta_j,\psi_t\rangle|^2\,dt, \qquad P(N,T) = \sum_{|j|>N} a(j,T), \qquad M_p(T) = \sum_{j\in\mathbb{Z}} |j|^p a(j,T). \tag{3}$$

$P(N,T)$ is the time-averaged probability of finding the particle beyond distance N, and $M_p(T)$ is the averaged p-th moment of the position. The upper transport exponents are

$$\beta^+(p) = \limsup_{T\to\infty} \frac{\log M_p(T)}{p\log T}, \qquad \alpha_u^+ = \sup_{p>0}\beta^+(p). \tag{4}$$

The quantity α_u^+ equals the fastest upper exponent that Damanik and Tcheremchantsev define through outside probabilities, since $\beta^+(p)$ increases to that exponent as $p\to\infty$ [11, Section 2, equation (10) and the display before it]. It sees even polynomially small tails: if $P(T^a,T)$ decays faster than every power of T, then $\alpha_u^+ \le a$; if $P(T_j^a,T_j) \ge cT_j^{-C}$ for some $c>0$, $C<\infty$ and a sequence $T_j\to\infty$, then $\alpha_u^+ \ge a$. Always $0\le\alpha_u^+\le 1$, since hopping is bounded.

Two endpoints were known. Damanik and Tcheremchantsev proved $\alpha_u^+ = 0$ for $\lambda > 1$ when $\beta(\alpha) = 0$ [11], and Jitomirskaya and Zhang proved $\alpha_u^+ = 1$ for every phase when $\beta(\alpha) = \infty$ [18]. Jitomirskaya and Krüger proved exponential dynamical localization in phase expectation when $\beta(\alpha)$ is sufficiently small, including positive values [16], and Bradshaw, de Jong, Liu, Wang, Wang and Yang refined the growth rates for weakly Liouville frequencies with $\beta(\alpha) = 0$ [7]. In between, Jitomirskaya and Zhang obtained lower bounds of the form $1 - C\log\lambda/\beta$, for every moment [18]. Haeming refined them, with an explicit constant, a logarithmic loss and one sequence of times

common to all phases [15]. Upper bounds on fractal dimensions of spectral measures near the transition were obtained by Jitomirskaya, Liu and Tcheremchantsev [17]. The exact value was open.

1.2 Main results

For the exact upper law we use an explicit full-measure set of phases. Begin with

$$G_\alpha = \{\theta \in \mathbb{T} : \ |\sin(2\pi q_n \theta)| \geq q_n^{-2} \text{ for all sufficiently large } n\}. \tag{5}$$

This set has full Lebesgue measure for every irrational α (Lemma 2.1). We shall in fact work with the larger set

$$G_\alpha^* = \Big\{\theta \in \mathbb{T} : \begin{array}{l} \text{for every } c, \zeta > 0 \text{ there is } n_0 \text{ such that} \\ |\sin(2\pi q_n \theta)| \geq e^{-\zeta q_n} \text{ whenever } n \geq n_0 \text{ and } b_n \geq c \end{array} \Big\}. \tag{6}$$

Here b_n is defined in Section 2; a condition on a finite set of indices is vacuous, so $G_\alpha^* = \mathbb{T}$ when $\beta(\alpha) = 0$. Equivalently, for every $c > 0$ such that $b_n \geq c$ for infinitely many n, $q_n^{-1} \log|\sin(2\pi q_n \theta)| \to 0$ along those n, with $\log 0 = -\infty$. Only scales with $b_n \geq c$ enter, and along them $|\sin(2\pi q_n \theta)|$ may tend to zero at any subexponential rate. The complement of G_α^* contains the set $\frac{1}{2}(\mathbb{Z} + \alpha\mathbb{Z})$ of phases at which $H_{\lambda,\alpha,\theta}$ is reflection symmetric whenever $\beta(\alpha) > 0$, in particular $\theta = 0$ (Lemma 2.2). Throughout, the number $\max\{0, 1 - \log\lambda/\beta(\alpha)\}$ is read as 0 when $\beta(\alpha) = 0$ and as 1 when $\beta(\alpha) = \infty$.

Theorem A (The transport law)**.** *Let $\lambda > 1$ and let α be irrational.*

1. *If $\beta(\alpha) < \infty$, then for every $\theta \in G_\alpha^*$ and every $p > 0$,*

$$\beta^+(p) = \alpha_u^+ = \max\Big\{0, \ 1 - \frac{\log\lambda}{\beta(\alpha)}\Big\}$$

   for $H_{\lambda,\alpha,\theta}$, where the right side is read as 0 when $\beta(\alpha) = 0$. Moreover, for every $a > \max\{0, 1 - \log\lambda/\beta(\alpha)\}$ and every $m > 0$, $P(T^a, T) = O(T^{-m})$ as $T \to \infty$.

2. *If $\beta(\alpha) = \infty$, then $\beta^+(p) = \alpha_u^+ = 1$ for every phase and every $p > 0$.*

The lower bound $\beta^+(p) \geq \max\{0, 1 - \log\lambda/\beta(\alpha)\}$ in part 1 holds for every phase and every $p > 0$.

Part 2 is due to Jitomirskaya and Zhang [18, Corollary 3], and the case $\beta(\alpha) = 0$ of part 1 to Damanik and Tcheremchantsev [11]; we include them so that the statement covers every frequency. The new content is part 1 for $0 < \beta(\alpha) < \infty$: the upper bound throughout, and the matching lower bound for every moment when $\beta(\alpha) > \log\lambda$. For $0 < \beta(\alpha) < \infty$, the lower bound for the fastest exponent α_u^+ also holds for a Lipschitz sampling function whose Lyapunov exponent is continuous on a neighborhood of a spectral minimizer, with $\log\lambda$ replaced by that minimum (Theorem 8.1).

For $\beta(\alpha) < \infty$ and $\theta \in G_\alpha^*$, Theorem A identifies the competition between the localization rate $\log\lambda$ and the arithmetic rate $\beta(\alpha)$. If $0 < \beta(\alpha) < \infty$, the common upper exponent is zero for $\lambda \geq e^{\beta(\alpha)}$, including equality, and increases continuously to 1 as $\lambda \downarrow 1$. If $\beta(\alpha) = 0$, it is zero for every $\lambda > 1$; if $\beta(\alpha) = \infty$, it is 1 for every $\lambda > 1$ and every phase. Thus, at almost every phase, the finite-positive-beta spectral transition of [2] is also a sharp transition for upper transport exponents. Figure 1 plots the law.

Here is a heuristic for the exponent in the regime $0 < \log\lambda < \beta(\alpha) < \infty$. Write $\delta_n = q_n\alpha - p_n$ and $b_n = q_n^{-1}\log|\delta_n|^{-1}$. A shift by $q = q_n$ sites changes the sampling phase by $\delta = \delta_n$, so the nearly periodic potential drifts slowly from block to block. The conserved polynomial constructed below has a diagonal cosine whose phase changes by δ per lattice site, giving the spatial scale $|\delta|^{-1}$.

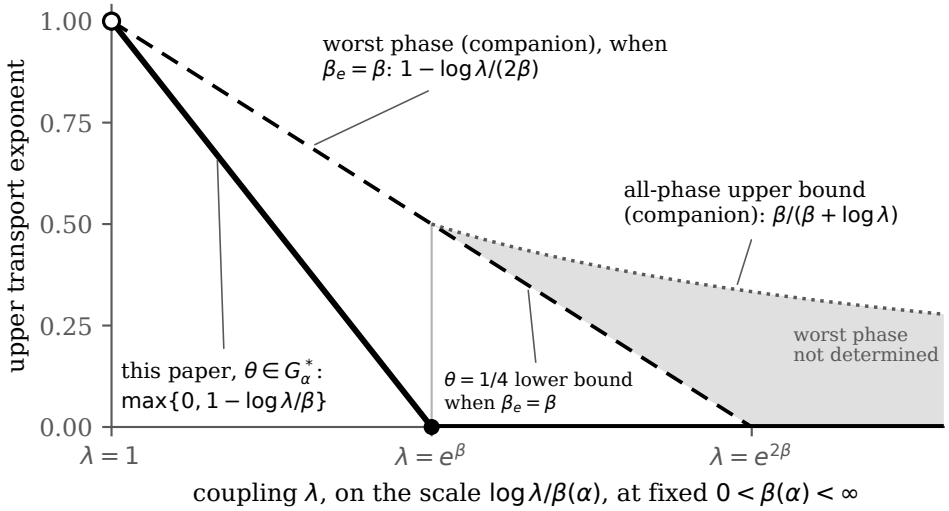


Figure 1: The exponent picture: a plot of proved formulas, not of numerical data. At a fixed $0 < \beta(\alpha) < \infty$ the horizontal coordinate is $\log\lambda/\beta(\alpha)$, so the marked points are $\lambda = 1$, $e^{\beta(\alpha)}$ and $e^{2\beta(\alpha)}$. Solid line: the law of this paper, $\max\{0, 1 - \log\lambda/\beta(\alpha)\}$, which for $\theta \in G_\alpha^*$ is the common value of every upper Abel-averaged transport exponent and of both packing dimensions of every nonzero spectral measure (Theorems A and C). It is positive for $1 < \lambda < e^{\beta(\alpha)}$ and zero at and above the threshold (closed point). The open point denotes the limit $\lambda \downarrow 1$; critical coupling is outside the theorems. For comparison, the other curves show fastest-exponent results of the worst-case companion [13], not of this paper. Dashed: $\max\{0, 1 - \log\lambda/(2\beta(\alpha))\}$, for frequencies with $\beta_e(\alpha) = \beta(\alpha)$, where β_e is the limsup of b_n along even denominators. For $1 < \lambda \le e^{\beta(\alpha)}$ it is then the exponent at the phase $1/4$ and the supremum over all phases [13, Theorem B(1)]; for $\lambda > e^{\beta(\alpha)}$ it is a lower bound at the phase $1/4$ [13, Theorem 7.5], while both that phase exponent and the supremum over phases are only bracketed by the released companion. Dotted: the all-phase upper bound $\beta(\alpha)/(\beta(\alpha) + \log\lambda)$ for $\lambda > e^{\beta(\alpha)}$; in the shaded region the worst phase is not determined. At $\beta(\alpha) = 0$ the upper exponents vanish for every $\lambda > 1$ at every phase, and at $\beta(\alpha) = \infty$ they equal 1 at every phase. These are upper exponents, not an instantaneous power law, and the zero branch does not assert bounded moments.

At such a scale the relevant band velocity is of exponential order λ^{-q}. Over the resonant time $T = |\delta|^{-1} = e^{b_n q}$, the corresponding distance is of exponential order $T\lambda^{-q} = T^{1-\log\lambda/b_n}$. Taking a subsequence with $b_n \to \beta(\alpha)$ suggests the formula in Theorem A. For phases in G_α^*, that theorem excludes faster polynomial tails at every sufficiently large averaging time. The next theorem shows, at every phase, that the Abel-averaged mass beyond Te^{-hq_n} tends to one in its stated resonant windows, for every fixed $h > \log\lambda$ and $c > h$.

Theorem B (Almost all of the packet moves)**.** *Let $\lambda > 1$ and let α be irrational. Let $h > \log\lambda$ and $c > h$, and assume $b_n \ge c$ for infinitely many n. Then*

$$\lim_{\substack{n\to\infty \\ b_n \ge c}} \inf_{\theta\in\mathbb{T}} \inf_{e^{hq_n} \le T \le |\delta_n|^{-1}} P_\theta(Te^{-hq_n}, T) = 1,$$

where P_θ is the outside probability for $H_{\lambda,\alpha,\theta}$ and the initial state δ_0. The same holds, uniformly in $j \in \mathbb{Z}$, for the initial state δ_j, with $|k| > Te^{-hq_n}$ replaced by $|k - j| > Te^{-hq_n}$.

At the resonant time $T = |\delta_n|^{-1}$ the radius in Theorem B is $e^{(b_n-h)q_n} = T^{1-h/b_n}$. For $\beta(\alpha) > \log\lambda$, take scales with $b_n \to \beta(\alpha)$ and then let $h \downarrow \log\lambda$; this gives the lower bound in Theorem A for every positive moment using the same sequence of averaging times. Quantitatively, for fixed $0 < \eta \le 1/4$, Theorem 9.8 bounds the outside probability below by

$$(1 - C_\eta q_n e^{-(h-\log\lambda)q_n/2})\mu^\theta(\{|\Delta_n| \le 2 - 3\eta\}),$$

and the second factor tends uniformly in θ to $\frac{2}{\pi}\arcsin(1-\frac{3\eta}{2})$. The estimate discards the spectral mass near the band edges and an error tending to zero; letting $\eta \downarrow 0$ after the scale limit gives probability tending to one. For $\theta \in G_\alpha^*$ and $\beta(\alpha) < \infty$, fix $c > h > \log\lambda$ and $\varepsilon, M > 0$. At all sufficiently large scales with $b_n \ge c$, uniformly for $e^{hq_n} \le T \le |\delta_n|^{-1}$,

$$\sum_{Te^{-hq_n} < |j| \le 2e^{\varepsilon q_n} T\lambda^{-q_n}} a(j,T) = 1 - o(1), \qquad P(2e^{\varepsilon q_n} T\lambda^{-q_n}, T) \le Ce^{-Mq_n}.$$

Thus every fixed quantile lies at radius $T\lambda^{-q_n}$ within arbitrarily small fixed positive exponential losses in q_n (Remark 9.10). These conclusions concern Abel averages and do not specify a packet profile or an instantaneous velocity.

The same number governs the spectral measures. For a finite Borel measure μ on $\mathbb{R}$ and $E \in \mathbb{R}$, the upper local dimension is

$$d_\mu^+(E) = \limsup_{r\downarrow 0} \frac{\log\mu([E-r, E+r])}{\log r}.$$

The lower and upper packing dimensions $\dim_P^-(\mu)$ and $\dim_P^+(\mu)$ are, respectively, the μ-essential infimum and supremum of d_μ^+. For $\varphi \in \ell^2(\mathbb{Z})$, let μ_φ be its spectral measure for $H_{\lambda,\alpha,\theta}$:

$$\mu_\varphi(B) = \langle \varphi, \mathbf{1}_B(H_{\lambda,\alpha,\theta})\varphi\rangle \qquad (B \subset \mathbb{R} \text{ Borel}).$$

Theorem C (Packing dimension)**.** *Let $\lambda > 1$ and let α be irrational.*

1. *For every phase θ and every nonzero $\varphi \in \ell^2(\mathbb{Z})$, $d_{\mu_\varphi}^+(E) \ge \max\{0, 1 - \log\lambda/\beta(\alpha)\}$ for μ_φ-almost every E, with the right side read as 0 when $\beta(\alpha) = 0$ and as 1 when $\beta(\alpha) = \infty$. In particular $\dim_P^-(\mu_\varphi)$ is at least this number.*

2. *If $\beta(\alpha) < \infty$ and $\theta \in G_\alpha^*$, then for every nonzero $\varphi \in \ell^2(\mathbb{Z})$,*

$$d_{\mu_\varphi}^+(E) = \max\Big\{0,\ 1 - \frac{\log\lambda}{\beta(\alpha)}\Big\} \quad \text{for } \mu_\varphi\text{-a.e. } E,$$

so that $\dim_P^-(\mu_\varphi) = \dim_P^+(\mu_\varphi) = \max\{0, 1 - \log\lambda/\beta(\alpha)\}$, read as 0 when $\beta(\alpha) = 0$.

When $\beta(\alpha) = \infty$, part 1 gives $d_{\mu_\varphi}^+ \ge 1$ almost everywhere at every phase. Every finite Borel measure on $\mathbb{R}$ has upper packing dimension at most 1, since this dimension is the infimum of the packing dimensions of its full-measure Borel sets [14, Appendix]. So all these packing dimensions equal 1; this endpoint is due to Jitomirskaya and Zhang [18, Corollary 3].

The set G_α^* has full measure and is invariant under $\theta \mapsto \theta + \alpha$ (Lemma 2.2). So for this model the fastest-tail exponent, every moment exponent and the packing dimension coincide: if $\beta(\alpha) < \infty$ and $\theta \in G_\alpha^*$, then for all $p > 0$ and all nonzero φ,

$$\alpha_u^+ = \beta^+(p) = \dim_P^-(\mu_\varphi) = \dim_P^+(\mu_\varphi) = \max\Big\{0,\ 1 - \frac{\log\lambda}{\beta(\alpha)}\Big\}.$$

For $0 < \log\lambda < \beta(\alpha) < \infty$, Jitomirskaya, Liu and Tcheremchantsev proved the upper packing bound $2(1 - \log\lambda/\beta(\alpha))$ for every α-Diophantine phase, a full-measure set, and noted that the factor 2 may not be sharp [17, Theorem 1.1 and Remark 1.1]. Theorem C shows that on G_α^* the sharp factor is 1, and that the bound is attained: the upper and lower packing dimensions both equal $1 - \log\lambda/\beta$. Their phase set is contained in ours. Let $\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \kappa|k|^{-\nu}$ for all $k \neq 0$. If $\beta(\alpha) = 0$ there is nothing to prove, since $G_\alpha^* = \mathbb{T}$. If $\beta(\alpha) > 0$, then $2\theta \notin \mathbb{Z}$, since $\|q_n\alpha\| = |\delta_n|$ is exponentially small along infinitely many n. For large n let j be the integer nearest to $2q_n\theta$, and let $k \in (-q_n/2, q_n/2]$ satisfy $kp_n \equiv j \pmod{q_n}$; then $k \neq 0$ (otherwise $q_n \mid j$ and $\|2\theta\|_{\mathbb{R}/\mathbb{Z}} \leq \|2q_n\theta\|_{\mathbb{R}/\mathbb{Z}}/q_n \leq 1/(2q_n)$), and $\kappa(q_n/2)^{-\nu} \leq \|2\theta - k\alpha\| \leq \|2q_n\theta\|/q_n + |\delta_n|/2$. Along scales with $b_n \geq c$ this gives $|\sin(2\pi q_n\theta)| \geq 2\|2q_n\theta\| \geq 2^\nu\kappa q_n^{1-\nu}$ for large n, so $\theta \in G_\alpha^*$. Hence the conclusion of Theorem C holds at every phase covered by their theorem. For $\lambda \geq e^{\beta(\alpha)}$ with $\beta(\alpha) < \infty$, their theorem already gives dimension zero at every α-Diophantine phase. In the other direction, Jitomirskaya and Zhang proved lower bounds of the form $1 - C\log\lambda/\beta$ for packing dimensions [18]. Part 1 gives this bound with $C = 1$ at every phase, and part 2 shows that $C = 1$ cannot be improved.

The upper bound in part 2 follows from Theorem A and the inequality between moment growth and packing dimension of Guarneri and Schulz-Baldes [14], in the form of Barbaroux, Germinet and Tcheremchantsev [3]. Part 1 does not follow from transport bounds, since those inequalities go the other way. It comes from a pointwise bound on the resolvent at the resonant scales (Lemma 9.13), which the proof of Theorem B also provides.

The last result concerns computability. The exponent in Theorem A is an explicit function of $\beta(\alpha)$, and $\beta(\alpha)$ is an asymptotic quantity of the continued fraction: it can be noncomputable even when α is computable. A real is computable if rational approximations to any prescribed accuracy can be produced by an algorithm. Two reals have the same Turing degree if each can be computed using the other as an oracle. We write $\mathbf{0}'$ for the halting problem and $\mathbf{0}''$ for the halting problem with an oracle for $\mathbf{0}'$. A real x is *right-c.e. relative to* $\mathbf{0}'$ if the set of rationals $r > x$ is computably enumerable relative to the halting problem.

Theorem D (Computability)**.** *1. Let $\lambda > 1$ be computable, let α be a computable irrational with $\beta(\alpha) < \infty$, and let $\theta \in G_\alpha^*$ be computable. Then $\alpha_u^+(H_{\lambda,\alpha,\theta})$ is computable from $\beta(\alpha)$. If moreover $\beta(\alpha) > \log\lambda$, then α_u^+ and $\beta(\alpha)$ have the same Turing degree.*

2. Take $\lambda = 2$ and $\theta = 1/4$. As α ranges over the computable irrationals whose convergent denominators are eventually odd, $\alpha_u^+(H_{2,\alpha,1/4})$ ranges over exactly the reals in $[0, 1]$ that are right-c.e. relative to $\mathbf{0}'$.

3. There is an explicit computable irrational α such that $\alpha_u^+(H_{2,\alpha,1/4}) \in [11/23, 1/2]$ has Turing degree $\mathbf{0}''$.

All Hamiltonian data in part 3 are computable: the coupling 2, the phase $1/4$ and every digit of α. Yet the transport exponent of this explicit operator cannot be computed even with an oracle for the halting problem. Undecidability of physical quantities is known for families of Hamiltonians, most prominently the spectral gap of Cubitt, Perez-Garcia and Wolf [9]. Here a single explicit instance of a textbook one-dimensional model has an exponent of exactly known, high Turing degree. Part 2 identifies the exact set of values, the reals that are right-c.e. relative to $\mathbf{0}'$. All of them are computable from $\mathbf{0}''$, and part 3 shows that degree $\mathbf{0}''$ is attained.

1.3 How the proof works

There are two lower-bound arguments. Section 8 treats the fastest exponent under the hypotheses of Theorem 8.1. For the almost Mathieu operator with $0 < \log\lambda < \beta(\alpha) < \infty$, choose $\log\lambda < h < \beta(\alpha)$ and a resonant subsequence with $b_n \to \beta(\alpha)$. On a short interval of energies, the complex-energy transfer matrices have norm at most Ce^{2hq_n} over a distance $N_n = e^{(b_n-h)q_n+O(\log q_n)}$, with averaging times $T_n = e^{b_nq_n+O(\log q_n)}$. Parseval's formula then gives a polynomially small Abel-averaged mass beyond $N_n/2$. This suffices for α_u^+, but not for the sharp lower bound on each fixed moment. Section 9 supplies that bound by proving that almost all of the Abel-averaged mass escapes beyond the radii of Theorem B.

The upper bound is the main work. Its key object is an exactly conserved quantity at each scale. Chambers' formula [8] says that the discriminant of the p/q-periodic approximant depends on the phase only through the energy-independent term $-2\lambda^q\cos(2\pi q\theta)$. Combined with the periodic magic formula of Damanik, Killip and Simon [10], this gives a degree-q polynomial P_q with

$$P_q(H) = V + K, \qquad V(j) = 2\lambda^q\cos\bigl(2\pi(q\theta + j\,\delta)\bigr), \qquad \delta = q\alpha - p,$$

where K has range q (Proposition 3.3). At every denominator with $|q\alpha - p| \le e^{-cq}$ for a fixed $c > 0$, K is exponentially small compared with λ^q (Lemma 3.4). The operator $P_q(H)$ commutes with H exactly, so no error accumulates in time. Its diagonal V is a cosine of amplitude $2\lambda^q$ that drifts slowly in space. Three estimates are built on it.

1. *Barriers* (Section 4). On the spectral subspace where $P_q(H)$ is near its initial value, the monotone part of the drifting cosine acts as a wall. A Sylvester-equation argument bounds the mass that crosses it, at all times. Dividing $P_q(H)$ by the size of K makes this work at every scale where $|q\alpha - p|$ is exponentially small, even where K itself is large.

2. *Exact slow dynamics* (Section 5). At scales with $|q\alpha - p| \le e^{-(\log\lambda+\eta)q}$, smooth filters on the inverse branches of P_q turn the energy into a smooth function of $P_q(H)/\lambda^q$. On each filtered piece the evolution is exactly generated by an operator whose position commutators give an upper velocity bound of order λ^{-q}, up to arbitrarily small fixed positive exponential losses in q, under the scale and sine conditions of Theorem 5.4. The filtered evolution identity holds for every time.

3. *A per-denominator estimate* (Section 6). For every sufficiently large denominator q_n, the complex-energy method of Damanik and Tcheremchantsev [11] bounds the Abel-averaged mass beyond C_Dq_n by $Ce^{-c_DC_Dq_n} + CT^3e^{-\kappa q_n}$. In particular, this mass is exponentially small in q_n for $1 \le T \le e^{\kappa_0 q_n}$ with any fixed $0 < \kappa_0 < \kappa/3$.

The final step (Section 7) is a covering argument. For each large time T, look at the first denominator $q_j \ge K\log T$. If it is not too large, the per-denominator estimate confines the packet. If it is very large, the preceding denominator has a definite Liouville ratio and T lies in a fixed logarithmic window of that scale, where the barrier and slow-dynamics estimates apply. No assumption on how the ratios $\log q_{n+1}/q_n$ behave between these scales is needed (Figure 2).

The bulk lower bound in Section 9 uses the conserved polynomial again, at scales with $b_n \ge c > h > \log\lambda$. Near the origin, $P_q(H) - 2\lambda^q\cos(2\pi q\theta)$ approximates $S^q + S^{-q}$, whose powers have diagonal matrix elements equal to the moments of the arcsine law on $[-2, 2]$. The image of the single-site spectral measure under the rational discriminant therefore converges to that law, uniformly in the phase (Lemma 9.1). After excluding a small spectral mass near the band edges, the first q-blocks on both sides are elliptic with a definite margin. The resolvent column, restricted to

either half-line, is the square-summable solution there, called the Weyl solution. For $e^{hq} \le T \le |\delta|^{-1}$ it remains spread over a long run of consecutive blocks. A pair inequality (Lemma 9.5) prevents any polarization from avoiding two adjacent blocks (Lemma 9.6). Comparing the few near blocks with the far mass shows that the mass within Te^{-hq} is negligible. Parseval's formula and Poisson smoothing convert this comparison into Abel-averaged outside probability tending to one, after the band-edge tolerance is sent to zero. At $T = |\delta|^{-1}$, the same runs bound the diagonal resolvent and hence the spectral mass in intervals of radius $|\delta|$, giving the lower bound on upper local dimensions.

1.4 Organization

Section 2 fixes notation and collects elementary facts. Section 3 constructs the conserved polynomial. Section 4 proves the barrier estimates and Section 5 the window theorem for slow dynamics. Section 6 records the per-denominator estimate. Section 7 proves the upper bound in Theorem A, and Section 8 a lower bound for α_u^+ that holds for Lipschitz potentials. Section 9 proves the bulk lower bound, in which the escaping fraction of the Abel-averaged packet tends to one, and completes the proofs of Theorems A, B and C. Section 10 determines generalized Rényi dimensions of orders between zero and one. Section 11 proves Theorem D and its coupling and frequency consequences. Section 12 gives transport at individual times and explicit finite-volume error bounds. Section 13 discusses what remains open.

2 Preliminaries

2.1 Notation

Throughout, $\lambda > 1$ is fixed, $L = \log \lambda > 0$, $v(x) = 2\lambda \cos(2\pi x)$, and $H = H_{\lambda,\alpha,\theta}$ is the operator (1). We write S for the shift $(Su)(n) = u(n+1)$, X for the position operator $X\delta_j = j\delta_j$, and $\mathbf{1}_A$ for multiplication by the indicator of $A \subset \mathbb{Z}$. For a frequency $\omega \in \mathbb{R}$ and a phase φ we also use $H_{\lambda,\omega,\varphi}$ with ω rational. The spectrum Σ of $H_{\lambda,\alpha,\theta}$ does not depend on θ when α is irrational, since lattice translations conjugate $H_{\lambda,\alpha,\theta}$ to $H_{\lambda,\alpha,\theta+k\alpha}$, the orbit $\theta + k\alpha$ is dense, and the operator depends norm-continuously on the phase. Matrix and operator norms are induced by the Hilbert-space norm unless indicated otherwise; $\|M\|_{\mathrm{HS}} = (\operatorname{tr} M^*M)^{1/2}$ is the Hilbert–Schmidt norm. We write $[A, B] = AB - BA$, $\operatorname{ad}_X^0 B = B$ and $\operatorname{ad}_X^{r+1} B = [X, \operatorname{ad}_X^r B]$.

Transfer matrices. For $E \in \mathbb{C}$ and $x \in \mathbb{T}$ put $T(E,x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix}$ and, for $r \ge 0$,

$$A_r(E,x) = T(E, x + (r-1)\alpha) \cdots T(E,x), \qquad A_0 = I.$$

A solution of $Hu = Eu$ satisfies $(u(r), u(r-1))^{\mathsf{T}} = A_r(E,\theta)(u(0), u(-1))^{\mathsf{T}}$. For a frequency ω other than α we write $A_r^{(\omega)}$. The Lyapunov exponent is $L(E) = \lim_r r^{-1} \int_{\mathbb{T}} \log \|A_r(E,x)\|\, dx = \inf_r r^{-1} \int_{\mathbb{T}} \log \|A_r(E,x)\|\, dx$. For $\lambda > 1$ it equals $L = \log \lambda$ at every $E \in \Sigma$ [6, Corollary 2].

Continued fractions. Let p_n/q_n be the convergents of α and put

$$\delta_n = q_n\alpha - p_n, \qquad D_n = |\delta_n|^{-1}, \qquad \beta_n = \frac{\log q_{n+1}}{q_n}, \qquad b_n = \frac{\log D_n}{q_n}.$$

The classical inequalities $(q_{n+1} + q_n)^{-1} < |\delta_n| < q_{n+1}^{-1}$ give

$$q_{n+1} < D_n < 2q_{n+1}, \qquad 0 < b_n - \beta_n < \frac{\log 2}{q_n}. \tag{7}$$

In particular $\limsup b_n = \limsup \beta_n = \beta(\alpha)$. Denominators grow at least like Fibonacci numbers, and $|\alpha - p_n/q_n| < q_n^{-2}$. When the scale n is fixed we drop the index and write q, p, δ, b, and $A = \lambda^q$. We also write $s = |\sin(2\pi q\theta)|$.

All constants below may depend on λ, on α and on parameters that are explicitly fixed. They never depend on the phase θ, the scale n or the time T, unless this is indicated. Thresholds of the form "for all sufficiently large n" are likewise uniform in the phase in Sections 3–6, among phases satisfying the stated sine condition at that scale. In Section 7 the phase is fixed and thresholds may depend on it.

2.2 The phase set

Lemma 2.1. *For every irrational α the set G_α in (5) has full Lebesgue measure.*

Proof. The map $\theta \mapsto q_n\theta$ preserves Lebesgue measure on $\mathbb{T}$, and $\{x \in \mathbb{T} : |\sin 2\pi x| < u\}$ has measure at most u. Hence $\{\theta : |\sin 2\pi q_n\theta| < q_n^{-2}\}$ has measure at most q_n^{-2}. Since $\sum_n q_n^{-2} < \infty$, the Borel–Cantelli lemma shows that almost every θ lies in only finitely many of these sets. □

Lemma 2.2. *Let α be irrational.*

1. *$G_\alpha \subset G_\alpha^*$; in particular G_α^* has full measure.*
2. *$G_\alpha^* + \alpha = G_\alpha^*$.*
3. *If $\beta(\alpha) > 0$, then $\frac{1}{2}(\mathbb{Z} + \alpha\mathbb{Z}) \cap G_\alpha^* = \emptyset$.*

Proof. Write $s_n(\theta) = |\sin(2\pi q_n\theta)| \le 1$.

1. If $s_n \ge q_n^{-2}$ then $0 \ge q_n^{-1} \log s_n \ge -2q_n^{-1} \log q_n \to 0$.

2. Let $\theta \in G_\alpha^*$ and $c > 0$. For $b_n \ge c$ we have $|\delta_n| = e^{-b_n q_n} \le e^{-cq_n}$, and $q_n(\theta \pm \alpha) = q_n\theta \pm p_n \pm \delta_n$, so

$$s_n(\theta \pm \alpha) \ge s_n(\theta) - 2\pi|\delta_n| \ge s_n(\theta) - 2\pi e^{-cq_n}.$$

Since $s_n(\theta) \ge e^{-cq_n/2}$ for large n with $b_n \ge c$, the right side is at least $s_n(\theta)/2$ for large such n. Hence $0 \ge q_n^{-1} \log s_n(\theta \pm \alpha) \ge q_n^{-1} \log s_n(\theta) - q_n^{-1} \log 2 \to 0$ along $b_n \ge c$. So $\theta \pm \alpha \in G_\alpha^*$.

3. Let $\theta = (m + k\alpha)/2$ with $m, k \in \mathbb{Z}$. Then $2q_n\theta = mq_n + kp_n + k\delta_n$, so $s_n(\theta) = |\sin(\pi k\delta_n)| \le \pi|k||\delta_n|$. If $\beta(\alpha) > 0$, choose any finite c with $0 < c < \beta(\alpha)$, for instance $c = 1$ if $\beta(\alpha) = \infty$; there are infinitely many n with $b_n \ge c$, and along them $q_n^{-1} \log s_n \le q_n^{-1} \log(\pi|k| + 1) - b_n \le -c/2$ for large n (if $k = 0$, then $s_n = 0$). So $\theta \notin G_\alpha^*$. □

2.3 Moments

Lemma 2.3.

1. *For every $p > 0$ there is C_p, depending only on p, such that $M_p(T) \le C_p(1+T)^p$ for all $T > 0$ and every bounded potential.*
2. *For $0 < a < 1$ and $T \ge 1$, $M_p(T) \le T^{ap} + \big(M_{2p}(T)\, P(T^a, T)\big)^{1/2}$.*
3. *If $0 < a < 1$ and $P(T^a, T) = O(T^{-m})$ for every $m > 0$, then $\alpha_u^+ \le a$.*

Proof. 1. Let $k \ge 1$ be an integer. We have $[X, H] = S^{-1} - S$ and $[X^k, H] = \sum_{i=0}^{k-1} X^i [X, H] X^{k-1-i}$. Moving the shifts to the right with $S^{\pm 1} X = (X \pm 1) S^{\pm 1}$ writes $[X^k, H]$ as $\sum_\pm c_\pm(X) S^{\pm 1}$, where $c_\pm$ are polynomials of degree at most $k - 1$ with coefficients bounded by a constant depending only on k. Since $X\delta_0 = 0$, Duhamel's formula $X^k \psi_t = -i \int_0^t e^{-i(t-s)H} [X^k, H] \psi_s \, ds$ and induction on k give

$\|X^k\psi_t\| \le C_k(1+|t|)^k$. For real $p>0$ choose an integer k with $2k \ge p$. Jensen's inequality for the probability distribution $|\psi_t(j)|^2$ gives

$$\sum_j |j|^p |\psi_t(j)|^2 \le \Big(\sum_j |j|^{2k}|\psi_t(j)|^2\Big)^{p/(2k)} = \|X^k\psi_t\|^{p/k} \le C(1+|t|)^p.$$

Averaging against $(2/T)e^{-2t/T}$ proves the claim.

2. Split $M_p(T)$ into $|j| \le T^a$ and $|j| > T^a$. The first part is at most T^{ap}, because $\sum_j a(j,T) = 1$. Apply the Cauchy–Schwarz inequality to the second part, for the probability measure $a(\cdot,T)$ on $\mathbb{Z}$.

3. By parts 1 and 2, $M_p(T) \le T^{ap} + C(1+T)^p T^{-m/2}$. Taking $m \ge 2p$ gives $M_p(T) \le CT^{ap}$, so $\beta^+(p) \le a$ for every p. □

2.4 Uniform growth near the spectrum

Lemma 2.4. *For every $h > L$ there are a bounded open set $U_h \supset \Sigma$ and a constant C_h such that*

$$\|A_r(E,x)\| \le C_h e^{hr} \qquad \text{for all real } E \in U_h,\ x \in \mathbb{T},\ r \ge 0.$$

Proof. Fix $E_0 \in \Sigma$ and $h_1 = (L+h)/2$. Since $L(E_0) = L < h_1$, there is r_0 with

$$\frac{1}{r_0}\int_{\mathbb{T}} \log\|A_{r_0}(E_0,x)\|\,dx < h_1.$$

The function $F_E(x) = \log\|A_{r_0}(E,x)\|$ is jointly continuous, so the same strict inequality holds for E in a compact neighborhood N_0 of E_0. Rotation by $r_0\alpha$ is uniquely ergodic, so Birkhoff averages of the continuous functions F_E converge to $\int F_E$ uniformly in x; by equicontinuity of $\{F_E\}_{E\in N_0}$ they converge uniformly in $E \in N_0$ as well. Hence there is m_0 with $\sum_{k<m} F_E(x+kr_0\alpha) \le mr_0h$ for all $m \ge m_0$, $x \in \mathbb{T}$ and $E \in N_0$. Writing a product of length $r = mr_0 + s$ with $0 \le s < r_0$ as m blocks of length r_0 and a remainder, submultiplicativity gives $\|A_r(E,x)\| \le Ce^{hr}$ on N_0. A finite cover of the compact set Σ by the interiors of such neighborhoods gives U_h and C_h. □

2.5 Rational spectra

For a rational p/q in lowest terms let $\Sigma_{p/q} = \bigcup_{\varphi\in\mathbb{T}} \sigma(H_{\lambda,p/q,\varphi})$.

Lemma 2.5. *Let H_0 be a bounded Schrödinger operator on $\ell^2(\mathbb{Z})$ with hopping one and real potential, and let $E \in \sigma(H_0)$. For every integer $M \ge 1$ there is a unit vector u supported in an interval of length $2M$ with $\|(H_0-E)u\| \le 8/M$.*

Proof. By Weyl's criterion there is a finitely supported vector w with $\|(H_0-E)w\| \le M^{-1}\|w\|$. For $k \in \mathbb{Z}$ let $\chi_k(n) = \sin(\pi(n-kM)/(2M))$ for $kM \le n \le (k+2)M$ and $\chi_k(n) = 0$ otherwise. Then $\sum_k \chi_k(n)^2 = 1$, because on $[kM,(k+1)M]$ the two windows χ_k and χ_{k-1} are a sine and a cosine of the same angle. Also $|\chi_k(n+1) - \chi_k(n)| \le \pi/(2M)$, and for each n at most three indices k have $\chi_k(n)$ or $\chi_k(n+1)$ nonzero. Since $([S,\chi_k]w)(n) = (\chi_k(n+1)-\chi_k(n))w(n+1)$, this gives $\sum_k \|[S^{\pm1},\chi_k]w\|^2 \le 3\pi^2(4M^2)^{-1}\|w\|^2$. Hence $\sum_k \|\chi_k w\|^2 = \|w\|^2$ and

$$\sum_k \|(H_0-E)\chi_k w\|^2 \le 2\sum_k \|\chi_k(H_0-E)w\|^2 + 2\sum_k \|[H_0,\chi_k]w\|^2 \le \frac{2+6\pi^2}{M^2}\,\|w\|^2.$$

Some k therefore has $\|(H_0-E)\chi_k w\| \le (8/M)\|\chi_k w\|$. □

Lemma 2.6. *There is C such that for every rational p/q:*

1. *every point of $\Sigma_{p/q}$ is within $C|\alpha - p/q|^{1/2}$ of Σ, and every point of Σ is within $C|\alpha - p/q|^{1/2}$ of $\Sigma_{p/q}$;*

2. *for every φ, every point of $\Sigma_{p/q}$ is within $2\pi\lambda/q$ of $\sigma(H_{\lambda,p/q,\varphi})$.*

Proof. 1. Let $E \in \sigma(H_{\lambda,p/q,\varphi})$ and take u from Lemma 2.5, supported in $[c-M, c+M]$. Choose θ' with $\theta' + c\alpha = \varphi + cp/q$. For $|n-c| \le M$ the potentials of $H_{\lambda,\alpha,\theta'}$ and $H_{\lambda,p/q,\varphi}$ differ by at most $4\pi\lambda M|\alpha - p/q|$. Hence $\operatorname{dist}(E,\Sigma) \le 8/M + 4\pi\lambda M|\alpha - p/q|$, and $M = \lceil |\alpha - p/q|^{-1/2} \rceil$ gives the first claim. The second claim is proved in the same way with the roles exchanged.

2. Translation by k sites conjugates $H_{\lambda,p/q,\varphi}$ to $H_{\lambda,p/q,\varphi+kp/q}$, and $\{kp/q\}$ is the full grid $q^{-1}\mathbb{Z}/\mathbb{Z}$. Every phase φ' is within $1/(2q)$ of a grid translate of φ, and the potential is $4\pi\lambda$-Lipschitz in the phase. □

3 The conserved polynomial

3.1 Rational input

For a rational p/q in lowest terms, the operator $H_{\lambda,p/q,\varphi}$ is q-periodic. Its discriminant is $\Delta_\varphi(E) = \operatorname{tr} A_q^{(p/q)}(E,\varphi)$, and its spectrum is $\{E : |\Delta_\varphi(E)| \le 2\}$.

Lemma 3.1 (Chambers' formula [8])**.** *There is a real monic polynomial $P_{p/q}$ of degree q such that*

$$\Delta_\varphi(E) = P_{p/q}(E) - 2\lambda^q \cos(2\pi q\varphi) \qquad \text{for all } E \in \mathbb{C},\ \varphi \in \mathbb{T}.$$

Proof. Each factor $T(E, \varphi + kp/q)$ is a trigonometric polynomial of degree one in φ, so $\Delta_\varphi(E)$ is a trigonometric polynomial of degree at most q in φ. The shift $\varphi \mapsto \varphi + p/q$ cyclically permutes the factors and leaves the trace unchanged. Since $\gcd(p,q) = 1$, the trace is invariant under $\varphi \mapsto \varphi + 1/q$, so only the Fourier modes 0 and $\pm q$ survive. The coefficient of $e^{2\pi i q\varphi}$ comes only from the product of the diagonal entries and equals $(-\lambda)^q e^{2\pi i p\, q(q-1)/(2q)} = (-1)^{q+p(q-1)}\lambda^q$. This is $-\lambda^q$ in all cases: if q is odd then $(-1)^q = -1$ and $p(q-1)$ is even, and if q is even then p is odd and $(-1)^{p(q-1)} = -1$. The coefficient of $e^{-2\pi i q\varphi}$ is its complex conjugate. The constant mode is a monic polynomial of degree q in E with real coefficients. □

Lemma 3.2 (Periodic magic formula [10])**.** *Let H_{per} be a q-periodic Schrödinger operator on $\ell^2(\mathbb{Z})$ with hopping one and discriminant Δ. Then $\Delta(H_{\mathrm{per}}) = S^q + S^{-q}$.*

This is the elementary direction of [10, Theorem 3.1], proved there by Floquet decomposition (equations (3.5)–(3.6)). The hypotheses hold: the operator is bounded, two-sided, periodic and has positive off-diagonal entries. Combining the two lemmas,

$$P_{p/q}(H_{\lambda,p/q,\varphi}) = S^q + S^{-q} + 2\lambda^q \cos(2\pi q\varphi). \tag{8}$$

3.2 The irrational operator

Proposition 3.3. *Fix $h > L$. There is C_h such that for all sufficiently large n and every phase θ the following holds, with $q = q_n$, $p = p_n$, $\delta = \delta_n$ and $P_q = P_{p/q}$.*

1. *The operator $Q = P_q(H_{\lambda,\alpha,\theta})$ commutes with $H_{\lambda,\alpha,\theta}$ and decomposes as*

$$Q = V + K, \qquad V(j) = 2\lambda^q \cos\bigl(2\pi(q\theta + j\delta)\bigr), \qquad K = S^q + S^{-q} + R,$$

where V is diagonal, R is self-adjoint, $\langle \delta_i, R\,\delta_j\rangle = 0$ for $|i-j| > q$, and

$$\|R\| \le C_h\, q^{5/2} e^{hq} |\delta|.$$

2. *For every fixed integer $r \geq 0$ there is $C_{r,h}$ with $\sup_{E\in U_h} |P_q^{(r)}(E)| \leq C_{r,h}\, q^r e^{hq}$, where U_h is the neighborhood of Lemma 2.4.*

Proof. Step 1: rational products. Let $d = 4\pi\lambda|\delta|$. For $0 \leq k < q$, the potentials $v(\varphi + kp/q)$ and $v(\varphi + k\alpha)$ differ by at most $4\pi\lambda k|\alpha - p/q| \leq d$. Write each rational transfer matrix as the irrational one plus an error of norm at most d, and expand the product of $r \leq q$ factors. A term with k errors contains at most $k+1$ maximal irrational segments, each bounded by Lemma 2.4. Hence for real $E \in U_h$ and every phase,

$$\|A_r^{(p/q)}(E,\varphi)\| \leq \sum_{k=0}^{r} \binom{r}{k} C_h^{k+1} e^{h(r-k)} d^k = C_h e^{hr}\big(1 + C_h d e^{-h}\big)^r \leq C_h' e^{hr}, \tag{9}$$

because $rd \leq 4\pi\lambda q|\delta| \leq 4\pi\lambda$. The same bound holds for every sub-block. Differentiating $A_q^{(p/q)}(E,\varphi)$ r times in E inserts the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ at r positions. There are at most q^r ordered choices, and each resulting product is bounded by $C^{r+1}e^{hq}$ by (9). Since $P_q(E) = \Delta_\varphi(E) + 2\lambda^q \cos(2\pi q\varphi)$ and $\lambda^q \leq e^{hq}$, this proves part 2.

Step 2: a smooth cutoff. Let U' be an open set with $\Sigma \subset U' \subset \overline{U'} \subset U_h$, and fix a smooth compactly supported χ with $\chi = 1$ on U' and $\operatorname{supp}\chi \subset U_h$. Put $f_q = \chi P_q$. By part 2, $\|f_q\|_{H^2(\mathbb{R})} \leq Cq^2 e^{hq}$. For bounded self-adjoint operators B, C, Fourier inversion and $\|e^{itB} - e^{itC}\| \leq |t|\,\|B - C\|$ give

$$\|f_q(B) - f_q(C)\| \leq \frac{\|B-C\|}{2\pi} \int_{\mathbb{R}} |t|\,|\hat{f}_q(t)|\,dt \leq c\,\|B-C\|\,\|f_q\|_{H^2(\mathbb{R})},$$

by the Cauchy–Schwarz inequality and $\int t^2(1+t^2)^{-2}\,dt < \infty$.

Step 3: rows. Fix $j \in \mathbb{Z}$. Let $\varphi_j = \theta + j(\alpha - p/q)$ and $B_j = H_{\lambda,p/q,\varphi_j}$, whose potential at site k is $v(\theta + j\alpha + (k-j)p/q)$. Let C_j be B_j with its potential on $[j-q, j+q]$ replaced by that of $H_{\lambda,\alpha,\theta}$. Those potentials differ by at most $4\pi\lambda|k-j|\,|\alpha - p/q| \leq d$, so $\|B_j - C_j\| \leq d$. By Lemma 2.6, $\sigma(B_j) \subset \Sigma_{p/q}$ lies within $C|\alpha - p/q|^{1/2}$ of Σ, and $\sigma(C_j)$ lies within d of $\sigma(B_j)$. For all large n both spectra lie in U', uniformly in j and θ, so $f_q(B_j) = P_q(B_j)$ and $f_q(C_j) = P_q(C_j)$.

A path of length at most q starting at j stays in $[j-q, j+q]$, so the j-th row of $P_q(C_j)$ equals the j-th row of $P_q(H_{\lambda,\alpha,\theta})$. By (8) the j-th row of $P_q(B_j)$ equals the j-th row of $S^q + S^{-q} + 2\lambda^q\cos(2\pi q\varphi_j)$, and $q\varphi_j = q\theta + j(q\alpha - p) = q\theta + j\delta$. Hence the j-th row of $R = Q - V - S^q - S^{-q}$ has ℓ^2 norm at most $\|f_q(C_j) - f_q(B_j)\| \leq Cq^2 e^{hq}|\delta|$. It has at most $2q+1$ nonzero entries, so its ℓ^1 norm is at most $\sqrt{2q+1}$ times that. Since Q, V and $S^q + S^{-q}$ are self-adjoint, so is R, and the Schur test gives $\|R\| \leq C_h q^{5/2} e^{hq}|\delta|$. Finally Q is a polynomial in H, so it commutes with H. □

Only rows are compared in Step 3. The full irrational and rational operators are not close in norm, and R need not have a sign.

The proposition gives two regimes. When $b_n \geq L + \eta$ for a fixed $\eta > 0$, choosing $h < L + \eta$ makes $\|R\| \to 0$, so $\|K\| \leq 3$ eventually. At all scales, the useful quantity is the size of K relative to λ^q:

$$\|K\| \leq J_n := 3 + C_h q^{5/2} e^{hq}|\delta|, \qquad \sigma_n := \frac{J_n}{\lambda^q}. \tag{10}$$

Lemma 3.4. *Fix $c > 0$ and $0 < \xi < \min(L, c)/2$, and use $h = L + \xi$ in (10). Then $|\delta_n| \leq \sigma_n \leq e^{-(\min(L,b_n)-2\xi)q_n}$ for all sufficiently large n with $b_n \geq c$.*

Proof. Enlarging C_h, we may assume $C_h \geq 1$. Then $\sigma_n \geq C_h q^{5/2} e^{\xi q}|\delta| \geq |\delta|$. For the upper bound, $3\lambda^{-q} = 3e^{-Lq}$ and $C_h q^{5/2} e^{hq}|\delta|\lambda^{-q} = C_h q^{5/2} e^{\xi q} e^{-b_n q}$. Each is at most $\frac{1}{2}e^{-(\min(L,b_n)-2\xi)q}$ once q is large, uniformly in $b_n \geq c$. □

Thus K may be exponentially large when $b_n < L$, but it is exponentially small compared with λ^q at every scale with $b_n \geq c$.

4 Spectral barriers

4.1 An abstract barrier lemma

In this subsection $Q = V + K$ is a bounded self-adjoint operator on $\ell^2(\mathbb{Z})$, V is a real diagonal operator, K has range at most q (that is, $\langle \delta_i, K\delta_j\rangle = 0$ for $|i-j| > q$), and $\|K\| \leq J$. Fix $d_0 \in \mathbb{R}$ and $W > 0$, and let $\Pi = \mathbf{1}_{[d_0-W,d_0+W]}(Q)$. For an interval $I \subset \mathbb{Z}$ write $\partial_q I = \{i \in I : \operatorname{dist}(i, \mathbb{Z} \setminus I) \leq q\}$.

Lemma 4.1 (Outer barrier)**.** *Let $I \subset \mathbb{Z}$ be an interval on which either $V - d_0 \geq V_0$ or $V - d_0 \leq -V_0$, where $V_0 \geq W + 2J$ and $V_0 > W$. If $B \subset I$ and $\operatorname{dist}(B, \mathbb{Z} \setminus I) > (\ell+1)q$ for an integer $\ell \geq 0$, then*

$$\|\mathbf{1}_B \Pi\| \leq 2\Big(\frac{J}{V_0 - W}\Big)^{\ell+2}.$$

Proof. Assume $V - d_0 \geq V_0$ on I; the other case follows by replacing Q and d_0 with $-Q$ and $-d_0$. Let $Q_I = \mathbf{1}_I Q \mathbf{1}_I$ acting on $\ell^2(I)$, let A be the restriction of Q to $\operatorname{Ran}\Pi$, and let $Y = -\mathbf{1}_I Q \mathbf{1}_{\mathbb{Z}\setminus I}\Pi$. Since Π commutes with Q, the operator $X = \mathbf{1}_I \Pi$ solves the Sylvester equation

$$(Q_I - d_0)X - X(A - d_0) = Y.$$

Here $Q_I - d_0 \geq V_0 - J$ and $\|A - d_0\| \leq W < V_0 - J$; the strict inequality follows from $V_0 \geq W + 2J$ if $J > 0$, and from $V_0 > W$ if $J = 0$. So

$$X = \int_0^\infty e^{-s(Q_I - d_0)}\, Y\, e^{s(A-d_0)}\, ds,$$

as one checks by differentiating the integrand. Since V is diagonal, $Y = -\mathbf{1}_I K \mathbf{1}_{\mathbb{Z}\setminus I}\Pi$, so $\|Y\| \leq J$ and the range of Y lies in $\ell^2(\partial_q I)$.

Expand $e^{-s(Q_I - d_0)}$ in a Dyson series around the diagonal part $V_I - d_0 \geq V_0$. The term with k factors of $K_I = \mathbf{1}_I K \mathbf{1}_I$ has norm at most $e^{-sV_0}(sJ)^k/k!$ and moves supports by at most kq. Since $\operatorname{dist}(B, \partial_q I) > \ell q$, only terms with $k \geq \ell + 1$ connect $\ell^2(\partial_q I)$ to $\ell^2(B)$. Therefore

$$\|\mathbf{1}_B X\| \leq J \int_0^\infty e^{-s(V_0 - W)} \sum_{k \geq \ell+1} \frac{(sJ)^k}{k!}\, ds = \sum_{k \geq \ell+1} \Big(\frac{J}{V_0 - W}\Big)^{k+1} \leq 2\Big(\frac{J}{V_0 - W}\Big)^{\ell+2},$$

using $J/(V_0 - W) \leq \frac{1}{2}$. □

Lemma 4.2 (Inner core)**.** *Let $I_0 \subset \mathbb{Z}$ be an interval with $|V - d_0| \leq W/4$ on I_0, and assume $W \geq 4J$. If $0 \in I_0$ and $\operatorname{dist}(0, \mathbb{Z} \setminus I_0) > (\ell+1)q$, then*

$$\|(1 - \Pi)\delta_0\| \leq 3\Big(\frac{4J}{3W}\Big)^{\ell+2}.$$

Proof. Let $\Pi_+ = \mathbf{1}_{(d_0+W,\infty)}(Q)$ and let A_+ be the restriction of Q to its range, so $A_+ - d_0 \geq W$. With $Q_{I_0} = \mathbf{1}_{I_0} Q \mathbf{1}_{I_0}$ we have $Q_{I_0} - d_0 \leq W/4 + J$, and $X_+ = \mathbf{1}_{I_0}\Pi_+$ solves $(Q_{I_0} - d_0)X_+ - X_+(A_+ - d_0) = Y_+$ with $Y_+ = -\mathbf{1}_{I_0} K \mathbf{1}_{\mathbb{Z}\setminus I_0}\Pi_+$. Hence

$$X_+ = -\int_0^\infty e^{s(Q_{I_0} - d_0)}\, Y_+\, e^{-s(A_+ - d_0)}\, ds.$$

Expanding $e^{s(Q_{I_0}-d_0)}$ around its diagonal part, which has norm at most $W/4$, only terms with at least $\ell+1$ factors of K connect $\partial_q I_0$ to the site 0. As in Lemma 4.1,

$$\|\Pi_+\delta_0\| = \|\delta_0^* X_+\| \le J\int_0^\infty e^{-3sW/4}\sum_{k\ge\ell+1}\frac{(sJ)^k}{k!}\,ds \le 2\Big(\frac{4J}{3W}\Big)^{\ell+2}.$$

The same bound holds for $\Pi_- = \mathbf{1}_{(-\infty,d_0-W)}(Q)$, applying the argument to $-Q$. Since $1-\Pi = \Pi_+ + \Pi_-$ with orthogonal ranges, the claim follows. □

Lemma 4.3 (Cutoff)**.** *Let Π be a spectral projection of an operator that commutes with H, let $N \ge 1$ be an integer, and put $B_N = \{-N-1, -N, N, N+1\}$. Then for every $T > 0$,*

$$P(N,T) \le 8\|(1-\Pi)\delta_0\|^2 + T^2\|\mathbf{1}_{B_N}\Pi\|^2.$$

Proof. Let $\chi = \mathbf{1}_{[-N,N]}$. The commutator $[H,\chi] = [S+S^{-1},\chi]$ is supported on the two bonds $\{N, N+1\}$ and $\{-N-1, -N\}$, has norm one, and satisfies $[H,\chi] = [H,\chi]\mathbf{1}_{B_N}$. Since Π commutes with H, Duhamel's formula gives

$$(1-\chi)e^{-itH}\Pi\delta_0 = e^{-itH}(1-\chi)\Pi\delta_0 - i\int_0^t e^{-i(t-s)H}[H,\chi]\,\Pi\,e^{-isH}\delta_0\,ds.$$

As $(1-\chi)\delta_0 = 0$, we have $\|(1-\chi)\Pi\delta_0\| \le \|(1-\Pi)\delta_0\|$. Hence

$$\|(1-\chi)\psi_t\| \le 2\|(1-\Pi)\delta_0\| + |t|\,\|\mathbf{1}_{B_N}\Pi\|.$$

Square, use $(x+y)^2 \le 2x^2+2y^2$, and average with $(2/T)e^{-2t/T}$, for which the mean of t^2 is $T^2/2$. □

The cutoff estimate contains a factor T before squaring because Duhamel's formula integrates the boundary commutator over time. It controls the accumulated crossing error; by itself it does not give a time-uniform confinement bound.

4.2 The local barrier at a scale

We now apply these lemmas to $Q = P_q(H)$ from Proposition 3.3, divided by J_n.

Proposition 4.4. *Fix $c > 0$ and $0 < \varepsilon < \min(L,c)/4$. There are C and n_0 such that the following holds for every $n \ge n_0$ with $b_n \ge c$ and* every *phase θ. Write $s = |\sin(2\pi q_n\theta)|$, $\mu_n = \min(L, b_n)$ and $\kappa_n = -q_n^{-1}\log s \in [0,\infty]$. There is an integer N_n with*

$$e^{\varepsilon q_n/2} \le N_n \le 3e^{\varepsilon q_n}\min\Big\{\frac{\sigma_n}{|\delta_n|s},\ \frac{\sigma_n^{1/2}}{|\delta_n|}\Big\} \le 3\exp\Big((b_n-\mu_n+\min\{\kappa_n,\mu_n/2\}+2\varepsilon)q_n\Big) \qquad (11)$$

(the first entry of the minimum is $+\infty$ if $s=0$) and

$$P(N_n,T) \le C(1+T^2)\exp\Big(-\frac{\varepsilon N_n}{900}\Big) \quad \text{for all } T>0.$$

In particular, if $s \ge e^{-\varepsilon q_n}$, then $N_n \le 3e^{((b_n-L)_+ + 3\varepsilon)q_n}$.

Proof. Write $q = q_n$, $\delta = \delta_n$, $b = b_n$, $A = \lambda^q$, $\mu = \mu_n \ge \mu_0 = \min(L, c)$, $x = 2\pi q\theta$ and $y_j = 2\pi j\delta$, so that $V(j) = 2A\cos(x + y_j)$. Take $\xi = \varepsilon/8$ in Lemma 3.4, so that $|\delta| \le \sigma \le e^{-(\mu-\varepsilon/4)q} \le 1$, where $\sigma = J_n/A$. Put $s_* = e^{\varepsilon q/2}\sigma^{1/2}$; since $\varepsilon/2 < (\mu - \varepsilon/4)/2$, $s_* \to 0$ uniformly in $b \ge c$. We construct N, an interval $I_0 = [-r, r]$ with $r = \lfloor N/500 \rfloor$, intervals $I_+ = [N/2, N]$, $I_- = [-N, -N/2]$ and a number $V_0 > 0$ such that

$$\begin{aligned} &V(j) - V(0) \text{ has a fixed sign on each of } I_+,\ I_-, \qquad |V(j) - V(0)| \ge V_0 \text{ on } I_+ \cup I_-, \\ &|V(j) - V(0)| \le V_0/32 \text{ on } I_0, \qquad V_0' := V_0/J_n \ge e^{\varepsilon q/2}, \qquad N \ge e^{\varepsilon q/2}. \end{aligned} \tag{12}$$

The signs on I_+ and I_- need not be opposite.

Case (i): $s \ge s_*$ (linear barrier). Put $N = \lceil e^{\varepsilon q/2}\sigma/(|\delta|s)\rceil$. Then $N \ge e^{\varepsilon q/2}$ because $\sigma \ge |\delta|$ and $s \le 1$, and

$$2\pi|\delta|N \le 2\pi\big(e^{\varepsilon q/2}\sigma/s + |\delta|\big) \le 2\pi(\sigma^{1/2} + \sigma) \le 4\pi\sigma^{1/2} \le s_*/50 \le s/50$$

for large q. For $|j| \le N$ the mean value theorem gives $V(j) - V(0) = -2Ay_j \sin(x + \vartheta_j y_j)$ with $\vartheta_j \in (0,1)$, and $|\sin(x+\vartheta_j y_j) - \sin x| \le |y_j| \le s/50$. So $\sin(x+\vartheta_j y_j)$ has the sign of $\sin x$ and modulus in $[0.98s, 1.02s]$. On $I_\pm$ we get a fixed sign and $|V(j) - V(0)| \ge 2A \cdot \pi|\delta|N \cdot 0.98s \ge A|\delta|Ns =: V_0$; on I_0, $|V(j) - V(0)| \le 4\pi \cdot 1.02\, A|\delta|rs \le V_0/32$. Finally $V_0' = A|\delta|Ns/J_n \ge e^{\varepsilon q/2}A\sigma/J_n = e^{\varepsilon q/2}$. Also $N \le 2e^{\varepsilon q/2}\sigma/(|\delta|s)$, and since $s \ge s_*$ this is at most $2\sigma^{1/2}/|\delta|$.

Case (ii): $s < s_*$ (quadratic barrier at a turning point). Put $N = \lceil 2e^{\varepsilon q/2}\sigma^{1/2}/|\delta|\rceil$. Then $N \ge 2e^{\varepsilon q/2}$, since $\sigma^{1/2} \ge \sigma \ge |\delta|$. For $|j| \le N$,

$$|y_j| \le 2\pi|\delta|N \le 2\pi(2s_* + |\delta|) \le 0.01$$

for large q, and $s < s_* \le 0.01$, so $|\cos x| \ge 0.99$. Taylor's formula gives

$$V(j) - V(0) = -2Ay_j \sin x - Ay_j^2 \cos(x + \vartheta_j y_j), \qquad \vartheta_j \in (0,1),$$

and $|\cos(x + \vartheta_j y_j) - \cos x| \le |y_j| \le 0.01$, so $\cos(x + \vartheta_j y_j)$ has the sign of $\cos x$ and modulus at least 0.98. On $I_\pm$ we have $|y_j| \ge \pi|\delta|N \ge 2\pi s_* > 4s$, hence $2A|y_j|s \le \frac{1}{2}Ay_j^2$, and therefore $V(j) - V(0)$ has the sign of $-\cos x$ on both I_+ and I_-, with

$$|V(j) - V(0)| \ge 0.48\, Ay_j^2 \ge 0.48\, A(\pi|\delta|N)^2 =: V_0.$$

On I_0, $|y_j| \le 2\pi|\delta|N/500$, and $s < s_* \le |\delta|N/2$, so

$$|V(j) - V(0)| \le 2A|y_j|s + Ay_j^2 \le A\pi^2\delta^2N^2\Big(\frac{2}{500\pi} + \frac{4}{500^2}\Big) \le 0.0013\, A\pi^2\delta^2N^2 \le V_0/32.$$

Finally $V_0' = 0.48\pi^2A\delta^2N^2/J_n \ge 1.92\pi^2e^{\varepsilon q}A\sigma/J_n \ge e^{\varepsilon q/2}$. Also $N \le 3e^{\varepsilon q/2}\sigma^{1/2}/|\delta|$, and since $s < s_*$ we have $\sigma^{1/2}/|\delta| < e^{\varepsilon q/2}\sigma/(|\delta|s)$, so $N \le 3e^{\varepsilon q}\sigma/(|\delta|s)$.

In both cases (12) holds and N satisfies the first two inequalities in (11). The third follows from $\sigma \le e^{-(\mu-\varepsilon/4)q}$ and $|\delta|^{-1} = e^{bq}$: the two entries of the minimum are at most $e^{(b-\mu+\kappa_n+\varepsilon/4)q}$ and $e^{(b-\mu/2+\varepsilon/8)q}$.

The barrier estimates. From here on the proof uses only (12). Apply the lemmas to $Q' = Q/J_n = V' + K'$, $\|K'\| \le 1$, with $d_0 = V'(0)$ and $W' = V_0'/8$. Since $V_0' \ge e^{\varepsilon q/2}$, $\log(7V_0'/8) \ge \varepsilon q/3$ and $\log(3V_0'/32) \ge \varepsilon q/3$ for large q, and $V_0' \ge W' + 2$, $W' \ge 4$. Let $R = \lfloor 3N/4 \rfloor$. Lemma 4.1, applied separately on I_+ and on I_- (on each of them $V' - d_0$ has a fixed sign and modulus at least V_0'), with $\ell = \lfloor (N/4 - 2)/q \rfloor - 2 \ge N/(5q)$, gives $\|\mathbf{1}_{B_R}\Pi\| \le 4(8/(7V_0'))^\ell \le 4e^{-\varepsilon N/15}$. On I_0, $|V' - d_0| \le V_0'/32 = W'/4$, and Lemma 4.2 with $\ell_0 = \lfloor r/q \rfloor - 2 \ge N/(600q)$ gives $\|(1 - \Pi)\delta_0\| \le 3(32/(3V_0'))^{\ell_0} \le 3e^{-\varepsilon N/1800}$. Lemma 4.3 gives $P(R, T) \le 72e^{-\varepsilon N/900} + 16T^2e^{-2\varepsilon N/15}$, and $P(N, T) \le P(R, T)$. The last assertion of the proposition follows from (11) with $\kappa_n \le \varepsilon$. All thresholds depend only on $\lambda, \alpha, c, \varepsilon$, not on θ. Take $N_n = N$. □

Near a turning point, case (ii) replaces the linear barrier by a quadratic barrier that confines on both sides with the same sign. Its radius is of order $\sigma^{1/2}/|\delta|$, up to the stated exponential losses. For comparison, at a phase with sine $s_0 > 0$ fixed, the linear radius is of order $\sigma/(|\delta|s_0)$, and the ratio of these two scales is $s_0\sigma^{-1/2}$. Thus for $b_n > L$ the turning-point radius has exponential order $e^{(b_n - L/2)q_n}$, compared with $e^{(b_n-L)q_n}$ at such a non-turning phase. Remark 4.5 gives finite-volume numerical illustrations of these scales.

Two features matter later. The physical evolution is never rescaled; only Q is divided by J_n, which changes neither its spectral projections nor its commutation with H. And the proposition needs only $b_n \geq c$: at scales with $b_n < L$ the available bound on $\|K\|$ may be large, but $\|K\|/\lambda^q$ is exponentially small.

Remark 4.5 (Numerical illustration; not used in any proof)**.** For $q = 3$, $p = 1$, $\alpha = (p + \delta)/q$ and $\lambda \in \{2, 3, 4\}$ we diagonalized $H_{\lambda,\alpha,\theta}$ on $[-M, M]$ with Dirichlet conditions and computed the infinite-time average $p_\infty(j) = \sum_E |\psi_E(0)|^2|\psi_E(j)|^2$ and its root mean square radius r_{rms}; the mass within 20 sites of the boundary was below 10^{-15} in all runs. At the turning points $\theta = 0$ and $\theta = 1/6$ (where $s = 0$) we found $r_{\mathrm{rms}}\,|\delta|\lambda^{q/2} = 0.136$–$0.143$ for all $\lambda \in \{2, 3, 4\}$ and $\delta \in \{10^{-3}, 2\cdot10^{-3}, 4\cdot10^{-3}\}$, that is, $r_{\mathrm{rms}} \approx 0.14\,\lambda^{-q/2}/|\delta|$, the radius $\sigma^{1/2}/|\delta|$ of Proposition 4.4(ii). At $\theta = 0.13$ ($s = 0.637$) and $\lambda \in \{3, 4\}$ we found $r_{\mathrm{rms}}\,|\delta|s\lambda^q = 0.160$–$0.172$ (the largest value at $\lambda = 4$, $\delta = 4 \cdot 10^{-3}$, where the radius is about one site), the generic radius $\sigma/(|\delta|s)$; at $\lambda = 4$, $\delta = 10^{-3}$ the turning-point radius is 17.2 against 3.9 at $\theta = 0.13$. The instantaneous radius at $\theta = 0$ grows linearly, about $(1$–$2)\lambda^{-q}t$, until it saturates near $t \approx 10^3$ for $\delta = 10^{-3}$. In the regime $\frac{1}{2}\log\lambda < b < \log\lambda$, where K is not small and the generic exponential radius exponent $(b - L)_+$ is zero, and where Proposition 4.4 at $s = 0$ proves only the radius $e^{(b/2+O(\varepsilon))q}$, which is larger than $\lambda^{-q/2}/|\delta| = e^{(b-L/2)q}$, we took $q = 20$, $p = 9$, $\lambda = 2$ and $\delta \in \{6, 3, 1.5\} \cdot 10^{-6}$, so $b = -\log|\delta|/q \in \{0.601, 0.636, 0.671\} < \log 2 = 0.693$. At $\theta = 0.013$ ($s = 0.998$) we found $r_{\mathrm{rms}} = 0.8$ in all three cases, while at $\theta = 0$ we found $r_{\mathrm{rms}} = 22.2$, 44.1, 89.4, against $0.14\,\lambda^{-q/2}/|\delta| = 22.8$, 45.6, 91.1. These are single-scale finite-volume computations at rational-plus-perturbation frequencies; they say nothing directly about exponents.

5 Exact slow dynamics at strongly resonant scales

This section treats scales with $b_n \geq L + \eta$ for a fixed $\eta > 0$, so $|\delta_n|$ is exponentially smaller than λ^{-q_n}. At phases satisfying $|\sin(2\pi q_n\theta)| \geq e^{-\zeta_1 q_n}$, with $\zeta_1 > 0$ chosen sufficiently small for the parameters of Theorem 5.4, spectral filters express most of the initial vector as pieces with an exact slow generator. Their commutator bounds give an upper velocity of exponential order λ^{-q_n}.

Throughout the section $q = q_n$, $A = \lambda^q$, $s = |\sin 2\pi q\theta| \geq e^{-\zeta_1 q}$, where $\zeta_1 \in (0, 1]$ is a parameter fixed before n, and $F = P_q/A$. Put

$$y_0 = 2\cos(2\pi q\theta), \qquad d = 2 - |y_0|, \qquad \rho = d/16, \qquad J_{\mathrm{big}} = [y_0 - 4\rho,\, y_0 + 4\rho].$$

Since $1 - |\cos x| \geq \frac{1}{2}\sin^2 x$ and $1 - |\cos x| \leq \sin^2 x$, we have $e^{-2\zeta_1 q} \leq s^2 \leq d \leq 2s^2$. We fix $h > L$ and write $E_h = e^{(h-L)q}$. By Proposition 3.3, $|F^{(k)}| \leq C_{k,h}q^k E_h$ on U_h for each fixed k. By Lemma 2.6, $\Sigma_{p/q} \subset U_h$ for all large n, and we assume this.

5.1 Inverse branches and filters

Lemma 5.1. *For all sufficiently large n and every phase with $s \geq e^{-\zeta_1 q}$, with ζ_1 fixed as above:*

1. *For each $y \in (-2, 2)$ the equation $F(E) = y$ has q simple real roots, all in $\Sigma_{p/q}$. They define q smooth monotone branches $e_1(y) < \cdots < e_q(y)$ on J_{big}.*

2. *For $b \neq b'$ the arcs $e_b(J_{\mathrm{big}})$ and $e_{b'}(J_{\mathrm{big}})$ are at distance at least $12\rho/(C_{1,h} q E_h)$.*

3. *$|e_b'(y)| \leq 2\lambda/(qs)$ on J_{big}, and for each $k \geq 1$ there are $C_{k,h}$, depending only on k, h, λ and α, and c_k, depending only on k, with $|e_b^{(k)}(y)| \leq C_{k,h}\, q^{c_k} e^{c_k \zeta_1 q} E_h^{k-1}$ on J_{big}.*

Proof. 1. Let $y = 2\cos(2\pi q\varphi)$. By Lemma 3.1, $F(E) = y$ exactly when $\Delta_\varphi(E) = 0 = 2\cos(\pi/2)$, that is, when E is an eigenvalue of the Floquet fiber of $H_{\lambda,p/q,\varphi}$ with Bloch phase $\pi/2$. This fiber is a $q \times q$ Hermitian matrix, so there are q real roots counted with multiplicity, all in $\sigma(H_{\lambda,p/q,\varphi}) \subset \Sigma_{p/q}$. A double root would give two independent Bloch solutions with multiplier i, forcing the real unimodular monodromy matrix to equal iI, which is impossible. Since the roots are simple, the implicit function theorem gives the smooth branches.

2. Simplicity for every $y \in (-2, 2)$ means that every critical value of P_q has modulus at least $2A$. Between the arcs $e_b(J_{\mathrm{big}})$ and $e_{b+1}(J_{\mathrm{big}})$ there is exactly one critical point E_c of P_q, because $P_q - Ay$ has q simple real roots for each $y \in J_{\mathrm{big}}$ and P_q' has degree $q-1$. Let E_1 be the endpoint of $e_b(J_{\mathrm{big}})$ facing E_c, with $F(E_1) = y_1 \in J_{\mathrm{big}}$. On $[E_1, E_c]$ the function P_q is monotone, and it moves from Ay_1 to a value of modulus at least $2A$. Let E' be the first point where $|P_q| = 2A$. On $[E_1, E']$ we have $|P_q| \leq 2A$, so this interval lies in $\Sigma_{p/q} \subset U_h$, where $|P_q'| \leq C_{1,h} q e^{hq}$. Since $2 - |y_1| \geq d - 4\rho = 12\rho$, the change of P_q over $[E_1, E']$ is at least $12A\rho$, so $|E' - E_1| \geq 12A\rho/(C_{1,h} q e^{hq})$.

3. Let $E = e_b(y)$ and $y = 2\cos(2\pi q\varphi)$. The eigenvalue $E(\varphi)$ of the Bloch-phase-$\pi/2$ fiber is simple, and differentiating $\Delta_\varphi(E(\varphi)) = 0$ gives $P_q'(E)E'(\varphi) = -4\pi q A \sin(2\pi q\varphi)$. By the Hellmann–Feynman formula, $|E'(\varphi)|$ is at most the norm of the phase derivative of the fiber, which is at most $4\pi\lambda$. Hence

$$|P_q'(E)| \geq \frac{qA|\sin(2\pi q\varphi)|}{\lambda}. \tag{13}$$

For $y \in J_{\mathrm{big}}$, $\sin^2(2\pi q\varphi) = 1 - y^2/4 \geq (2 - |y|)/2 \geq 6\rho \geq 3s^2/8$, so $|\sin(2\pi q\varphi)| \geq s/2$ and $|e_b'(y)| = 1/|F'(E)| \leq 2\lambda/(qs) \leq 2\lambda e^{\zeta_1 q}$. Differentiating $F(e_b(y)) = y$ repeatedly expresses $F'(e_b)e_b^{(k)}$ as a sum of products of $F^{(j)}(e_b)$, $2 \leq j \leq k$, with lower derivatives of e_b whose orders add up to k. Each such product contains exactly one factor $F^{(j)}$, bounded by $Cq^j E_h$, and induction on k gives the stated bound. □

Fix a smooth $f_0 : \mathbb{R} \to [0,1]$ with $f_0 = 1$ on $[-1,1]$ and $f_0 = 0$ outside $[-2,2]$, and put $f(y) = f_0((y - y_0)/\rho)$. For each branch b choose a smooth $\chi_b : \mathbb{R} \to [0,1]$ equal to 1 on $e_b([y_0 - 2\rho, y_0 + 2\rho])$ and supported in $e_b((y_0 - 3\rho, y_0 + 3\rho))$. By Lemma 5.1, the transition regions have length at least $\rho/\max|F'| \geq \rho/(C_{1,h} q E_h)$, and the supports of different χ_b are disjoint. Define

$$g_b(E) = \chi_b(E)\, f(F(E)), \qquad G_b(y) = \kappa(y)\, e_b(y),$$

where κ is a smooth cutoff equal to 1 on $[y_0 - 2\rho, y_0 + 2\rho]$ and supported in $[y_0 - 3\rho, y_0 + 3\rho]$, with $|\kappa^{(k)}| \leq C_k \rho^{-k}$.

Lemma 5.2. 1. *$\sum_b g_b(E) = f(F(E))$ for every real E.*

2. *$G_b(F(E)) = E$ for every $E \in \operatorname{supp} g_b$.*

3. *For every $r \geq 0$ there are $C_{r,h}$ and c_r with $\|g_b\|_{C^r} + \|G_b\|_{C^r} \leq C_{r,h} q^{c_r} e^{c_r \zeta_1 q} E_h^r$. All supports lie in a fixed bounded interval.*

4. *For every $r \geq 1$, $\|X^r g_b(H)\delta_0\| \leq C_{r,h}\, q^{c_{r+2}} e^{c_{r+2}\zeta_1 q} E_h^{r+2}$.*

Proof. 1. If $f(F(E)) \neq 0$ then $F(E) \in (y_0 - 2\rho, y_0 + 2\rho)$, so by Lemma 5.1 $E = e_b(F(E))$ for exactly one b. On that arc $\chi_b = 1$ and $\chi_{b'} = 0$ for $b' \neq b$.

2. On $\operatorname{supp} g_b$, E lies on the b-th arc and $F(E) \in \operatorname{supp} f$, where $\kappa = 1$; hence $G_b(F(E)) = e_b(F(E)) = E$.

3. This follows from $|f^{(k)}| \leq C_k \rho^{-k} \leq C_k 16^k e^{2k\zeta_1 q}$, the bounds on $F^{(k)}$ on U_h, the length of the transition regions of χ_b, and Lemma 5.1(3).

4. Since $X\delta_0 = 0$, $X^r g_b(H)\delta_0 = \operatorname{ad}_X^r(g_b(H))\delta_0$. Write $g(H) = (2\pi)^{-1} \int \hat{g}(t) e^{itH}\, dt$ and $H_u = e^{iuX} H e^{-iuX}$. Then $\operatorname{ad}_X^r(e^{itH}) = (-i\partial_u)^r e^{itH_u}|_{u=0}$, and all u-derivatives of H_u have norm at most 2. Repeated Duhamel expansion gives $\|\operatorname{ad}_X^r(e^{itH})\| \leq C_r(1+|t|)^r$. Hence $\|\operatorname{ad}_X^r(g(H))\| \leq C_r \int (1+|t|)^r |\hat{g}(t)|\, dt \leq C_r \|g\|_{C^{r+2}}$ for g supported in a fixed bounded interval. Apply part 3. □

5.2 The exact slow propagator

Proposition 5.3. *Assume $\|K\| \leq 3$ and $9q^{r+1} \leq A$. Put $B = Q/A = F(H)$ and $H_b = G_b(B)$. Then:*

1. *$e^{-itH} g_b(H) = e^{-itH_b} g_b(H)$ for every $t \in \mathbb{R}$.*
2. *$\|\operatorname{ad}_X^k B\| \leq 3(2q+1)q^k/A$ for every $k \geq 1$.*
3. *$\|\operatorname{ad}_X^k H_b\| \leq C_k\, q^{k+1} A^{-1} \int (1+|t|)^k |\hat{G}_b(t)|\, dt$ for $1 \leq k \leq r$.*
4. *If $\omega \geq \max_{1\leq k\leq r} \|\operatorname{ad}_X^k H_b\|$ and $M_0 \geq \max_{0\leq k\leq r} \|X^k g_b(H)\delta_0\|$, then*

$$\|X^r e^{-itH_b} g_b(H)\delta_0\| \leq C_r M_0 (1+\omega|t|)^r \qquad \text{for all } t.$$

Proof. 1. By Lemma 5.2(2), $\phi(E) g_b(E) = \phi\big(G_b(F(E))\big) g_b(E)$ for every function ϕ and every real E. Taking $\phi(E) = e^{-itE}$ and applying the functional calculus of H gives the identity, since $G_b(F(H)) = H_b$. No error term appears: $B = P_q(H)/A$ includes the residual R exactly.

2. Write $K = \sum_{m=-q}^{q} K_m$, where K_m keeps the entries $\langle \delta_i, K\delta_j\rangle$ with $i - j = m$. Each

$$K_m = \frac{1}{2\pi} \int_0^{2\pi} e^{-imu}\, e^{iuX} K e^{-iuX}\, du$$

has $\|K_m\| \leq \|K\|$, and $\operatorname{ad}_X K_m = mK_m$. The diagonal V commutes with X. Hence $\|\operatorname{ad}_X^k B\| = \|\operatorname{ad}_X^k K\|/A \leq 3(2q+1)q^k/A$.

3. Write $G_b(B) = (2\pi)^{-1} \int \hat{G}_b(t) e^{itB}\, dt$ and $B_u = e^{iuX} B e^{-iuX}$, so that $\|\partial_u^j B_u\| = \|\operatorname{ad}_X^j B\| \leq 9q^{j+1}/A \leq 1$. The derivative $\partial_u^k e^{itB_u}$ is a sum, over ordered partitions of $\{1,\dots,k\}$ into blocks, of time-ordered integrals containing one factor $\partial_u^{|\beta|} B_u$ per block β. A partition into m blocks contributes at most $|t|^m \prod_\beta \|\partial_u^{|\beta|} B_u\| \leq |t|^m\, 9q^{k+1}/A$, since every factor is at most 1 and one factor is at most $9q^{k+1}/A$. Summing gives $\|\operatorname{ad}_X^k e^{itB}\| \leq C_k(1+|t|)^k q^{k+1}/A$, and integrating against $\hat{G}_b$ proves part 3.

4. Let $\phi_t = e^{-itH_b} g_b(H)\delta_0$. The identity $X^k Y = \sum_{j=0}^{k} \binom{k}{j} (\operatorname{ad}_X^j Y) X^{k-j}$ gives $[X^k, H_b] = \sum_{j=1}^{k} \binom{k}{j} (\operatorname{ad}_X^j H_b) X^{k-j}$. Duhamel's formula then yields

$$\|X^k \phi_t\| \leq \|X^k \phi_0\| + \omega \sum_{j=1}^{k} \binom{k}{j} \int_0^{|t|} \|X^{k-j}\phi_\tau\|\, d\tau,$$

and induction on $k \leq r$ gives the claim. Truncating X to bounded regions and passing to strong limits justifies these manipulations. □

5.3 The window theorem

Theorem 5.4. *Fix $\eta > 0$, $\varepsilon > 0$, $\Lambda \geq 1$ and $M > 0$. There are $\zeta_1 > 0$, C and n_0 such that for every $n \geq n_0$ with $b_n \geq L + \eta$, every phase with $|\sin 2\pi q_n\theta| \geq e^{-\zeta_1 q_n}$, and every $1 \leq T \leq e^{\Lambda q_n}$,*

$$P\big(R_n(T), T\big) \leq Ce^{-Mq_n}, \qquad R_n(T) = e^{\varepsilon q_n}\Big[1 + \min\Big(\frac{T}{\lambda^{q_n}}, \frac{1}{\lambda^{q_n}|\delta_n|}\Big)\Big].$$

Proof. Write $q = q_n$, $A = \lambda^q$, $\delta = \delta_n$, $b = b_n$, and note $A|\delta| = e^{(L-b)q} \leq e^{-\eta q}$.

The barrier radius. Apply Proposition 4.4 with $c = L + \eta$ and $\varepsilon' = \min(\varepsilon/4, L/8)$, and require $\zeta_1 \leq \varepsilon'$. Then $\mu_n = L$, and since $|\sin 2\pi q\theta| \geq e^{-\varepsilon' q}$, the last assertion of Proposition 4.4 gives $N_n \leq 3e^{(b-L+3\varepsilon')q} \leq e^{\varepsilon q}/(A|\delta|) =: R_a$ for large q, and

$$P(R_a, T) \leq P(N_n, T) \leq C(1+T^2)e^{-\varepsilon' N_n/900} \leq Ce^{2\Lambda q}\exp(-\varepsilon' e^{\varepsilon' q/2}/900) \leq e^{-Mq}.$$

The filtered evolution. Fix $\zeta = \varepsilon/8$ and integers $k > M/L + 1$ and $r > M/\varepsilon + 1$. Then fix $\zeta_1 = \min\{\varepsilon',\, L/(2k),\, \zeta/c_*\}$, where c_* is the largest exponent of $e^{\zeta_1 q}$ appearing in Lemmas 5.1 and 5.2 and Proposition 5.3(3) for orders up to $r+2$. Choose $h \in (L, L+\eta)$ so close to L that the bounds of Lemma 5.2 and Proposition 5.3 for orders up to $r+2$ are at most $e^{2\zeta q}$ after absorbing powers of q and the factors $e^{c\zeta_1 q}$, $c \leq c_*$; this is possible since they are polynomial in q times powers of $E_h = e^{(h-L)q}$ of bounded order. Since $h < L + \eta$, Proposition 3.3 gives $\|R\| \leq C_h q^{5/2} e^{(h-L-\eta)q} \to 0$, so $\|K\| \leq 3$ and $9q^{r+1} \leq \lambda^q$ for large q, as Proposition 5.3 requires.

By Lemma 5.2(1), $f(B) = \sum_b g_b(H)$, so $\delta_0 = \sum_b g_b(H)\delta_0 + (1 - f(B))\delta_0$.

The remainder. Since $0 \leq 1 - f \leq 1$ and $1 - f(y) = 0$ for $|y - y_0| < \rho$, the spectral theorem gives $\|(1 - f(B))\delta_0\| \leq \rho^{-k}\|(B - y_0)^k\delta_0\|$. If v is supported in $[-jq, jq]$ with $j < k$, then $(B - y_0)v$ is supported in $[-(j+1)q, (j+1)q]$, and on that set $|V(i)/A - y_0| \leq 4\pi kq|\delta|$. Hence $\|(B - y_0)v\| \leq (4\pi kq|\delta| + 3/A)\|v\|$. For large q, $q|\delta| \leq qe^{-(L+\eta)q} \leq 1/A$, so

$$\|(1 - f(B))\delta_0\| \leq \Big(\frac{C_k}{\rho A}\Big)^k \leq \big(16C_k e^{(2\zeta_1 - L)q}\big)^k \leq e^{-Mq},$$

This vector evolves unitarily, so its contribution to $P(\cdot, T)$ is at most $2e^{-2Mq}$ at every radius.

The retained part. By Proposition 5.3(1), $\sum_b e^{-itH} g_b(H)\delta_0 = \sum_b e^{-itH_b} g_b(H)\delta_0 =: \Phi_t$. By Lemma 5.2 and Proposition 5.3, each branch satisfies the hypotheses of Proposition 5.3(4) with $M_0 \leq e^{2\zeta q}$ and $\omega \leq e^{3\zeta q}/A$. Summing over q branches,

$$\|X^r \Phi_t\| \leq C_r q e^{2\zeta q}(1 + |t|e^{3\zeta q}/A)^r.$$

Let $R_b = e^{\varepsilon q}(1 + T/A)$. By Markov's inequality $\|\mathbf{1}_{|X|>R_b}\Phi_t\| \leq R_b^{-r}\|X^r\Phi_t\|$. Averaging against $(2/T)e^{-2t/T}$, whose moments satisfy $\int t^j (2/T)e^{-2t/T}dt \leq C_j T^j$,

$$\frac{2}{T}\int_0^\infty e^{-2t/T}\|\mathbf{1}_{|X|>R_b}\Phi_t\|^2\, dt \leq C\, q^2 e^{4\zeta q}\,\frac{(1 + Te^{3\zeta q}/A)^{2r}}{R_b^{2r}} \leq Cq^2 e^{(4\zeta + 6r\zeta - 2r\varepsilon)q} \leq e^{-Mq},$$

Combining the two parts with $\|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$ gives $P(R_b, T) \leq Ce^{-Mq}$.

Conclusion. $R_n(T) \geq \min(R_a, R_b)$, and $P(\cdot, T)$ is nonincreasing, so $P(R_n(T), T) \leq \max\big(P(R_a, T), P(R_b, T)\big) \leq Ce^{-Mq}$. Every constant was fixed before n, θ and T vary. □

For the phases and scales in Theorem 5.4, the radius $R_n(T)$ bounds the location of all but an exponentially small Abel-averaged mass, for $1 \leq T \leq e^{\Lambda q_n}$ with fixed Λ. Up to the factor $e^{\varepsilon q_n}$ and the additive 1, this upper radius grows at rate λ^{-q_n} until it saturates at the barrier radius $1/(\lambda^{q_n}|\delta_n|)$. The theorem bounds the packet from outside; it does not by itself prove motion at that speed or stopping at that radius.

6 A per-denominator estimate

The barrier and slow-dynamics estimates need a definite Liouville ratio at the scale used. Times governed by scales with small ratios are handled by the complex-energy method of Damanik and Tcheremchantsev. We use it only in the following form.

Proposition 6.1. *There are constants $C_D \geq 1$ and $C, c_D, \kappa > 0$ such that for every phase θ, every sufficiently large n, every $T \geq 1$ and every $N \geq C_D q_n$,*

$$P(N,T) \leq Ce^{-c_D N} + CT^3 e^{-\kappa q_n}.$$

No assumption on q_{n+1} or on $\beta(\alpha)$ enters. For every fixed $0 < \kappa_0 < \kappa/3$, the estimate makes $P(C_D q_n, T)$ exponentially small in q_n, uniformly for $1 \leq T \leq e^{\kappa_0 q_n}$.

The proof combines two ingredients. The first is the general transport bound of Damanik and Tcheremchantsev [11, Theorem 7]. For a Schrödinger operator with bounded potential, choose $K_E \geq 4$ with $\sigma(H) \subset [-K_E + 1, K_E - 1]$; here $K_E = \max(4, 2\lambda + 3)$ works, since $\|H\| \leq 2 + 2\lambda$. Their theorem bounds the right outside probability, for the initial state δ_1 and the same exponential time average, by

$$C\exp(-cN) + CT^3 \int_{-K_E}^{K_E} \Big(\max_{1 \leq k \leq N} \|\Phi(k, E + i/T)\|^2 \Big)^{-1} dE,$$

where $\Phi(k,z)$ is the transfer matrix from site 1 to site k, and the left outside probability by the analogous expression for negative k. The constants depend only on K_E and the bound on the potential. Changing the initial site from δ_1 to δ_0 shifts the phase by α and the radius by one, which is absorbed by enlarging C_D and C.

The second ingredient is a uniform lower bound for transfer matrices at complex energies. It follows the argument of [11, Section 5], with a slightly smaller auxiliary block, which gives an orbit-hitting step with room to spare.

Lemma 6.2. *There is $\gamma > 0$ such that for all sufficiently large n, every phase x and every z with $|\operatorname{Re} z| \leq K_E$ and $0 \leq \operatorname{Im} z \leq 1$,*

$$\max_{1 \leq k \leq 2q_n} \|A_k(z,x)\| \geq e^{\gamma q_n}.$$

Proof. Let $\Gamma > L$ satisfy $\|A_r(z,x)\|_{\mathrm{HS}} \leq e^{\Gamma r}$ for all $r \geq 1$, $x \in \mathbb{T}$ and z in the strip; this is possible because the one-step matrices are uniformly bounded there.

Herman's bound. With $w = e^{2\pi i x}$, the entry $(A_r(z,x))_{11}$ is a Laurent polynomial in w. Its term of lowest degree, w^{-r}, comes only from the product of the diagonal entries and has coefficient $\prod_{k<r}(-\lambda e^{-2\pi i k\alpha})$, of modulus λ^r. So $p(w) = w^r (A_r(z,x))_{11}$ is a polynomial with $|p(0)| = \lambda^r$, and Jensen's formula gives

$$\int_{\mathbb{T}} \log \|A_r(z,x)\|\, dx \geq \int_{\mathbb{T}} \log |p(e^{2\pi i x})|\, dx \geq rL.$$

A long good interval. Let $\rho_0 = L/(2\Gamma - L)$ and $G_r = \{x : \|A_r(z,x)\|_{\mathrm{HS}} > e^{rL/2}\}$. Splitting the integral over G_r and its complement gives $rL \leq |G_r|\Gamma r + (1 - |G_r|)rL/2$, so $|G_r| \geq \rho_0$. The function $\|A_r(z,x)\|_{\mathrm{HS}}^2$ is a real trigonometric polynomial of degree at most $2r$ in x, so unless $G_r = \mathbb{T}$ it has at most $4r$ components, and one of them is an interval of length at least $\rho_0/(4r)$.

Orbit hitting. Let $q = q_n$ and $r = \lfloor \rho_0 q/32 \rfloor$. Then that interval has length at least $8/q$. The points $x + j\alpha$, $0 \leq j < q$, are within $1/q$ of a translate of the grid $q^{-1}\mathbb{Z}$, because $|j\alpha - jp_n/q| < j/q^2 < 1/q$. Hence they are $3/(2q)$-dense, and some $j < q$ has $x + j\alpha \in G_r$, so $\|A_r(z, x + j\alpha)\| \geq 2^{-1/2} e^{rL/2}$.

Conclusion. Since $A_{j+r}(z,x) = A_r(z, x+j\alpha)A_j(z,x)$ and $\|M^{-1}\| = \|M\|$ for a 2×2 matrix of determinant one,

$$2^{-1/2}e^{rL/2} \le \|A_{j+r}(z,x)\|\,\|A_j(z,x)\|.$$

So one of $\|A_j\|$ (if $j \ge 1$) or $\|A_{j+r}\|$ is at least $2^{-1/4}e^{rL/4}$, with $1 \le j+r \le 2q$. Since $r \ge \rho_0 q/32 - 1$, this proves the lemma with any $\gamma < \rho_0 L/128$. □

Proof of Proposition 6.1. For $T \ge 1$ and $|E| \le K_E$, the point $E + i/T$ lies in the strip of Lemma 6.2. For the right half-line, the transfer matrices from site 1 are $A_k(z, \theta+\alpha)$, so the integrand in [11, Theorem 7] is at most $e^{-2\gamma q_n}$ once $N \ge 2q_n$. For the left half-line, reflection $n \mapsto -n$ maps $H_{\lambda,\alpha,\theta}$ to $H_{\lambda,\alpha,-\theta}$, because the cosine is even; so the left transfer matrices are right transfer matrices for another phase, and the same bound holds. With the matrix conventions of [11] one can equally use the identity $\Phi(n,z,\alpha,\theta) = R\,\Phi(-n,z,-\alpha,\alpha+\theta)\,R$ with $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which preserves norms. Integration over $|E| \le K_E$ gives the claim with $\kappa = 2\gamma$ and $C_D = 3$, the extra unit absorbing the change of initial site. Finitely many small n are absorbed into C, since $P \le 1 \le T^3$. □

7 The upper bound

In this section $\beta = \beta(\alpha) < \infty$ and $\theta \in G_\alpha^*$ is fixed. For $c, \zeta > 0$ let $n_G(\theta, c, \zeta)$ be an index such that $|\sin(2\pi q_n\theta)| \ge e^{-\zeta q_n}$ for all $n \ge n_G(\theta,c,\zeta)$ with $b_n \ge c$. The thresholds n_0 below may depend on θ through n_G, applied with parameters fixed before n and T vary, and the time threshold in Lemma 7.2 may depend on θ as well. The constants C do not, since Proposition 4.4 and Theorem 5.4 are uniform over phases satisfying the sine condition at the scale used. All of these are fixed before n and T vary.

7.1 Logarithmic windows at scales with a definite ratio

Proposition 7.1. *Let $0 < a < 1$, $m > 0$, $c > 0$ and $0 < d < B$, and let $B_0 > \beta$ satisfy $f_0 := \max\{0, 1 - L/B_0\} < a$. There are C and n_0 such that for every $n \ge n_0$ with $b_n \ge c$ and every T with $e^{dq_n} \le T \le e^{Bq_n}$,*

$$P(T^a, T) \le CT^{-m}.$$

Proof. Choose $\eta > 0$ with $2\eta < ad/4$, then $\varepsilon > 0$ with $\varepsilon < ad/4$ and $\varepsilon/d < (a - f_0)/2$. Since $\limsup b_n = \beta < B_0$, we may take $n_0 \ge \max\{n_G(\theta,c,\zeta_1), n_G(\theta,c,\varepsilon_2)\}$, where ζ_1 is given by Theorem 5.4 for the parameters below and ε_2 is defined in the second case, so large that $b_n \le B_0$ for $n \ge n_0$. Write $q = q_n$, $b = b_n$ and $\tau = \log T/q \in [d, B]$.

Case $b \ge L + 2\eta$. Apply Theorem 5.4 with this η and ε, $\Lambda = \max(1, B)$ and $M = mB$. It gives $P(R_n(T), T) \le Ce^{-mBq} \le CT^{-m}$. Moreover

$$\frac{\log R_n(T)}{\log T} \le \frac{\varepsilon}{d} + \frac{\log 2}{dq} + \frac{\max\{0, \min(\tau - L,\, b - L)\}}{\tau}.$$

The last term is at most $1 - L/b \le f_0$, since $b > L$. Indeed, it equals 0 if $\tau \le L$, equals $1 - L/\tau \le 1 - L/b$ if $L < \tau \le b$, and equals $(b-L)/\tau \le (b-L)/b$ if $\tau \ge b$. Hence $\log R_n(T)/\log T < (a + f_0)/2 + o(1) < a$ for large n, so $R_n(T) \le T^a$ and $P(T^a, T) \le P(R_n(T), T) \le CT^{-m}$.

Case $c \le b < L + 2\eta$. Apply Proposition 4.4 with $\varepsilon_2 = \min\{\varepsilon/3, \min(L, c)/8\}$. Since $|\sin 2\pi q\theta| \ge e^{-\varepsilon_2 q}$, its last assertion gives $N_n \le 3e^{(2\eta + 3\varepsilon_2)q} \le T^a$ for large q, since $2\eta + 3\varepsilon_2 < ad/4 + ad/4 = ad/2$ and $T \ge e^{dq}$. Hence

$$P(T^a, T) \le P(N_n, T) \le C(1 + e^{2Bq}) \exp\bigl(-\varepsilon_2 e^{\varepsilon_2 q/2}/900\bigr) \le e^{-mBq} \le T^{-m}$$

for large q. All parameters were fixed before n and T vary. $\square$

When $\beta \le L$ one may take $B_0 < L + 2\eta$, and then only the second case occurs.

7.2 Covering every time

Lemma 7.2. *Let $a \in (0,1)$ with $a > \max\{0, 1 - L/\beta\}$ and let $m > 0$. Then $P(T^a, T) = O(T^{-m})$ as $T \to \infty$.*

Proof. Let C_D, κ be as in Proposition 6.1. Fix $K \ge 1$ with $\kappa K > m + 3$, and put $d = 1/K$ and $c = a/(2K)$. Fix $B_0 > \beta$ with $\max\{0, 1 - L/B_0\} < a$, and put $B = \max\{2B_0/a, 2d\}$. Let C and n_0 be given by Proposition 7.1 for these parameters and the fixed phase θ.

For large T let j be the least index with $q_j \ge K \log T$. Two cases exhaust all possibilities.

Case A: $q_j \le T^a/(2C_D)$. Let $N = \lfloor T^a \rfloor$, so $N \ge C_D q_j$ for large T. Proposition 6.1 gives

$$P(T^a, T) \le P(N, T) \le Ce^{-c_D(T^a - 1)} + CT^3 e^{-\kappa q_j} \le Ce^{-c_D(T^a-1)} + CT^{3-\kappa K} = O(T^{-m}).$$

This case uses no information about the phase and allows arbitrarily long stretches of scales with small ratios.

Case B: $q_j > T^a/(2C_D)$. Let $q = q_{j-1}$. By the choice of j, $q < K \log T$, so $\tau = \log T/q > d$. By (7), $q_j < D_{j-1} = e^{b_{j-1}q}$, so

$$a \log T - \log(2C_D) < \log q_j < b_{j-1} q.$$

Along a sequence of times in this case, $q_j \to \infty$, hence $j \to \infty$ and $q \to \infty$. Dividing by $q < K \log T$ gives $b_{j-1} > a/K - \log(2C_D)/q \ge a/(2K) = c$ for large T. Also $b_{j-1} \le B_0$ for large j, so $\tau < (B_0 + \log(2C_D)/q)/a \le B$. Thus $T \in [e^{dq}, e^{Bq}]$ and $b_{j-1} \ge c$. Since $\theta \in G_\alpha^*$ and $j - 1 \ge n_0$ for large T, Proposition 7.1 gives $P(T^a, T) \le CT^{-m}$. $\square$

The covering needs no information about the scales between q_{j-1} and q_j other than the two inequalities above. In particular it allows $\liminf_n \beta_n = 0$. Figure 2 shows how the two cases assign every large time to a single scale.

Theorem 7.3. *Let $\lambda > 1$, let α be irrational with $\beta(\alpha) < \infty$, and let $\theta \in G_\alpha^*$. For every $a > \max\{0, 1 - L/\beta(\alpha)\}$ and every $m > 0$, $P(T^a, T) = O(T^{-m})$. Consequently*

$$\alpha_u^+(H_{\lambda,\alpha,\theta}) \le \max\Big\{0,\ 1 - \frac{\log \lambda}{\beta(\alpha)}\Big\}.$$

Proof. For $a < 1$ this is Lemma 7.2. For $a \ge 1$ choose $a' \in (\max\{0, 1 - L/\beta\}, 1)$ and use $P(T^a, T) \le P(T^{a'}, T)$. The bound on α_u^+ follows from Lemma 2.3(3), letting a decrease to $\max\{0, 1 - L/\beta\}$. $\square$

8 A lower bound for Lipschitz potentials

This section proves a lower bound for the fastest exponent α_u^+, for every phase, under the arithmetic and Lyapunov-continuity hypotheses of Theorem 8.1. The sampling function may be any Lipschitz function satisfying those hypotheses. For the almost Mathieu operator, Section 9 strengthens the lower bound to every positive moment. In this section only, let $v : \mathbb{T} \to \mathbb{R}$ be Lipschitz, let $H_\theta u(n) = u(n+1) + u(n-1) + v(\theta + n\alpha)u(n)$, and let $T(E, x)$, $A_r(E, x)$, $L(E)$ and Σ be defined as in Section 2 with this v.

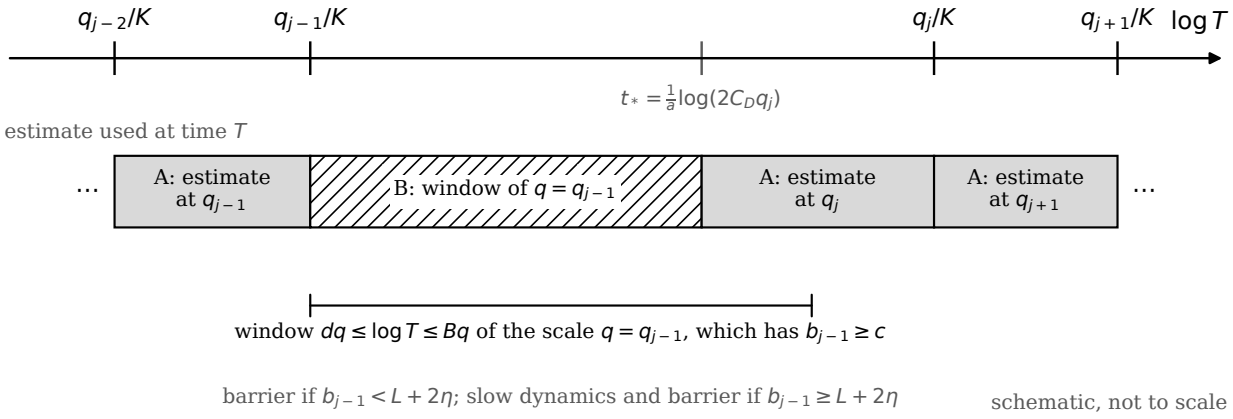


Figure 2: Schematic of the time covering in the proof of Lemma 7.2; not to scale, and the positions of the denominators are arbitrary. Here $\beta(\alpha) < \infty$ and $\theta \in G_\alpha^*$ is fixed. Fix $a \in (\max\{0, 1-L/\beta(\alpha)\}, 1)$ and $m > 0$, choose $K \ge 1$ with $\kappa K > m + 3$, and for a large time T let q_j be the first denominator with $q_j \ge K \log T$, so that $\log T \in (q_{j-1}/K, q_j/K]$. If $q_j \le T^a/(2C_D)$, that is, if $\log T \ge t_*$ (case A, shaded), Proposition 6.1 at the scale q_j gives $P(T^a, T) = O(T^{-m})$. Otherwise (case B, hatched), which can happen only after a jump from q_{j-1} to a denominator $q_j > e^{aq_{j-1}/K}/(2C_D)$, the scale $q = q_{j-1}$ has $b_{j-1} \ge c = a/(2K)$ and $\log T/q \in [d, B]$, where $d = 1/K$, $B_0 > \beta(\alpha)$ is chosen with $\max\{0, 1 - L/B_0\} < a$, and $B = \max\{2B_0/a, 2d\}$. Proposition 7.1 then applies, through the barrier of Proposition 4.4 or through Theorem 5.4, according as b_{j-1} is below or at least $L + 2\eta$, for the small $\eta > 0$ fixed in its proof. Every sufficiently large time falls in one of the two cases, whatever the ratios b_n at other scales; no regularity of these ratios and no overlap of windows at consecutive denominators is assumed.

Theorem 8.1. *Let α be irrational with $0 < \beta(\alpha) < \infty$. Assume that $L(\cdot)$ is continuous on a neighborhood of a point $E_* \in \Sigma$ with $L(E_*) = \ell := \min_\Sigma L$. Then for every phase θ,*

$$\alpha_u^+(H_\theta) \ge \max\Big\{0,\ 1 - \frac{\ell}{\beta(\alpha)}\Big\}.$$

For the almost Mathieu operator with $\lambda > 1$, $L(E)$ is continuous in E and equals $\log\lambda$ on Σ [6, Theorem 1 and Corollary 2], so $\ell = \log\lambda$ and, for $0 < \beta(\alpha) < \infty$, Theorem 8.1 gives $\alpha_u^+ \ge \max\{0, 1 - \log\lambda/\beta(\alpha)\}$ at every phase.

8.1 Elliptic blocks

Let $e_1 = (1,0)^\mathsf{T}$ and $e_2 = (0,1)^\mathsf{T}$. A real determinant-one matrix is elliptic when its trace has modulus less than 2; it is then conjugate over $\mathbb{R}$ to a rotation. We call the coordinates after this conjugation the rotation frame. The direction of a nonzero vector in that frame is its polarization.

For a finite sequence of energies $E - v_r$, $0 \le r < q$, let $T_r = \begin{pmatrix} E - v_r & -1 \\ 1 & 0 \end{pmatrix}$, $C_0 = I$, $C_r = T_{r-1}\cdots T_0$ and $\mathcal{A} = C_q$. Suppose $\mathcal{A}$ is real and elliptic, $|\operatorname{tr}\mathcal{A}| < 2$. Choose $B \in SL(2,\mathbb{R})$ with $B^{-1}\mathcal{A}B = R_\varrho$, the rotation by angle ϱ, and put $u_r = e_1^\mathsf{T} C_r B$, the first row of $C_r B$, and

$$S_\mathcal{A} = \sum_{r=0}^{q-1} \|u_r\|^2, \qquad N_\kappa = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 8.2. *1.* $S_{\mathcal{A}} = |\partial_E \operatorname{tr} \mathcal{A}|/|\sin \varrho|$.

2. For every $z \in \mathbb{C}$ *and* $0 \le r \le q$, *with* $\mathcal{A}(E+z)$ *and* $C_r(E+z)$ *the products at the energy* $E+z$,

$$\|B^{-1}\mathcal{A}(E+z)B\| \le e^{|z|S_{\mathcal{A}}}, \qquad \|C_r(E+z)B\| \le \|C_r(E)B\|\, e^{|z|S_{\mathcal{A}}}.$$

3. If $|\operatorname{tr}\mathcal{A}| \le 3/2$ *and the entries* v_r *depend on a parameter* ϕ *with* $|\partial_\phi v_r| \le \mathrm{Lip}$, *then* $\|B^{-1}(\partial_\phi \mathcal{A})B\| \le C \,\mathrm{Lip}\, |\partial_E \operatorname{tr}\mathcal{A}|$ *and* $\|B^{-1}(\partial_E \mathcal{A})B\| \le C|\partial_E \operatorname{tr}\mathcal{A}|$, *with an absolute constant* C.

4. Let $z_{\mathcal{A}}$ *be the fixed point of* $\mathcal{A}$ *in the upper half-plane. Any* $B \in SL(2,\mathbb{R})$ *with* $B(i) = z_{\mathcal{A}}$ *conjugates* $\mathcal{A}$ *to a rotation, and* $\|B\|^2 = \|B^{-1}\|^2 = e^{d(i,z_{\mathcal{A}})}$, *where* d *is the hyperbolic distance. If* $|\operatorname{tr}\mathcal{A}| \le 3/2$, *then* $e^{d(i,z_{\mathcal{A}})} \le C\|\mathcal{A}\|$, *and* $z_{\mathcal{A}}$ *moves with hyperbolic speed at most* $C\|B^{-1}\mathcal{A}'B\|$ *under any parameter derivative* $\mathcal{A}'$.

Proof. The insertion identity is the common source of the derivative and complex-energy estimates in parts 1–3; part 4 controls how the conjugating frame changes. Put $Q_r = B^{-1}C_r^{-1}N_\kappa C_r B$. Since $T_r(E+z) = T_r(E)(I + zN_\kappa)$, telescoping gives

$$B^{-1}\mathcal{A}(E+z)B = R_\varrho \prod_{r=q-1}^{0} (I + zQ_r),$$

with the factors ordered from $r = q-1$ on the left to $r = 0$ on the right. A direct computation gives $Q_r = -J_0 u_r^{\mathsf{T}} u_r$, a rank-one matrix with $\|Q_r\| = \|u_r\|^2$. Hence $\|B^{-1}\mathcal{A}(E+z)B\| \le \prod_r (1+|z|\|u_r\|^2) \le e^{|z|S_{\mathcal{A}}}$, and the same computation truncated at r gives the second inequality in part 2. Differentiating at $z=0$, $B^{-1}(\partial_E\mathcal{A})B = R_\varrho \sum_r Q_r$, and the trace of $R_\varrho(-J_0 u_r^{\mathsf{T}} u_r)$ equals $\sin\varrho\,\|u_r\|^2$, because the symmetric part of $R_\varrho J_0$ is $-\sin\varrho\, I$ and the antisymmetric part has zero quadratic form. This proves part 1. A change of v_r at rate c_r gives in the same way $B^{-1}\mathcal{A}'B = R_\varrho \sum_r c_r Q_r$, so $\|B^{-1}\mathcal{A}'B\| \le \max_r |c_r|\, S_{\mathcal{A}}$. When $|\operatorname{tr}\mathcal{A}| \le 3/2$, $|\sin\varrho| \ge \sqrt{7}/4$, and part 3 follows from part 1. For part 4, recall that every $M \in SL(2,\mathbb{R})$ satisfies $\|M\|_{\mathrm{HS}}^2 = 2\cosh d(i, M(i))$; since its singular values are σ and σ^{-1}, this gives $\|M\|^2 = e^{d(i,M(i))}$. Apply this to B and B^{-1}. If $B(i) = z_{\mathcal{A}}$, then $B^{-1}\mathcal{A}B$ fixes i and is a rotation. Conjugating to that rotation and differentiating $\mathcal{A}(z_{\mathcal{A}}) = z_{\mathcal{A}}$ shows that the fixed point moves at hyperbolic speed at most $|1 - e^{-2i\varrho}|^{-1}\|B^{-1}\mathcal{A}'B\|$. Finally $\|\mathcal{A}\|_{\mathrm{HS}}^2 = 2 + 4\sinh^2\big(d(i,z_{\mathcal{A}})\big)\sin^2\varrho$, which with $|\sin\varrho| \ge \sqrt{7}/4$ gives $e^{d(i,z_{\mathcal{A}})} \le C\|\mathcal{A}\|$. □

8.2 Proof of Theorem 8.1

If $\beta(\alpha) \le \ell$ there is nothing to prove, so assume $\ell < \beta(\alpha)$. Fix h with $\ell < h < \beta(\alpha)$.

Uniform real growth. By continuity of L near E_* there is a compact neighborhood U of E_* on which $L < h$. The argument of Lemma 2.4 gives C_h with $\|A_r(E,x)\| \le C_h e^{hr}$ for $E \in U$, $x \in \mathbb{T}$ and $r \ge 0$. By the product rule, with $D_q = Cqe^{hq}$,

$$|\partial_E \operatorname{tr} A_q(E,x)| \le D_q, \qquad |\partial_x \operatorname{tr} A_q(E,x)| \le \mathrm{Lip}(v)\, D_q \qquad (E \in U), \tag{14}$$

the second almost everywhere in x.

A band near E_*. Choose convergents with $\beta_n \to \beta(\alpha)$; write $q = q_n$, $\delta = \delta_n$. Fix a phase x_0 and let H^{per} be the q-periodic operator whose potential repeats $v(x_0 + k\alpha)$, $0 \le k < q$. Its monodromy is $A_q(E, x_0)$, and it differs in norm by at most $\mathrm{Lip}(v)|\delta|$ from the p/q-periodic operator at phase x_0. The proof of Lemma 2.6, which uses only the Lipschitz bound on v, shows that the phase-union spectrum of the p/q-periodic operators converges to Σ and lies within $\mathrm{Lip}(v)/(2q)$ of

each fixed-phase spectrum. So $\sigma(H^{\mathrm{per}})$ has a point converging to E_*. Every band of a q-periodic operator with hopping one has width at most $2\pi/q$: distributing the Bloch phase over the q bonds makes the Floquet matrices $2/q$-Lipschitz in the Bloch phase. Hence, for large n, an entire band of H^{per} lies in U. On that band $\operatorname{tr} A_q(E, x_0)$ is monotone and takes all values in $[-2, 2]$, so the set

$$S_q = \{E \text{ in the band} : |\operatorname{tr} A_q(E, x_0)| \le 1\}$$

is an interval of length at least $2/D_q$.

Slowly drifting blocks. The blocks of the actual operator starting at sites mq, for $\theta = x_0$, are $A^{(m)}(E) = A_q(E, x_0 + mq\alpha) = A_q(E, x_0 + m\delta)$, since $q\alpha = p + \delta$. Let

$$K_q = \Big\lfloor \frac{c_0}{D_q|\delta|} \Big\rfloor, \qquad N_q = qK_q, \qquad T_q = C_0 K_q D_q,$$

with c_0 small and C_0 large fixed constants. Since $\log(1/|\delta|) = b_n q$ and $b_n \to \beta(\alpha) > h$, $K_q \to \infty$. For $E \in S_q$ and $0 \le m < K_q$, (14) gives $|\operatorname{tr} A^{(m)}(E) - \operatorname{tr} A_q(E, x_0)| \le \operatorname{Lip}(v) D_q K_q |\delta| \le \operatorname{Lip}(v) c_0 \le 1/2$, so every block satisfies $|\operatorname{tr} A^{(m)}| \le 3/2$.

By Lemma 8.2(3)–(4), the fixed points z_m of the blocks $A^{(m)}$ move a total hyperbolic distance $\sum_m d(z_m, z_{m+1}) \le C \operatorname{Lip}(v) D_q K_q |\delta| \le C$. Choose any $B_m \in SL(2, \mathbb{R})$ with $B_m(i) = z_m$. Since $B_{m+1}^{-1} B_m$ maps i to a point at distance $d(z_m, z_{m+1})$ from i, Lemma 8.2(4) gives $\|B_{m+1}^{-1} B_m\| = e^{d(z_m, z_{m+1})/2}$, so $\prod_m \|B_{m+1}^{-1} B_m\| \le e^C$. By Lemma 8.2(1)–(2), each block at the energy $E + i/T_q$ satisfies $\|B_m^{-1} A^{(m)}(E + i/T_q) B_m\| \le e^{CD_q/T_q}$, and the product of K_q such factors is at most $e^{CK_q D_q/T_q} \le e^{C/C_0}$. The endpoint lifts cost at most Ce^{hq} by Lemma 8.2(4). A partial last block is handled by the second inequality in Lemma 8.2(2). Hence, writing $\Phi(x, z)$ for the transfer matrix of H_{x_0} from site 0 to site x,

$$\max_{0 \le x \le N_q} \|\Phi(x, E + i/T_q)\| \le Ce^{2hq} \qquad (E \in S_q). \tag{15}$$

Resolvent. Let $G(n; z) = \langle \delta_n, (H_{x_0} - z)^{-1} \delta_0 \rangle$. For $n \ge 1$ the vector $(G(n+1; z), G(n; z))$ is obtained from $(G(1; z), G(0; z))$ by the transfer matrices starting at site 1, which are bounded as in (15) after a shift of the starting site that changes only constants. Since these matrices have determinant one and $|G(0; E + i/T)| \ge \operatorname{Im} G(0; E + i/T) \ge c/T$ for bounded E,

$$|G(n; E + i/T_q)|^2 + |G(n+1; E + i/T_q)|^2 \ge cT_q^{-2} e^{-4hq} \qquad (E \in S_q,\ 0 \le n \le N_q).$$

By Parseval's formula, $a(n, T) = (\pi T)^{-1} \int_{\mathbb{R}} |G(n; E + i/T)|^2 \, dE$. Summing over the pairs of sites between $N_q/2$ and N_q gives

$$P(N_q/2, T_q) \ge c\, N_q\, |S_q|\, T_q^{-3} e^{-4hq} \ge T_q^{-C_*} \tag{16}$$

for a finite constant C_* depending only on h and $\beta(\alpha)$, because $\log T_q = b_n q + O(\log q)$ and $|S_q| \ge 2/D_q$.

The exponent. From $\log N_q = (b_n - h)q + O(\log q)$ and $\log T_q = b_n q + O(\log q)$ we get $\log N_q / \log T_q \to 1 - h/\beta(\alpha)$. By (16), $M_p(T_q) \ge (N_q/2)^p P(N_q/2, T_q) \ge c(N_q/2)^p T_q^{-C_*}$, so $\beta^+(p) \ge 1 - h/\beta(\alpha) - C_*/p$. Letting $p \to \infty$ and then $h \downarrow \ell$ proves the theorem for the phase x_0. The phase was arbitrary. □

The mass in (16) is only polynomially small, so this argument bounds α_u^+ but not each fixed $\beta^+(p)$. The losses are the pointwise bound $|G(0)| \ge c/T$, the two frame changes at the ends of the run, and an energy window of small spectral weight. Section 9 removes all three for the almost Mathieu operator.

9 Every moment and the exact packing dimension

The lower bound of Section 8 gives only a polynomially small Abel-averaged mass near the critical distance. For the almost Mathieu operator, along sufficiently large scales with $b_n \ge c > h > \log\lambda$, this section proves that the mass beyond Te^{-hq_n} tends to one uniformly in phase and in $e^{hq_n} \le T \le |\delta_n|^{-1}$. Two facts drive the proof. First, the rational discriminant is asymptotically arcsine distributed under the single-site spectral measure, so most of that measure lies away from the parabolic band edges (Lemma 9.1). Second, at such energies the square-summable half-line solution of the equation with energy $E + i/T$—the Weyl solution—remains spread over a long run of elliptic blocks (Lemma 9.6). These facts give Theorem 9.8, Theorem B and the lower moment bounds. A diagonal resolvent estimate from the same runs gives the lower local-dimension bound; the upper transport bound gives the upper packing-dimension bound. The dependencies are: Lemmas 9.1 and 9.6 give Theorem 9.8, which gives Corollary 9.12 and Theorems A and B; Lemma 9.6 gives Lemma 9.13 and part 1 of Theorem C; Theorem A gives Proposition 9.14 and part 2 of Theorem C.

Throughout, $\lambda > 1$, $v(x) = 2\lambda\cos 2\pi x$ and $L = \log\lambda$, except in Subsection 9.2, where v is any Lipschitz function. For a phase θ we write μ^θ for the spectral measure of δ_0 for $H_{\lambda,\alpha,\theta}$, and

$$G(k;z) = \langle \delta_k, (H_{\lambda,\alpha,\theta} - z)^{-1}\delta_0\rangle \qquad (k \in \mathbb{Z},\ \operatorname{Im} z > 0).$$

Two identities connect these objects with transport. The first is Parseval's formula, already used in Section 8:

$$a(k,T) = \frac{1}{\pi T}\int_{\mathbb{R}} |G(k;E+i/T)|^2\, dE. \tag{17}$$

It follows from $(H - E - i/T)^{-1} = i\int_0^\infty e^{-t/T}e^{itE}e^{-itH}\,dt$ and Plancherel's theorem. The second is the spectral theorem: for $z = E + i\varepsilon$ with $\varepsilon > 0$,

$$\sum_{k\in\mathbb{Z}} |G(k;z)|^2 = \frac{\operatorname{Im} G(0;z)}{\varepsilon}, \qquad \frac{1}{\pi}\operatorname{Im} G(0;z) = \int_{\mathbb{R}} P_\varepsilon(E-E')\, d\mu^\theta(E'), \qquad P_\varepsilon(x) = \frac{\varepsilon}{\pi(x^2+\varepsilon^2)}. \tag{18}$$

9.1 The arcsine law

Fix a scale n and write $q = q_n$, $p = p_n$, $\delta = \delta_n$. For the phase θ put

$$\Delta_n(E) = P_q(E) - 2\lambda^q\cos(2\pi q\theta),$$

the discriminant of the periodic operator $H_{\lambda,p/q,\theta}$ (Lemma 3.1). For that periodic operator the distribution of Δ_n under the spectral measure of δ_0 is exactly the arcsine law: by (8), $\Delta_n(H_{\lambda,p/q,\theta}) = S^q + S^{-q}$. The next lemma transfers this to the irrational operator at every scale where $|\delta_n|$ is much smaller than λ^{-q_n}.

Let σ be the arcsine law on $[-2,2]$, with density $1/(\pi\sqrt{4-x^2})$. Its moments are $m_k = \binom{k}{k/2}$ for even k and $m_k = 0$ for odd k, and $\sigma([-a,a]) = \frac{2}{\pi}\arcsin(a/2)$ for $0 \le a \le 2$.

Lemma 9.1 (Arcsine law)**.** *Let $c > L$ and let $\mathcal{N}$ be an infinite set of indices with $b_n \ge c$ for $n \in \mathcal{N}$. For every phase θ let ν_n be the image of μ^θ under Δ_n. Then $\int x^k\, d\nu_n \to m_k$ for every integer $k \ge 0$, uniformly in θ, as $n \to \infty$ in $\mathcal{N}$. Consequently ν_n converges weakly to σ, and for every $0 \le \eta < 2$, uniformly in θ,*

$$\lim_{n\in\mathcal{N}} \mu^\theta\big(\{E : |\Delta_n(E)| \le 2-\eta\}\big) = \frac{2}{\pi}\arcsin\Big(1 - \frac{\eta}{2}\Big).$$

Proof. Fix $h \in (L, c)$. By Proposition 3.3, for large n and every phase,

$$\Delta_n(H) = Q - V(0) = D + F + R, \qquad D = V - V(0), \quad F = S^q + S^{-q},$$

where D is diagonal and R has range at most q and norm at most $C_h q^{5/2} e^{hq}|\delta|$. Each of D, F and R maps a vector supported in $[-jq, jq]$ to a vector supported in $[-(j+1)q, (j+1)q]$.

Fix $k \geq 1$. In the expansion of $\Delta_n(H)^k \delta_0$ the operator D only acts on vectors supported in $[-kq, kq]$, where

$$|D(i)| = 2\lambda^q |\cos(2\pi(q\theta + i\delta)) - \cos(2\pi q\theta)| \leq 4\pi\lambda^q \, kq|\delta|.$$

So $\Delta_n(H)^k \delta_0 = (D_k + F + R)^k \delta_0$ with $D_k = D\mathbf{1}_{[-kq,kq]}$. Put $\varepsilon_n = 4\pi kq\lambda^q|\delta| + C_h q^{5/2} e^{hq}|\delta|$, so that $\|D_k + R\| \leq \varepsilon_n$, and recall $\|F\| \leq 2$. Expanding the product,

$$\big\|\Delta_n(H)^k \delta_0 - F^k \delta_0\big\| \leq (2 + \varepsilon_n)^k - 2^k.$$

Since $b_n \geq c > h > L$, both $q\lambda^q|\delta| = qe^{(L-b_n)q}$ and $q^{5/2}e^{hq}|\delta| = q^{5/2}e^{(h-b_n)q}$ tend to 0, uniformly in θ, so $\varepsilon_n \to 0$. Next, $F^k\delta_0 = \sum_i \binom{k}{(k+i)/2}\delta_{iq}$, the sum running over $|i| \leq k$ with $i \equiv k \pmod 2$, so $\langle \delta_0, F^k\delta_0\rangle = m_k$. Since $\int x^k \, d\nu_n = \langle \delta_0, \Delta_n(H)^k\delta_0\rangle$, we obtain $|\int x^k \, d\nu_n - m_k| \leq (2+\varepsilon_n)^k - 2^k \to 0$.

The law σ has compact support, so it is determined by its moments [5, Theorem 30.1], and convergence of all moments implies $\nu_n \Rightarrow \sigma$ [5, Theorem 30.2]. Since σ has no atoms, weak convergence gives $\nu_n(J) \to \sigma(J)$ for every interval J. Take $J = [-2+\eta, 2-\eta]$. The convergence is uniform in θ: otherwise there would be $\varepsilon_0 > 0$, indices $n_k \to \infty$ in $\mathcal{N}$ and phases θ_k with $|\nu_{n_k}^{\theta_k}(J) - \sigma(J)| \geq \varepsilon_0$; but the moments of $\nu_{n_k}^{\theta_k}$ still converge to m_j for every j, so $\nu_{n_k}^{\theta_k} \Rightarrow \sigma$ and $\nu_{n_k}^{\theta_k}(J) \to \sigma(J)$, a contradiction. □

Lemma 9.2 (Uniform arcsine bound)**.** *Let c and $\mathcal{N}$ be as in Lemma 9.1, and let $0 \leq \eta < 2$. Then*

$$\liminf_{n\in\mathcal{N}} \inf_{\theta\in\mathbb{T}} \mu^\theta(\{E : |\Delta_n(E)| \leq 2 - \eta\}) \geq \frac{2}{\pi}\arcsin\Big(1 - \frac{\eta}{2}\Big).$$

Proof. Put $a = 2-\eta$ and fix $\tau \in (0, a)$. Let f be the continuous function that equals 1 on $[-a+\tau, a-\tau]$, vanishes outside $(-a, a)$ and is linear on the two remaining intervals, so $0 \leq f \leq \mathbf{1}_{[-a,a]}$. By the Weierstrass theorem there is a polynomial p_0 with $|p_0 - f| \leq \tau$ on $[-3, 3]$. Let d be its degree and A a constant with $|p_0(x)| \leq A(|x|/3)^d$ for $|x| \geq 3$. For an integer K with $2K \geq d$ put

$$p(x) = p_0(x) - \tau - A(x/3)^{2K}.$$

Then $p \leq \mathbf{1}_{[-a,a]}$ on $\mathbb{R}$: on $[-3, 3]$ we have $p \leq p_0 - \tau \leq f \leq \mathbf{1}_{[-a,a]}$, and for $|x| \geq 3$ we have $p(x) \leq |p_0(x)| - A(|x|/3)^{2K} \leq 0$. Hence, with $\nu_n = \nu_n^\theta$ the image of μ^θ under Δ_n,

$$\mu^\theta(\{|\Delta_n| \leq a\}) = \int \mathbf{1}_{[-a,a]} \, d\nu_n \geq \int p \, d\nu_n.$$

The right side is a fixed finite combination of the moments $\int x^k \, d\nu_n$, which converge to m_k uniformly in θ by the proof of Lemma 9.1. So $\liminf_n \inf_\theta \mu^\theta(\{|\Delta_n| \leq a\}) \geq \int p \, d\sigma \geq \int f \, d\sigma - 2\tau - A(2/3)^{2K} \geq \sigma([-a+\tau, a-\tau]) - 2\tau - A(2/3)^{2K}$, since σ lives on $[-2, 2]$. Let $K \to \infty$ (A does not depend on K), then $\tau \to 0$, and use $\sigma([-a, a]) = \frac{2}{\pi}\arcsin(a/2)$. □

The proof uses no condition on the phase. The only arithmetic input is $b_n \geq c > L$, which makes the residual ε_n small on the sites that k steps from the origin can reach.

9.2 One elliptic run

In this subsection $v : \mathbb{T} \to \mathbb{R}$ is Lipschitz with constant $\mathrm{Lip}(v)$, and $T(E,x)$ and $A_r(E,x)$ are defined with this v as in Section 8. Fix a phase θ and a convergent p/q of α, and put $\delta = q\alpha - p$, so that $x + q\alpha = x + \delta$ in $\mathbb{T}$. Let $z = E + i\varepsilon$ with $E \in \mathbb{R}$ and $\varepsilon > 0$, and let $g \in \ell^2(\mathbb{Z}_{\geq 0})$ satisfy

$$g(k+1) + g(k-1) + v(\theta + k\alpha)\, g(k) = z\, g(k) \qquad (k \geq 1). \tag{19}$$

Cut the half-line into blocks of q sites. Put $x_m = \theta + (mq+1)\alpha$, $\Psi_m = (g(mq+1), g(mq))^{\mathsf{T}}$ and

$$\mathcal{M}_m = \sum_{r=0}^{q-1} |g(mq+1+r)|^2.$$

Then $(g(mq+1+r), g(mq+r))^{\mathsf{T}} = A_r(z, x_m)\Psi_m$ for $0 \leq r \leq q$, and in particular $\Psi_{m+1} = A_q(z, x_m)\Psi_m$. Consecutive blocks see almost the same potential: $x_{m+1} = x_m + \delta$, so the potentials at corresponding sites differ by at most $\mathrm{Lip}(v)|\delta|$. We use the notation T_r, C_r, N_κ, J_0 and R_ϱ of Lemma 8.2, and d for the hyperbolic distance in the upper half-plane, normalized as there.

The first lemma is the telescoping identity behind Lemma 8.2, for site-dependent shifts.

Lemma 9.3 (Insertion)**.** *Let T_r and C_r be as in Lemma 8.2 for energies $E - v_r$, $0 \leq r < q$, let $B \in SL(2,\mathbb{R})$, and put $u_r = e_1^{\mathsf{T}} C_r B$. Let $w_0, \dots, w_{q-1} \in \mathbb{C}$ and let C_r' be the products for the energies $E - v_r + w_r$. Then for $0 \leq r \leq q$,*

$$C_r' B = C_r B\, \Pi_r, \qquad \Pi_r = (I + w_{r-1}Q_{r-1}) \cdots (I + w_0 Q_0), \qquad Q_j = -J_0 u_j^{\mathsf{T}} u_j,$$

with $\det \Pi_r = 1$ and $\|\Pi_r - I\| \leq \exp(\sum_{j<r} |w_j|\, \|u_j\|^2) - 1$.

Proof. The new one-step matrices are $T_j(I + w_j N_\kappa)$. If $C_j' B = C_j B \Pi_j$, then

$$C_{j+1}' B = T_j(I + w_j N_\kappa) C_j B \Pi_j = C_{j+1} B\, (I + w_j B^{-1} C_j^{-1} N_\kappa C_j B)\, \Pi_j,$$

which gives the product formula by induction. The computation in the proof of Lemma 8.2, which uses only $\det B = \det C_j = 1$, gives $B^{-1}C_j^{-1}N_\kappa C_j B = -J_0 u_j^{\mathsf{T}} u_j$. This matrix has norm $\|u_j\|^2$, and it has trace $-u_j J_0 u_j^{\mathsf{T}} = 0$ and rank one, so $\det(I + w_j Q_j) = 1$. Expanding the product gives the norm bound. □

Lemma 9.4 (Perturbed rotations)**.** *Let $|\sin \varrho| \geq s > 0$ and let $M = R_\varrho(I + Y) \in SL(2,\mathbb{R})$ with $\|Y\| \leq y \leq s/8$. Then M is elliptic, and its fixed point w in the upper half-plane satisfies $d(i,w) \leq 12y/s$.*

Proof. The unitary matrix $\mathcal{C} = 2^{-1/2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ acts as the Cayley map from the upper half-plane to the unit disc, sends i to 0, and is an isometry for the hyperbolic metrics. It conjugates R_ϱ to $\mathrm{diag}(e^{-i\varrho}, e^{i\varrho})$ and M to a matrix $\begin{pmatrix} a & b \\ \bar b & \bar a \end{pmatrix}$ with $|a|^2 - |b|^2 = 1$. Since $\mathcal{C}$ is unitary, $\|\mathcal{C}M\mathcal{C}^{-1} - \mathcal{C}R_\varrho\mathcal{C}^{-1}\| = \|R_\varrho Y\| \leq y$, so $|b| \leq y$ and $|\operatorname{Im} a| \geq s - y \geq \frac{7}{8}s > |b|$. Hence M is elliptic. The fixed point of $\zeta \mapsto (a\zeta + b)/(\bar b\zeta + \bar a)$ in the disc solves $\bar b\zeta^2 - 2i(\operatorname{Im} a)\zeta - b = 0$ and has modulus

$$t = \frac{|b|}{|\operatorname{Im} a| + \sqrt{(\operatorname{Im} a)^2 - |b|^2}} \leq \frac{8y}{7s} \leq \frac{1}{4}.$$

Its hyperbolic distance from 0 is $\log \frac{1+t}{1-t} \leq \frac{2t}{1-t} \leq \frac{8}{3}t \leq 12y/s$. □

Lemma 9.5 (Pair inequality)**.** *Let G be a real symmetric positive semidefinite 2×2 matrix with trace S, and let R_ϱ be a rotation. Then $\Phi^*(G+R_\varrho^{\mathsf T}GR_\varrho)\Phi \ge S(1-|\cos\varrho|)\|\Phi\|^2$ for every $\Phi\in\mathbb{C}^2$.*

Proof. Write $G=\frac{S}{2}I+gF_\phi$ with $F_\phi=\begin{pmatrix}\cos\phi & \sin\phi\\ \sin\phi & -\cos\phi\end{pmatrix}$ and $0\le g\le S/2$. Since $R_\varrho^{\mathsf T}F_\phi R_\varrho=F_{\phi-2\varrho}$ and $F_\phi+F_{\phi-2\varrho}=2\cos\varrho\,F_{\phi-\varrho}$, the real symmetric matrix $G+R_\varrho^{\mathsf T}GR_\varrho=SI+2g\cos\varrho\,F_{\phi-\varrho}$ has eigenvalues $S\pm 2g|\cos\varrho|\ge S(1-|\cos\varrho|)$. A real symmetric matrix bounds its Hermitian form on $\mathbb{C}^2$ below by its smallest eigenvalue. □

The point of Lemma 9.5 is that it holds for every polarization Φ. A single block may carry little mass, for instance when the solution is a standing wave with a node there, but two adjacent blocks cannot both do so.

Lemma 9.6 (Elliptic propagation)**.** *Let $0<\eta\le 1$ and assume $|\operatorname{tr}A_q(E,x_0)|\le 2-\eta$. Let $B_0\in SL(2,\mathbb{R})$ map i to the fixed point of $A_q(E,x_0)$ in the upper half-plane, and put*

$$S_0=\sum_{r=0}^{q-1}\|e_1^{\mathsf T}A_r(E,x_0)B_0\|^2,\qquad \Phi_0=B_0^{-1}\Psi_0,$$

$$\xi=(\mathrm{Lip}(v)+1)\max(\varepsilon,|\delta|),\qquad m_*=\Big\lfloor\frac{\eta^{3/2}}{2^{10}\xi S_0}\Big\rfloor.$$

If $m_*\ge 8$, then:

1. *for $0\le m<m_*$, $\mathcal{M}_m+\mathcal{M}_{m+1}\ge 2^{-8}\eta\,S_0\|\Phi_0\|^2$ and $\mathcal{M}_m\le 16S_0\|\Phi_0\|^2$;*
2. *with $m_0=\lceil m_*/2\rceil$,*
$$\sum_{k>m_0q}|g(k)|^2\ge\frac{\eta^{5/2}}{2^{22}}\cdot\frac{\|\Phi_0\|^2}{\xi};$$
3. *$\sum_{k>m_*q}|g(k)|^2\le 2\|\Phi_0\|^2/\varepsilon$, $\sum_{k\ge1}|g(k)|^2\le(1/\xi+2/\varepsilon)\|\Phi_0\|^2$, and $|g(0)|\le\|B_0\|\,\|\Phi_0\|$.*

By Lemma 8.2(1), $S_0=|\partial_E\operatorname{tr}A_q(E,x_0)|/|\sin\varrho_0|$, where ϱ_0 is the rotation angle of $A_q(E,x_0)$. For fixed η, the run consists of m_* blocks and has length comparable to $q/(\xi S_0)$, with comparison constants depending on η. Its length is set by the larger of the damping ε and the drift $|\delta|$; it is not a fixed function of the scale. When $\varepsilon\ge|\delta|$, the total mass in part 3 is at most $3(\mathrm{Lip}(v)+1)\|\Phi_0\|^2/\xi$, so part 2 says that the far half of the run carries a fixed fraction of the mass of g on $\mathbb{Z}_{>0}$. No arithmetic condition on θ enters, and nothing is assumed about the solution beyond the run.

Proof. Frames. Write $A_m=A_q(E,x_m)$ for the real-energy block monodromies. Suppose A_m is elliptic and $B_m\in SL(2,\mathbb{R})$ maps i to its fixed point, so that $B_m^{-1}A_mB_m=R_{\varrho_m}$ by Lemma 8.2(4). Put

$$u_r^{(m)}=e_1^{\mathsf T}A_r(E,x_m)B_m,\qquad G_m=\sum_{r=0}^{q-1}(u_r^{(m)})^{\mathsf T}u_r^{(m)},\qquad S_m=\operatorname{tr}G_m,\qquad \Phi_m=B_m^{-1}\Psi_m.$$

One step. Apply Lemma 9.3 with $B=B_m$ and the shifts $w_r=v(x_m+r\alpha)-v(x_{m+1}+r\alpha)$, which satisfy $|w_r|\le\mathrm{Lip}(v)|\delta|$. It gives, for $0\le r\le q$,

$$A_r(E,x_{m+1})B_m=A_r(E,x_m)B_m\,\Pi_r^{(m)},\qquad \|\Pi_r^{(m)}-I\|\le y_m:=e^{\mathrm{Lip}(v)|\delta|S_m}-1.$$

In particular $B_m^{-1}A_{m+1}B_m = R_{\varrho_m}\Pi_q^{(m)}$ and $|\operatorname{tr} A_{m+1} - \operatorname{tr} A_m| \le 2y_m$. Put $s_0 = \sqrt{\eta}/2$. If $|\sin\varrho_m| \ge s_0$ and $y_m \le s_0/8$, Lemma 9.4 shows that A_{m+1} is elliptic with fixed point $B_m(w_m)$, where $d(i,w_m) \le d_m := 12y_m/s_0$. Choose $N_m \in SL(2,\mathbb{R})$ with $N_m(i) = w_m$; by Lemma 8.2(4), $\|N_m\|^2 = \|N_m^{-1}\|^2 = e^{d(i,w_m)} \le e^{d_m}$. Then $B_{m+1} = B_m N_m$ maps i to the fixed point of A_{m+1}.

The damped step. Lemma 9.3 with $B = B_m$ and the constant shift $w_r = i\varepsilon$ gives $A_q(z,x_m)B_m = B_m R_{\varrho_m}\Pi'_m$ with $\det\Pi'_m = 1$ and $\|\Pi'_m - I\| \le e^{\varepsilon S_m} - 1$. Hence

$$\Phi_{m+1} = N_m^{-1} R_{\varrho_m}\Pi'_m\,\Phi_m.$$

Bootstrap. We show by induction that for $0 \le m \le m_*$ the frame B_m is defined and

$$(a_m) \qquad |\operatorname{tr} A_m| \le 2 - \eta/2, \qquad S_0/2 \le S_m \le 2S_0, \qquad \|\Phi_0\|/2 \le \|\Phi_m\| \le 2\|\Phi_0\|.$$

(a_0) holds. Assume (a_j) for all $j < m$, where $m \le m_*$. Note that $\varepsilon \le \xi$ and $\operatorname{Lip}(v)|\delta| \le \xi$, and that $m_* \ge 1$ forces $\xi S_0 \le 2^{-10}\eta^{3/2}$. For $j < m$ we have $|\cos\varrho_j| \le 1 - \eta/4$, hence $\sin^2\varrho_j \ge \eta/4$ and $|\sin\varrho_j| \ge s_0$. Also $\operatorname{Lip}(v)|\delta|S_j \le 2\xi S_0 \le 1$, and $e^x - 1 \le 2x$ for $0 \le x \le 1$, so

$$y_j \le 4\xi S_0 \le 2^{-8}\eta^{3/2} \le s_0/8.$$

So the frames $B_1, \dots, B_m$ are defined. Summing the one-step estimates over $j < m$:

- $|\operatorname{tr} A_m - \operatorname{tr} A_0| \le 2\sum_{j<m} y_j \le 8m_*\xi S_0 \le 2^{-7}\eta^{3/2} \le \eta/2$, so $|\operatorname{tr} A_m| \le 2 - \eta/2$.
- $u_r^{(j+1)} = u_r^{(j)}\Pi_r^{(j)}N_j$, so, since $\|u\Pi_r^{(j)}\| \ge \|u\| - \|u(\Pi_r^{(j)} - I)\|$ and $\|uN_j\| \ge \|u\|/\|N_j^{-1}\|$, $S_{j+1}/S_j \in [(1-y_j)^2e^{-d_j}, (1+y_j)^2e^{d_j}]$ and $|\log(S_{j+1}/S_j)| \le 3y_j + d_j \le 27y_j/\sqrt{\eta}$. Hence $|\log(S_m/S_0)| \le 108\, m_*\xi S_0/\sqrt{\eta} \le 108\eta/2^{10} < \log 2$.
- A 2×2 matrix of determinant one has $\|M^{-1}\| = \|M\|$, and $\|\Pi'_j\| \le e^{\varepsilon S_j}$. So $|\log(\|\Phi_{j+1}\|/\|\Phi_j\|)| \le \varepsilon S_j + d_j/2 \le 2\xi S_0 + 12y_j/\sqrt{\eta} \le 50\xi S_0/\sqrt{\eta}$, and $|\log(\|\Phi_m\|/\|\Phi_0\|)| \le 50\eta/2^{10} < \log 2$.

This proves (a_m).

Pairs of blocks. Fix $0 \le m < m_*$ and put $\varepsilon_2 = e^{3\xi S_m} - 1$. Since $3\xi S_m \le 6\xi S_0 \le 1$, we have $\varepsilon_2 \le 12\xi S_0 \le 12\cdot 2^{-10}\eta^{3/2}$. By Lemma 9.3 with the shift $i\varepsilon$,

$$g(mq+1+r) = e_1^{\mathsf T}A_r(z,x_m)B_m\Phi_m = u_r^{(m)}(\Phi_m + e_r), \qquad \|e_r\| \le \varepsilon_2\|\Phi_m\|.$$

For the next block, $\Psi_{m+1} = B_m R_{\varrho_m}\Pi'_m\Phi_m$. Apply Lemma 9.3 first with the phase shifts, which gives $A_r(E,x_{m+1})B_m = A_r(E,x_m)B_m\Pi_r^{(m)}$, and then with the shift $i\varepsilon$, for which the rows $u_r^{(m)}\Pi_r^{(m)}$ have squared norms at most $(1+y_m)^2\|u_r^{(m)}\|^2 \le 2\|u_r^{(m)}\|^2$. This gives

$$g((m+1)q+1+r) = u_r^{(m)}\,\Pi_r^{(m)}\Pi''_r\,R_{\varrho_m}\Pi'_m\,\Phi_m = u_r^{(m)}(R_{\varrho_m}\Phi_m + e'_r),$$

where $\|\Pi''_r - I\| \le e^{2\varepsilon S_m} - 1$. The three factors $\Pi_r^{(m)}$, Π''_r and Π'_m differ from I by at most $e^{\operatorname{Lip}(v)|\delta|S_m} - 1$, $e^{2\varepsilon S_m} - 1$ and $e^{\varepsilon S_m} - 1$, and R_{ϱ_m} is orthogonal. Hence $\|\Pi_r^{(m)}\Pi''_r R_{\varrho_m}\Pi'_m - R_{\varrho_m}\| \le e^{(\operatorname{Lip}(v)|\delta| + 3\varepsilon)S_m} - 1 \le \varepsilon_2$, because $\operatorname{Lip}(v)|\delta| + 3\varepsilon \le 3\xi$. So $\|e'_r\| \le \varepsilon_2\|\Phi_m\|$.

For every $e \in \mathbb{C}^2$, $\sum_r |u_r^{(m)}e|^2 = e^*G_m e \le S_m\|e\|^2$, and $\Phi^*G_m\Phi = \sum_r |u_r^{(m)}\Phi|^2$. Apply $\|a+b\|^2 \ge \frac12\|a\|^2 - \|b\|^2$ to vectors indexed by $0 \le r < q$:

$$\mathcal{M}_m \ge \tfrac12\Phi_m^*G_m\Phi_m - S_m\varepsilon_2^2\|\Phi_m\|^2, \qquad \mathcal{M}_{m+1} \ge \tfrac12\Phi_m^*R_{\varrho_m}^{\mathsf T}G_m R_{\varrho_m}\Phi_m - S_m\varepsilon_2^2\|\Phi_m\|^2.$$

By (a_m), $|\cos\varrho_m| \le 1-\eta/4$, so Lemma 9.5 and $2\varepsilon_2^2 \le \eta/16$ give

$$\mathcal{M}_m + \mathcal{M}_{m+1} \ge \Big(\frac{\eta}{8} - 2\varepsilon_2^2\Big) S_m \|\Phi_m\|^2 \ge \frac{\eta}{16} S_m \|\Phi_m\|^2 \ge \frac{\eta}{2^7} S_0 \|\Phi_0\|^2.$$

Also $\mathcal{M}_m \le (1+\varepsilon_2)^2 S_m\|\Phi_m\|^2 \le 16 S_0\|\Phi_0\|^2$. This proves part 1.

Flux. Multiply (19) by $\overline{g(k)}$, sum over $k > n$ for some $n \ge 0$, and take imaginary parts. Since $g \in \ell^2$ and $\sum_{k>n} 2\operatorname{Re}\big(g(k+1)\overline{g(k)}\big)$ is real,

$$\varepsilon \sum_{k>n} |g(k)|^2 = \operatorname{Im}\big(g(n)\overline{g(n+1)}\big).$$

If $(g(n+1), g(n))^{\mathsf T} = B\Phi$ with B real and $\det B = 1$, then $2|\operatorname{Im}(g(n)\overline{g(n+1)})| = |\det(B\Phi, B\bar\Phi)| = |\det(\Phi,\bar\Phi)| \le \|\Phi\|^2$. At $n = m_* q$ this gives $\sum_{k>m_*q}|g(k)|^2 \le \|\Phi_{m_*}\|^2/(2\varepsilon) \le 2\|\Phi_0\|^2/\varepsilon$.

Counting. Let $X = \eta^{3/2}/(2^{10}\xi S_0)$, so that $X \ge m_* \ge 8$ and $m_* \ge X/2$. The blocks $m_0, \dots, m_*$ lie in $\{k > m_0 q\}$ and contain at least $\lfloor m_*/2\rfloor/2 \ge m_*/8$ disjoint pairs $(m, m+1)$ with $m < m_*$. By part 1,

$$\sum_{k>m_0q} |g(k)|^2 \ge \frac{m_*}{8}\cdot\frac{\eta}{2^8} S_0\|\Phi_0\|^2 \ge \frac{X}{16}\cdot\frac{\eta}{2^8} S_0\|\Phi_0\|^2 = \frac{\eta^{5/2}}{2^{22}}\cdot\frac{\|\Phi_0\|^2}{\xi}.$$

For part 3, $\sum_{k\ge1}|g(k)|^2 = \sum_{m<m_*}\mathcal{M}_m + \sum_{k>m_*q}|g(k)|^2$, and $16 m_* S_0 \le 16 X S_0 = 2^{-6}\eta^{3/2}/\xi \le 1/\xi$. Finally $g(0) = e_2^{\mathsf T}\Psi_0 = e_2^{\mathsf T} B_0 \Phi_0$. □

9.3 Almost all of the packet at resonant times

From now on $v(x) = 2\lambda\cos 2\pi x$ again, so $\operatorname{Lip}(v) = 4\pi\lambda$. We first compare the traces of the actual q-blocks with the rational discriminant.

Lemma 9.7 (Block traces)**.** *For every $h > L$ there is C such that for all large n, all real $E \in U_h$ and all $x \in \mathbb{T}$,*

$$\big|\operatorname{tr} A_{q_n}(E,x) - P_{q_n}(E) + 2\lambda^{q_n}\cos(2\pi q_n x)\big| \le C\, q_n e^{hq_n}|\delta_n|.$$

Proof. By Lemma 3.1, $P_q(E) - 2\lambda^q\cos(2\pi q x) = \operatorname{tr} A_q^{(p/q)}(E,x)$. As in Step 1 of the proof of Proposition 3.3, write each rational factor as the irrational one plus an error of norm at most $d = 4\pi\lambda|\delta|$ and expand. The difference $A_q^{(p/q)}(E,x) - A_q(E,x)$ is the sum of the terms with at least one error, so its norm is at most

$$\sum_{k=1}^{q}\binom{q}{k} C_h^{k+1} e^{h(q-k)} d^k = C_h e^{hq}\big[(1 + C_h d e^{-h})^q - 1\big] \le C_h^2 q d\, e^{hq} e^{C_h q d},$$

and $qd \le 4\pi\lambda$. The trace of a 2×2 matrix is at most twice its norm. □

Theorem 9.8 (Almost all of the packet moves out)**.** *Let $\lambda > 1$, let α be irrational and let θ be any phase. Fix $h > L$, $c > h$ and $0 < \eta \le 1/4$. There are C and n_0, depending only on λ, α, h, c and η, such that for every $n \ge n_0$ with $b_n \ge c$ and every T with $e^{hq_n} \le T \le |\delta_n|^{-1}$,*

$$P(Te^{-hq_n}, T) \ge \big(1 - Cq_n e^{-(h-L)q_n/2}\big)\,\mu^\theta\big(\{E : |\Delta_n(E)| \le 2 - 3\eta\}\big).$$

If $b_n \ge c$ for infinitely many n, then along those n the factor $\mu^\theta(\cdot)$ is at least $\frac{2}{\pi}\arcsin(1-\frac{3\eta}{2}) - o(1)$, uniformly in θ, by Lemma 9.2; since η is arbitrary, the Abel-averaged probability beyond Te^{-hq_n} tends to 1 (Theorem B). At the resonant time $T = |\delta_n|^{-1} = e^{b_n q_n}$ the distance is $e^{(b_n-h)q_n} = T^{1-h/b_n}$.

The mechanism is visible in the proof: at an energy whose first q-blocks on both sides are elliptic, the resolvent $G(\cdot; E+i/T)$ has, on each side, comparable mass on every pair of adjacent blocks among at least $c'T/(qe^{h_0 q})$ consecutive q-blocks, and at most $16S_0\|\Phi_0\|^2$ on each of them (Lemma 9.6(1)); so the at most $Te^{-hq}/q+1$ blocks within distance Te^{-hq} carry only a fraction $O(qe^{-(h-h_0)q})$ of it, with $h_0=(L+h)/2$. The mass beyond the run, which the earlier version of this argument counted against the fraction, is itself beyond Te^{-hq}.

Proof. Let $h_0=(L+h)/2$ and $U=U_{h_0}$. By Lemma 2.4 and Proposition 3.3(2) there is C with $\|A_r(E,x)\|\le Ce^{h_0 r}$ for real $E\in U$, all x and $r\ge 0$, and $\sup_U|P_q'|\le Cqe^{h_0 q}$. Write $q=q_n$, $\delta=\delta_n$, $b=b_n$ and $\varepsilon=1/T$, so that $|\delta|\le\varepsilon\le e^{-hq}$, and put

$$N=Te^{-hq},\qquad I=\{E\in U:\ |\Delta_n(E)|\le 2-2\eta\}.$$

Below, C and c' denote positive constants depending only on λ, α, h and η, and "for large n" means for n beyond a threshold depending only on λ, α, h, c and η; nothing depends on θ or T. Note that $h_0-h=-(h-L)/2$.

Step 1: two elliptic first blocks. Fix $E\in I$ and $z=E+i\varepsilon$. For $k\ge 0$ put $g_+(k)=G(k;z)$ and $g_-(k)=G(-k;z)$. Both lie in $\ell^2(\mathbb{Z}_{\ge 0})$. The function g_+ satisfies (19) for the phase θ. Since the cosine is even, $v(\theta-k\alpha)=v(-\theta+k\alpha)$, so g_- satisfies (19) for the phase $-\theta$. The first blocks are $A_q(E,\pm\theta+\alpha)$. Since $q(\pm\theta+\alpha)=\pm q\theta+p+\delta$, Lemma 9.7 and $|\cos(2\pi(\pm q\theta+\delta))-\cos(2\pi q\theta)|\le 2\pi|\delta|$ give

$$\bigl|\operatorname{tr}A_q(E,\pm\theta+\alpha)-\Delta_n(E)\bigr|\le Cqe^{h_0 q}|\delta|+4\pi\lambda^q|\delta|.$$

The right side tends to 0 because $b\ge c>h_0>L$. So for large n both first blocks satisfy $|\operatorname{tr}A_q|\le 2-\eta$.

Step 2: the run is long and the near blocks are few. Apply Lemma 9.6 to $g_\pm$, with this η, the damping ε and $\xi=(4\pi\lambda+1)\varepsilon$ (recall $\varepsilon\ge|\delta|$), and let $B_0^\pm$, $S_0^\pm$, $\Phi_0^\pm$, $m_*^\pm$, $m_0^\pm$ be the corresponding data. From $|\operatorname{tr}A_q|\le 2-\eta$ we get $|\sin\varrho_0^\pm|\ge\sqrt{\eta}/2$, and the product rule gives $|\partial_E\operatorname{tr}A_q(E,x)|\le 2qC^2e^{h_0 q}$. By Lemma 8.2(1), $S_0^\pm\le Cqe^{h_0 q}$. Hence

$$\frac{\eta^{3/2}}{2^{10}\xi S_0^\pm}\ge\frac{c'T}{qe^{h_0 q}}\ge\frac{c'e^{(h-h_0)q}}{q},$$

so for large n we have $m_*^\pm\ge 8$, and $m_0^\pm q\ge m_*^\pm q/2\ge c'Te^{-h_0 q}/4\ge Te^{-hq}=N$.

Far mass. Lemma 9.6(2) and $m_0^\pm q\ge N$ give

$$\sum_{k>N}|g_\pm(k)|^2\ge\frac{\eta^{5/2}}{2^{22}}\cdot\frac{\|\Phi_0^\pm\|^2}{\xi}.\tag{20}$$

Near mass. The sites $1\le k\le N$ lie in the blocks $\{mq+1,\dots,mq+q\}$ with $0\le m\le(N-1)/q$, which are at most $N/q+1$ blocks, and $(N-1)/q<m_0^\pm\le m_*^\pm-1$ because $m_*^\pm\ge 8$. So Lemma 9.6(1) applies to each of them, and with (20),

$$\sum_{k=1}^{N}|g_\pm(k)|^2\le\Bigl(\frac{N}{q}+1\Bigr)16S_0^\pm\|\Phi_0^\pm\|^2\le 2^{26}\eta^{-5/2}\Bigl(\frac{N}{q}+1\Bigr)S_0^\pm\xi\sum_{k>N}|g_\pm(k)|^2.$$

Here, using $S_0^\pm\le Cqe^{h_0 q}$, $\xi=(4\pi\lambda+1)/T$, $N=Te^{-hq}$ and $T\ge e^{hq}$,

$$\Bigl(\frac{N}{q}+1\Bigr)S_0^\pm\xi\le C(4\pi\lambda+1)\Bigl(e^{(h_0-h)q}+\frac{qe^{h_0 q}}{T}\Bigr)\le C(4\pi\lambda+1)(1+q)e^{-(h-L)q/2}.$$

The site 0. By Lemma 8.2(4), $\|B_0^+\|^2 = e^{d(i,z_+)}$, where z_+ is the fixed point of $\mathcal{A} = A_q(E, \theta+\alpha)$. The identity $\|\mathcal{A}\|_{\mathrm{HS}}^2 = 2 + 4\sinh^2(d(i,z_+))\sin^2\varrho_0^+$ from the proof of Lemma 8.2(4) gives $\|B_0^+\|^2 \le 1 + 4\|\mathcal{A}\|/\sqrt{\eta} \le Ce^{h_0 q}$. With part 3 of Lemma 9.6 and (20),

$$|G(0;z)|^2 \le \|B_0^+\|^2\|\Phi_0^+\|^2 \le Ce^{h_0 q}\xi \sum_{k>N} |g_+(k)|^2 \le C(4\pi\lambda+1)e^{-(h-L)q/2}\sum_{k>N}|g_+(k)|^2.$$

Conclusion of Step 2. Since $g_+(k) = G(k;z)$ and $g_-(k) = G(-k;z)$ for $k \ge 1$, adding the last three displays gives, for large n,

$$\sum_{|k|\le N} |G(k;z)|^2 \le \kappa_n \sum_{|k|>N} |G(k;z)|^2, \qquad \kappa_n = Cqe^{-(h-L)q/2},$$

and therefore

$$\sum_{|k|>N} |G(k;z)|^2 \ge (1-\kappa_n)\sum_{k\in\mathbb{Z}} |G(k;z)|^2 \qquad (E \in I). \tag{21}$$

Step 3: Parseval. By (17), (21) and (18),

$$P(N,T) = \frac{1}{\pi T}\int_{\mathbb{R}} \sum_{|k|>N} |G(k;E+i\varepsilon)|^2\,dE \ge (1-\kappa_n)\int_I \frac{1}{\pi}\operatorname{Im} G(0;E+i\varepsilon)\,dE,$$

and the right side equals $(1-\kappa_n)\int_{\mathbb{R}}\big(\int_I P_\varepsilon(E-E')\,dE\big)\,d\mu^\theta(E')$.

Step 4: Poisson smoothing. Let $r_* > 0$ be the distance from Σ to $\mathbb{R}\setminus U$, and $r = \min(r_*/2, \eta/\sup_U|P_q'|)$, so that $r \ge \eta/(Cqe^{h_0 q})$ for large n. If $E' \in \Sigma$ and $|\Delta_n(E')| \le 2-3\eta$, then $[E'-r, E'+r] \subset U$, and $\Delta_n' = P_q'$ gives $|\Delta_n| \le 2-2\eta$ there. So $[E'-r, E'+r] \subset I$ and

$$\int_I P_\varepsilon(E-E')\,dE \ge \frac{2}{\pi}\arctan\frac{r}{\varepsilon} \ge 1 - \frac{2\varepsilon}{\pi r}.$$

Since $\varepsilon/r \le Cqe^{h_0 q}/(\eta T) \le Cqe^{-(h-L)q/2}$ and μ^θ is supported on Σ, we obtain $P(N,T) \ge (1-\kappa_n)(1-Cqe^{-(h-L)q/2})\,\mu^\theta(\{|\Delta_n| \le 2-3\eta\})$, which is the claim after enlarging C. □

Remark 9.9 (Where the old constant was lost)**.** An earlier version compared the far mass (20) with the whole mass $\sum_{k\ge1}|g_\pm(k)|^2 \le 3(4\pi\lambda+1)\|\Phi_0^\pm\|^2/\xi$ of Lemma 9.6(3), which produced the factor $\eta^{5/2}2^{-24}/(4\pi\lambda+1)$, and then took $\eta = 1/4$ and $\mu^\theta(\{|\Delta_n| \le 5/4\}) \ge 1/4$, which gives $2^{-31}/(1+4\pi\lambda)$; an extra factor $1/2$, taken in the eventual Poisson-smoothing bound, produced $2^{-32}/(1+4\pi\lambda)$. The upper bound of Lemma 9.6(3) counts the mass beyond the run, but that mass is also beyond N. The flux identity, the Parseval step and the Poisson smoothing lose only factors $1-o(1)$; the only genuine loss is the spectral mass near the band edges, $1 - \frac{2}{\pi}\arcsin(1-\frac{3\eta}{2}) \approx \frac{2}{\pi}\sqrt{3\eta}$, which disappears as $\eta \to 0$.

Remark 9.10 (An annulus at strongly resonant scales)**.** Theorem 9.8 has a matching upper half. Let $\theta \in G_\alpha^*$ and $\beta(\alpha) < \infty$, and fix $B_0 > \beta(\alpha)$, $h > L$, $c > h$, $\varepsilon > 0$ and $M > 0$. Let ζ_1 be given by Theorem 5.4 for $\eta = c - L$, this ε and M, and $\Lambda = \max(1, B_0)$. For large n with $b_n \ge c$ we have $b_n \le B_0$ and, since $\theta \in G_\alpha^*$, $|\sin(2\pi q_n\theta)| \ge e^{-\zeta_1 q_n}$; this threshold in n may depend on the fixed phase. For every T with $e^{hq_n} \le T \le |\delta_n|^{-1}$ put

$$N_- = Te^{-hq_n} = T\lambda^{-q_n}e^{-(h-L)q_n}, \qquad N_+ = e^{\varepsilon q_n}\big(1+T\lambda^{-q_n}\big) \le 2e^{\varepsilon q_n}T\lambda^{-q_n}.$$

Theorem 5.4, whose time range contains $T \le |\delta_n|^{-1} \le e^{B_0 q_n}$, gives $P(N_+,T) \le Ce^{-Mq_n}$, and Theorem B, which is deduced from Theorem 9.8 and Lemma 9.2 in Section 9.4, gives $P(N_-,T) \ge$

$1-o(1)$ as $n\to\infty$ with $b_n\ge c$, uniformly in T. So the annulus $N_-<|j|\le N_+$ carries Abel-averaged mass $1-o(1)$: asymptotically the whole Abel-averaged packet lies at distance $T\lambda^{-q_n}$, up to the factors $e^{(h-L)q_n}$ and $2e^{\varepsilon q_n}$, whose exponents can be fixed as small as we like before n is taken large. In particular, for each fixed $s\in(0,1)$, the s-quantile of the Abel-averaged distribution $j\mapsto a(j,T)$ of $|j|$ (the median for $s=1/2$) lies in $[N_-,N_+]$ for all large n with $b_n\ge c$, uniformly in T. The window begins at $T=e^{hq_n}=\lambda^{q_n}e^{(h-L)q_n}$, later than λ^{q_n} by the factor $e^{(h-L)q_n}$ for the fixed $h>L$, because Theorem 9.8 requires $T\ge e^{hq_n}$. All of this concerns time averages, not the instantaneous position of the packet.

Remark 9.11 (Numerical illustration; not used in any proof)**.** We propagated δ_0 for $H_{\lambda,\alpha,\theta}$, $\alpha=p/q+\eta_\alpha$, by a Chebyshev expansion of e^{-itH} on a box of half-width at least $6T\lambda^{-q}+300$ (the mass ever seen within 30 sites of the box edge was below 10^{-150}), and computed $a(j,T)$ by the trapezoid rule in t on $[0,7T]$ with step 25 (the neglected Abel weight is e^{-14}; the computed total mass was within $1.3\cdot10^{-3}$ of 1). Each probability was checked against the finite-box Abel average evaluated from the eigendecomposition, using $P(N,T)=\sum_{|j|>N}a(j,T)$ with integer thresholds evaluated exactly. The table reports the latter values rounded to the displayed precision; powers of ten indicate orders of magnitude. The step-25 trapezoid approximation can differ at the 10^{-2} level for the shorter averaging times. With $\delta=q\alpha-p$ and $\ell=T\lambda^{-q}$ we found:

λ	p/q	θ	T	$P(0.01\ell,T)$	$P(0.1\ell,T)$	$P(\ell,T)$	median of $\lvert j\rvert$
1.5	2/5	0.2137	$\lvert\delta\rvert^{-1}=2000$	0.945	0.674	0.049	$0.21\,\ell$
1.5	2/5	0	$\lvert\delta\rvert^{-1}=2000$	0.891	0.473	10^{-8}	$0.09\,\ell$
1.5	2/5	0.2137	$0.2\lvert\delta\rvert^{-1}=400$	0.921	0.688	0.115	$0.27\,\ell$
2	2/5	0.2137	$\lvert\delta\rvert^{-1}\approx6667$	0.951	0.740	0.002	$0.26\,\ell$
2.5	2/5	0.2137	$\lvert\delta\rvert^{-1}=20000$	0.954	0.747	10^{-6}	$0.27\,\ell$
1.5	3/8	0.2137	$\lvert\delta\rvert^{-1}=6250$	0.869	0.498	10^{-25}	$0.10\,\ell$
1.2	2/5	0.2137	$\lvert\delta\rvert^{-1}=300\lambda^5$	0.918	0.548	10^{-23}	$0.12\,\ell$
1.2	3/8	0.2137	$\lvert\delta\rvert^{-1}=300\lambda^8$	0.859	0.453	10^{-11}	$0.08\,\ell$
1.2	5/13	0.2137	$\lvert\delta\rvert^{-1}=300\lambda^{13}$	0.772	0.356	10^{-19}	$0.08\,\ell$
1.2	8/21	0.2137	$\lvert\delta\rvert^{-1}=300\lambda^{21}$	0.928	0.698	0.114	$0.24\,\ell$

So the Abel-averaged distribution of $|j|$ lives on the scale $T\lambda^{-q}$, with a profile that depends on λ, q and θ (for $\lambda=1.2$ the median moved non-monotonically in q, between $0.08\,\ell$ and $0.24\,\ell$), and the fraction beyond a fixed multiple $cT\lambda^{-q}$ is far from 1 unless c is small. This is consistent with Theorem B, whose threshold Te^{-hq_n} is smaller than $T\lambda^{-q_n}$ by the factor $e^{-(h-\log\lambda)q_n}\to0$, and indicates that some such vanishing factor is needed for a fraction tending to 1. At these denominators $e^{-(h-\log\lambda)q/2}$ is not small, so the computation illustrates the theorem but does not test its rate.

9.4 Every moment

Corollary 9.12. *Let $\lambda>1$ and let α be irrational. For every phase θ and every $p>0$,*

$$\beta^+(p)\ge\max\Big\{0,\ 1-\frac{\log\lambda}{\beta(\alpha)}\Big\},$$

where the right side is read as 0 when $\beta(\alpha)=0$ and as 1 when $\beta(\alpha)=\infty$. If $\beta(\alpha)>\log\lambda$, moreover $\limsup_{T\to\infty}P(T^a,T)>0$ for every $0<a<1-\log\lambda/\beta(\alpha)$.

Proof. Nonnegativity. The vector δ_0 is not an eigenvector of H, since $H\delta_0$ has nonzero entries at ±1. So μ^θ is not a point mass and $\sum_E\mu^\theta(\{E\})^2<1$. Now

$$a(0,T)=\iint\frac{2/T}{2/T+i(E-E')}\,d\mu^\theta(E)\,d\mu^\theta(E'),$$

and the integrand has modulus at most one and tends to $\mathbf{1}_{E=E'}$, so $a(0,T) \to \sum_E \mu^\theta(\{E\})^2$ by dominated convergence. Hence $M_p(T) \geq P(0,T) = 1 - a(0,T)$ is bounded below for large T, and $\beta^+(p) \geq 0$.

The main bound. Assume $\beta(\alpha) > L$. Fix $h \in (L, \beta(\alpha))$ and $c \in (h, \beta(\alpha))$. Since $\limsup b_n = \beta(\alpha)$, there are $n_j \to \infty$ with $b_{n_j} \geq c$ and $b_{n_j} \to \beta(\alpha)$. Apply Theorem 9.8 with $\eta = 1/4$ at $T_j = |\delta_{n_j}|^{-1}$, and put $N_j = T_j e^{-hq_{n_j}} = e^{(b_{n_j}-h)q_{n_j}}$. By Lemma 9.1, $\mu^\theta(\{|\Delta_{n_j}| \leq 5/4\}) \to \frac{2}{\pi} \arcsin \frac{5}{8} > \frac{2}{5}$, and the factor $1 - Cq_{n_j} e^{-(h-L)q_{n_j}/2}$ tends to 1, so $P(N_j, T_j) \geq c_0 := 1/4$ for large j. Hence

$$M_p(T_j) \geq N_j^p \, P(N_j, T_j) \geq c_0 N_j^p, \qquad \frac{\log N_j}{\log T_j} = \frac{b_{n_j} - h}{b_{n_j}} \to 1 - \frac{h}{\beta(\alpha)},$$

with limit 1 if $\beta(\alpha) = \infty$. So $\beta^+(p) \geq 1 - h/\beta(\alpha)$, and letting $h \downarrow L$ proves the bound. If $a < 1 - L/\beta(\alpha)$, choose h with $a < 1 - h/\beta(\alpha)$; then $T_j^a \leq N_j$ for large j, so $P(T_j^a, T_j) \geq c_0$. □

Damanik and Tcheremchantsev [11] also use the exponent α_l^+, the supremum of the a for which $P(T^a, T)$ decays at no polynomial rate along some sequence of times; always $\alpha_l^+ \leq \alpha_u^+$. The second statement of Corollary 9.12 gives $\alpha_l^+ \geq 1 - \log\lambda/\beta(\alpha)$. Thus, for $\theta \in G_\alpha^*$ and $\beta(\alpha) < \infty$, $\alpha_l^+ = \alpha_u^+$. When $\gamma := 1 - \log\lambda/\beta(\alpha) > 0$, Theorem B in fact gives, for every $0 < a < \gamma$, Abel-averaged outside probability tending to one beyond T^a along a sequence of averaging times. This does not assert a positive limsup for $P(T^\gamma, T)$ at the exact critical radius.

Proof of Theorem A. Part 1. Theorem 7.3 gives $P(T^a, T) = O(T^{-m})$ for every $a > \max\{0, 1 - L/\beta(\alpha)\}$ and every $m > 0$, and $\alpha_u^+ \leq \max\{0, 1 - L/\beta(\alpha)\}$. Since $\beta^+(p) \leq \alpha_u^+$ by (4), the same bound holds for every $\beta^+(p)$. Corollary 9.12 gives the reverse inequality $\beta^+(p) \geq \max\{0, 1 - L/\beta(\alpha)\}$ at every phase, and $\alpha_u^+ \geq \beta^+(p)$.

Part 2. Corollary 9.12 gives $\beta^+(p) \geq 1$ at every phase, and Lemma 2.3(1) gives $\beta^+(p) \leq 1$. Hence $\beta^+(p) = \alpha_u^+ = 1$. This recovers [18, Corollary 3]. □

Proof of Theorem B. Let $\mathcal{N} = \{n : b_n \geq c\}$; if it is finite there is nothing to prove. Fix $\tau \in (0,1)$ and choose $\eta \in (0, 1/4]$ with $\frac{2}{\pi} \arcsin(1 - \frac{3\eta}{2}) \geq 1 - \tau/2$. By Lemma 9.2, for large $n \in \mathcal{N}$ and every phase, $\mu^\theta(\{|\Delta_n| \leq 2 - 3\eta\}) \geq 1 - 3\tau/4$. Theorem 9.8 with this η gives, for large $n \in \mathcal{N}$, every phase and every $T \in [e^{hq_n}, |\delta_n|^{-1}]$,

$$P_\theta(Te^{-hq_n}, T) \geq (1 - Cq_n e^{-(h-L)q_n/2})(1 - 3\tau/4) \geq 1 - \tau.$$

For a packet started at δ_j, the translation U^j of the proof of Proposition 9.14 conjugates $H_{\lambda,\alpha,\theta}$ to $H_{\lambda,\alpha,\theta+j\alpha}$ and maps δ_j to δ_0 and $\{|k-j| > N\}$ to $\{|k| > N\}$; so the bound for the phase $\theta + j\alpha$ is the bound for the start δ_j. □

9.5 Local dimensions

A lower bound on transport does not by itself bound packing dimensions from below: the inequalities of Guarneri and Schulz-Baldes go the other way. The lower bound on dimensions comes instead from a pointwise estimate on the resolvent, which the elliptic propagation estimates also provide.

Lemma 9.13 (Local mass)**.** *Let θ, h, c and η be as in Theorem 9.8, and put $h_0 = (L+h)/2$. There are C and n_0, depending only on λ, α, h, c and η, such that for every $n \geq n_0$ with $b_n \geq c$ and every $E \in U_{h_0}$ with $|\Delta_n(E)| \leq 2 - 2\eta$,*

$$|G(0; E + i|\delta_n|)| \leq Ce^{h_0 q_n}, \qquad \mu^\theta([E - |\delta_n|, E + |\delta_n|]) \leq 2Ce^{h_0 q_n}|\delta_n|.$$

Proof. Use Steps 1 and 2 of the proof of Theorem 9.8 with $T = |\delta_n|^{-1}$, so $\varepsilon = |\delta_n|$ and $\xi = (4\pi\lambda+1)\varepsilon$, on the right half-line only. By (18) and Lemma 9.6(2),

$$\operatorname{Im} G(0;z) = \varepsilon \sum_k |G(k;z)|^2 \geq \varepsilon \cdot \frac{\eta^{5/2}}{2^{22}} \cdot \frac{\|\Phi_0^+\|^2}{\xi} = c_1 \|\Phi_0^+\|^2, \qquad c_1 = \frac{\eta^{5/2}}{2^{22}(4\pi\lambda+1)}.$$

By Lemma 9.6(3), $\operatorname{Im} G(0;z) \leq |G(0;z)| \leq \|B_0^+\|\,\|\Phi_0^+\|$. Hence $\|\Phi_0^+\| \leq \|B_0^+\|/c_1$ and $|G(0;z)| \leq \|B_0^+\|^2/c_1 \leq Ce^{h_0 q}$, by the bound on $\|B_0^+\|^2$ in Step 2. Finally $\operatorname{Im} G(0;E+i\varepsilon) \geq \int_{|E'-E|\leq\varepsilon} \frac{\varepsilon}{2\varepsilon^2}\, d\mu^\theta(E')$, so $\mu^\theta([E-\varepsilon, E+\varepsilon]) \leq 2\varepsilon\,|G(0;E+i\varepsilon)|$. □

Proof of Theorem C, part 1. Let $\beta = \beta(\alpha)$. Upper local dimensions of a finite measure are nonnegative, so assume $\beta > L$. Fix $h \in (L,\beta)$, $c \in (h,\beta)$ and $n_j \to \infty$ with $b_{n_j} \geq c$ and $b_{n_j} \to \beta$, and put $h_0 = (L+h)/2$, $U = U_{h_0}$ and $r_j = |\delta_{n_j}|$.

Two canonical measures at the same radii. Let $\theta' = \theta + \alpha$. Translation by one site conjugates $H_{\lambda,\alpha,\theta}$ to $H_{\lambda,\alpha,\theta'}$ and maps δ_1 to δ_0, as in the proof of Proposition 9.14 below. So $\mu_{\delta_1} = \mu^{\theta'}$, and

$$\nu := \mu_{\delta_0} + \mu_{\delta_1} = \mu^\theta + \mu^{\theta'}.$$

Write Δ_n and Δ_n' for the discriminants at the phases θ and θ'. Since $q_n\theta' = q_n\theta + p_n + \delta_n$, we have $|\Delta_n' - \Delta_n| \leq 4\pi\lambda^{q_n}|\delta_n| \to 0$. For $0 < \eta \leq 1/8$ let $J_j(\eta) = \{E \in U : |\Delta_{n_j}(E)| \leq 2 - 4\eta\}$. For large j, every $E \in J_j(\eta)$ satisfies $|\Delta_{n_j}(E)| \leq 2-2\eta$ and $|\Delta_{n_j}'(E)| \leq 2-2\eta$. Lemma 9.13, applied at the phases θ and θ', then gives

$$\nu([E-r_j, E+r_j]) \leq 4Ce^{h_0 q_{n_j}} r_j \qquad (E \in J_j(\eta),\ j \text{ large}).$$

A full-measure set. Let $\Lambda_\eta = \limsup_j J_j(\eta)$. Each $E \in \Lambda_\eta$ lies in $J_j(\eta)$ for infinitely many j, and $\log r_j = -b_{n_j} q_{n_j}$, so

$$\frac{\log \nu([E-r_j, E+r_j])}{\log r_j} \geq 1 - \frac{h_0}{b_{n_j}} - \frac{\log(4C)}{b_{n_j} q_{n_j}} \to 1 - \frac{h_0}{\beta}$$

along those j, with limit 1 if $\beta = \infty$. Thus $d_\nu^+ \geq 1 - h/\beta$ on Λ_η. For a finite measure, $\nu(\limsup_j J_j) \geq \limsup_j \nu(J_j)$. The measures μ^θ and $\mu^{\theta'}$ are supported on $\Sigma \subset U$, and for large j the set $J_j(\eta)$ contains $\{E \in \Sigma : |\Delta_{n_j}'(E)| \leq 2 - 5\eta\}$. Lemma 9.1 at the two phases gives

$$\nu(\Lambda_\eta) \geq \frac{2}{\pi}\arcsin(1-2\eta) + \frac{2}{\pi}\arcsin\Big(1 - \frac{5\eta}{2}\Big),$$

which tends to $2 = \nu(\mathbb{R})$ as $\eta \to 0$. So $\bigcup_{k\geq 8} \Lambda_{1/k}$ has full ν-measure, and $d_\nu^+ \geq 1 - h/\beta$ holds ν-almost everywhere. Letting $h \downarrow L$ along a sequence gives $d_\nu^+ \geq \max\{0, 1 - L/\beta\}$ ν-almost everywhere.

Every spectral measure. Since $\delta_{k+1} = (H - v(\theta + k\alpha))\delta_k - \delta_{k-1}$, the vectors δ_0, δ_1 generate $\ell^2(\mathbb{Z})$ under H, and every spectral measure μ_φ is absolutely continuous with respect to ν. Let $f = d\mu_\varphi/d\nu$. By differentiation of integrals for Radon measures [21, Theorem 2.12 and Corollary 2.14(2)], $\mu_\varphi([E-r, E+r])/\nu([E-r, E+r]) \to f(E)$ as $r \downarrow 0$ for ν-almost every E, and $0 < f(E) < \infty$ for μ_φ-almost every E. At such points $\log \mu_\varphi([E-r,E+r]) = \log \nu([E-r,E+r]) + O(1)$, so $d_{\mu_\varphi}^+(E) = d_\nu^+(E)$. Since $\mu_\varphi \ll \nu$, the bound $d_\nu^+ \geq \max\{0, 1 - L/\beta\}$ holds μ_φ-almost everywhere. □

The upper bound in part 2 is a consequence of the upper transport bound, Theorem 7.3.

Proposition 9.14. *Let $\lambda > 1$ and let α be irrational with $\beta(\alpha) < \infty$. For every $\theta \in G_\alpha^*$ and every nonzero $\varphi \in \ell^2(\mathbb{Z})$, the spectral measure μ_φ of $H_{\lambda,\alpha,\theta}$ satisfies $\dim_P^+(\mu_\varphi) \leq \max\{0, 1 - \log\lambda/\beta(\alpha)\}$.*

Proof. Guarneri and Schulz-Baldes [14, Theorem 1] proved that for every $q > 0$ the upper growth exponent of the q-th moment, computed with the Cesàro average $T^{-1}\int_0^T \cdot\, dt$, is at least the upper packing dimension $\dim_P^+(\mu_{\delta_0})$, defined as the essential supremum of the upper local dimension. The same inequality holds for every unit vector $\psi \in \ell^2(\mathbb{Z})$ whose spectral measure has compact support, by [3, Theorem 1 and Proposition 1(iii)]: with $\langle\langle |X|^p\rangle\rangle_\psi(T) = T^{-1}\int_0^T \||X|^{p/2}e^{-itH}\psi\|^2\, dt$,

$$\limsup_{T\to\infty} \frac{\log\langle\langle |X|^p\rangle\rangle_\psi(T)}{\log T} \geq p\, D^+_{\mu_\psi}\Big(\frac{1}{1+p}\Big) \geq p\dim_P^+(\mu_\psi) \qquad (p>0),$$

where D^+_μ is the upper generalized Rényi dimension. Since $(2/T)e^{-2t/T} \geq (2/T)e^{-2}$ for $0 \leq t \leq T$, the Abel-averaged moment $M_q(T)$ dominates $2e^{-2}$ times the Cesàro-averaged one. So the Cesàro exponents are at most $\beta^+(q) \leq \alpha_u^+$, and Theorem 7.3 gives $\dim_P^+(\mu_{\delta_0}) \leq \max\{0, 1 - L/\beta\}$ for $\theta \in G_\alpha^*$.

The translation $(Uu)(n) = u(n+1)$ satisfies $UH_{\lambda,\alpha,\theta}U^{-1} = H_{\lambda,\alpha,\theta+\alpha}$ and $U\delta_1 = \delta_0$. Hence the spectral measure of δ_1 for the phase θ equals that of δ_0 for the phase $\theta+\alpha$, and the same bound holds because $\theta + \alpha \in G_\alpha^*$ by Lemma 2.2(2). For a Jacobi operator every spectral measure μ_φ is absolutely continuous with respect to $\nu = \mu_{\delta_0} + \mu_{\delta_1}$. The upper packing dimension of a measure is the infimum of the packing dimensions of its full-measure Borel sets [14, Appendix]. Packing dimension of sets is countably stable, so $\dim_P^+(\nu) = \max\{\dim_P^+(\mu_{\delta_0}), \dim_P^+(\mu_{\delta_1})\}$, and absolute continuity gives $\dim_P^+(\mu_\varphi) \leq \dim_P^+(\nu)$. □

Proof of Theorem C, part 2. By Proposition 9.14, the μ_φ-essential supremum of $d^+_{\mu_\varphi}$ is at most $\max\{0, 1 - L/\beta(\alpha)\}$. By part 1, $d^+_{\mu_\varphi}$ is at least this number μ_φ-almost everywhere. So $d^+_{\mu_\varphi}$ equals it μ_φ-almost everywhere, and its essential infimum $\dim_P^-(\mu_\varphi)$ and essential supremum $\dim_P^+(\mu_\varphi)$ both equal it. The set G_α^* has full measure by Lemma 2.2. □

10 Generalized Rényi dimensions

The equality of every upper transport exponent is stronger than the packing statement alone. It determines an entire interval of generalized Rényi dimensions. These dimensions describe the spectral measure; they are not entanglement entropies.

For a compactly supported probability measure μ and $0 < r < 1$, put

$$I_\mu(r,\epsilon) = \int \mu([E-\epsilon, E+\epsilon])^{r-1}\, d\mu(E), \qquad D^\pm_\mu(r) = \limsup_{\epsilon\downarrow 0} \frac{\log I_\mu(r,\epsilon)}{(r-1)\log\epsilon},$$

where the lim sup is replaced by lim inf in the definition of $D^-_\mu(r)$. We use the integral convention of Barbaroux, Germinet and Tcheremchantsev [3]; for positive orders it agrees with the usual partition definition.

Lemma 10.1. *Let μ be a nonzero finite Borel measure on $\mathbb{R}$ and $0 < r < 1$. Then $D^+_\mu(r) \geq \dim_P^+(\mu)$, where $\dim_P^+(\mu)$ is the μ-essential supremum of d^+_μ.*

Proof. Write $B(E,\rho) = [E-\rho, E+\rho]$ and $d = \dim_P^+(\mu)$; we may assume $d > 0$. Fix $0 < \kappa < d$. The set $A = \{E : d^+_\mu(E) > d - \kappa\}$ has $\mu(A) > 0$. For $E \in A$ there are arbitrarily small ϵ with $\mu(B(E,\epsilon)) < \epsilon^{d-\kappa}$; if $2^{-k-1} \leq \epsilon \leq 2^{-k}$, then $\mu(B(E,\rho_k)) \leq (2\rho_k)^{d-\kappa}$ with $\rho_k = 2^{-k-1}$. So $A \subset \limsup_k A_k$, where $A_k = \{E \in A : \mu(B(E,\rho_k)) \leq (2\rho_k)^{d-\kappa}\}$ is Borel. By the Borel–Cantelli lemma $\sum_k \mu(A_k) = \infty$, so $\mu(A_k) \geq k^{-2}$ for infinitely many k. For those k, since $r - 1 < 0$,

$$I_\mu(r,\rho_k) \geq \int_{A_k} \mu(B(E,\rho_k))^{r-1}\, d\mu(E) \geq k^{-2}(2\rho_k)^{(d-\kappa)(r-1)},$$

and dividing $\log I_\mu(r,\rho_k)$ by $(r-1)\log\rho_k > 0$ gives a quantity at least $(d-\kappa)\log(2\rho_k)/\log\rho_k - 2\log k/((r-1)\log\rho_k) \to d-\kappa$. Hence $D_\mu^+(r) \ge d-\kappa$ for every κ. □

Theorem 10.2. *Let $\lambda > 1$, let α be irrational, and let $\mu = \mu_{\delta_0}$. For every phase,*

$$D_\mu^-(r) = 0 \qquad (0 < r < 1).$$

For $\theta \in G_\alpha^*$ when $\beta(\alpha) < \infty$, and for every phase when $\beta(\alpha) = \infty$,

$$D_\mu^+(r) = \max\{0, 1 - \log\lambda/\beta(\alpha)\} \qquad (0 < r < 1),$$

with the usual endpoint conventions.

Proof. Write

$$C_p(T) = \frac{1}{T}\int_0^T \sum_j |j|^p |\langle \delta_j, e^{-itH}\delta_0\rangle|^2\, dt.$$

Theorem 1 of [3] and Lemma 10.1 give, with $r = 1/(1+p)$,

$$D_\mu^\pm(r) \le \limsup_{T\to\infty} \frac{\log C_p(T)}{p\log T}, \qquad D_\mu^+(r) \ge \dim_P^+(\mu),$$

where the first lim sup is replaced by lim inf for the minus sign. The compact support of μ permits every $p > 0$. On $0 \le t \le T$, the Abel weight is at least $2e^{-2}/T$, so $C_p(T) \le (e^2/2)M_p(T)$. Hence

$$\dim_P^+(\mu) \le D_\mu^+(r) \le \beta^+(p).$$

For $\theta \in G_\alpha^*$ and $\beta(\alpha) < \infty$, Theorem A(1) identifies the right side with γ. The first paragraph of the proof of Proposition 9.14, which holds for $\theta \in G_\alpha^*$, together with Theorem C(1), identifies the left side with γ as well. The shifted phase condition is unnecessary for this single canonical measure. The endpoint $\beta = \infty$ follows in the same way from Theorems A(2) and C(1). Finally, Damanik and Tcheremchantsev [11, Theorem 4] prove $\beta^-(p) = 0$ for every supercritical almost Mathieu operator and every phase. The lower version of the same inequality gives $0 \le D_\mu^-(r) \le \beta^-(p) = 0$. □

The lower identity is an existing consequence of [11, 3]. The new upper transport and packing formulas supply the upper identity. We make no assertion of continuity at $r = 0$ or $r = 1$, and do not identify the upper dimensions for $r > 1$.

11 Computability

We use standard notions for reals: a real is computable from an oracle if the oracle computes a sequence of rationals converging to it with a known rate, and two reals have the same Turing degree if each is computable from the other. A real x is right-c.e. relative to $\mathbf{0}'$ if $\{r \in \mathbb{Q} : r > x\}$ is c.e. relative to $\mathbf{0}'$. We use one external lemma.

Lemma 11.1 (Becher, Bugeaud and Slaman [4, Lemma 4])**.** *A real x is the* lim sup *of a computable sequence of rationals if and only if x is right-c.e. relative to $\mathbf{0}'$.*

Proof of Theorem D. 1. By Theorem A, $\alpha_u^+ = \max\{0, 1 - L/\beta\}$, a computable function of β when λ is computable. If $\beta > L$ then $\beta = L/(1-\alpha_u^+)$, so β is computable from α_u^+.

2. Let $\lambda = 2$, so $L = \log 2$, and $\theta = 1/4$. If the denominators of α are eventually odd, then $|\sin(2\pi q_n/4)| = 1$ eventually, so $1/4 \in G_\alpha$ and Theorem A applies.

Every value is right-c.e. relative to $\mathbf{0}'$. For computable α the ratios β_n form a computable sequence of reals, so rationals within 2^{-n} of them form a computable sequence with the same lim sup. If $\beta = \infty$ then $\alpha_u^+ = 1$. Otherwise β is right-c.e. relative to $\mathbf{0}'$ by Lemma 11.1, and so is $x = \max\{0, 1 - L/\beta\}$: for rational $r \in (0,1)$, $x < r$ holds if and only if some rational s satisfies $\beta < s < L/(1-r)$, which is a $\mathbf{0}'$-c.e. condition; $x < r$ never holds for $r \le 0$ and always holds for $r \ge 1$.

Every such value occurs. For $x = 1$ take $a_1 = 1$ and $a_{n+1} = 2^{q_n^2}$, so $\beta = \infty$. For $x = 0$ take $a_1 = 1$ and $a_n = 2$ for $n \ge 2$, so $\beta = 0$. Let $x \in (0,1)$ be right-c.e. relative to $\mathbf{0}'$. Then $s = 1/(1-x) > 1$ is right-c.e. relative to $\mathbf{0}'$, by the same monotonicity argument. Lemma 11.1 gives a computable sequence of rationals v_n with $\limsup v_n = s$; replacing v_n by $\min\{\max\{v_n, 1\}, B\}$ for an integer $B > s$ does not change the lim sup. Define

$$a_1 = 1, \qquad a_{n+1} = 2^{\lceil v_n q_n \rceil} \quad (n \ge 1).$$

Since $a_{n+1} q_n \le q_{n+1} \le 2 a_{n+1} q_n$,

$$0 \le \beta_n - (\log 2) v_n \le \frac{\log(4 q_n)}{q_n},$$

so $\beta = s \log 2 > L$ and $\alpha_u^+ = 1 - L/\beta = 1 - 1/s = x$. In all three constructions every partial quotient after a_1 is even, so $q_0 = q_1 = 1$ and the recurrence $q_{n+1} = a_{n+1} q_n + q_{n-1}$ keeps every q_n odd. The partial quotients are computable, and $|\alpha - p_n/q_n| < q_n^{-2}$ with $q_n \to \infty$, so α is computable.

3. Let K_2 be the halting problem relative to $\mathbf{0}'$, a set of degree $\mathbf{0}''$ that is c.e. relative to $\mathbf{0}'$, and put

$$s = 2 - \sum_{e \in K_2} 4^{-(e+2)} \in \Big[\frac{23}{12}, 2\Big].$$

The sum is left-c.e. relative to $\mathbf{0}'$, so s is right-c.e. relative to $\mathbf{0}'$, and part 2 produces a computable α with $\alpha_u^+(H_{2,\alpha,1/4}) = x = 1 - 1/s \in [11/23, 1/2]$. The oracle K_2 computes s. Conversely, from s one recovers K_2: the base-4 digits of $2 - s$ are 0 or 1, and after the digits at positions below $e+2$ are subtracted, the remainder is at most $\frac{1}{3} \cdot 4^{-(e+2)}$ if $e \notin K_2$ and at least $4^{-(e+2)}$ if $e \in K_2$, so a sufficiently good rational approximation decides membership. Thus s, and hence $x = 1 - 1/s$, has degree exactly $\mathbf{0}''$. □

Remark 11.2. When the ratios β_n converge, their limit is computable from $\mathbf{0}'$, and every nonnegative real t computable from $\mathbf{0}'$ arises as such a limit for a computable α with odd denominators. By the limit lemma there are computable rationals $w_n \to t/\log 2$; put $a_1 = 1$ and $a_{n+1} = 2^{\max\{1, \lceil q_n \max\{w_n, 0\} \rceil\}}$, so that the partial quotients are even and $\beta_n \to t$, including $t = 0$. In this regular class the exponent therefore has degree at most $\mathbf{0}'$, and degree $\mathbf{0}'$ occurs. For instance, with $h_K = \sum_{j \in K} 2 \cdot 4^{-(j+1)}$ for the halting set K and v_n the stage-n approximations of $3/2 + h_K/4$, the exponent of $H_{2,\alpha,1/4}$ is $1 - 1/s \in [1/3, 2/5]$ with $s = 3/2 + h_K/4$, of degree $\mathbf{0}'$. Part 3 of Theorem D needs ratios that oscillate: a convergent computable sequence cannot have a limit of degree $\mathbf{0}''$.

Remark 11.3. All three parts of Theorem D hold for every moment exponent: by Theorem A, $\beta^+(p) = \alpha_u^+$ for every $p > 0$ in parts 1–3. They also hold for packing dimensions, with the following phase conditions.

In part 1, $\theta \in G_\alpha^*$. The first paragraph of the proof of Proposition 9.14 gives $\dim_P^+(\mu_{\delta_0}) \le \alpha_u^+$, and Theorem C(1) gives $d_{\mu_{\delta_0}}^+ \ge \alpha_u^+$ almost everywhere. So the spectral measure of δ_0 has $\dim_P^- = \dim_P^+ = \alpha_u^+$. For every nonzero spectral measure, Theorem C(2) gives the same conclusion.

In parts 2 and 3, $\theta = 1/4$ and the denominators of α are eventually odd. Then $1/4 \in G_\alpha \subset G_\alpha^*$, as shown in the proof of part 2 and by Lemma 2.2(1). So, when $\beta(\alpha) < \infty$, Theorem C(2) shows that the exponent is also the packing dimension of every nonzero spectral measure of $H_{2,\alpha,1/4}$. When $\beta(\alpha) = \infty$, the exponent is 1, and so is every such packing dimension: Theorem C(1) gives $d^+ \geq 1$ almost everywhere, and every finite Borel measure on $\mathbb{R}$ has upper packing dimension at most 1, as noted after Theorem C. Hence in parts 2 and 3 the packing dimensions of all nonzero spectral measures of $H_{2,\alpha,1/4}$ take exactly the values stated for α_u^+. In particular the explicit operator of part 3 has spectral measures whose packing dimension has Turing degree $\mathbf{0}''$.

The noncomputability concerns infinite-time asymptotics of an infinite system. Finite-volume propagator entries and finite-time probabilities, at computable times and with specified computable data, can be approximated effectively to any prescribed accuracy. For the explicit operator in Theorem D(3), however, no algorithm can compute the long-time exponent, even with an oracle for the halting problem.

11.1 The degree persists under changes of coupling

The noncomputable exponent is not tied to a single coupling value.

Corollary 11.4. *There exists one computable irrational α, with eventually odd convergent denominators, such that for every computable $\lambda \in (1, 2^{23/12})$ and phase $\theta = 1/4$, every upper moment exponent and both packing dimensions of every nonzero spectral measure have Turing degree $\mathbf{0}''$. The same is true of $D^+_{\mu_{\delta_0}}(r)$ for each $0 < r < 1$.*

Proof. Use the frequency in Theorem D(3). Its resonance parameter is $\beta = s \log 2$, where $s \in [23/12, 2]$ has degree $\mathbf{0}''$. Since $\log \lambda < \frac{23}{12} \log 2 \leq \beta$, the exponent is positive for every indicated coupling and equals $\gamma_\lambda = 1 - \log \lambda / \beta$. The computable positive number $\log \lambda$ gives reductions in both directions: $\gamma_\lambda = 1 - \log \lambda / \beta$ and $\beta = \log \lambda / (1 - \gamma_\lambda)$. The phase conditions do not depend on λ, and Theorem 10.2 supplies the last assertion. □

Remark 11.5. For the frequency of Corollary 11.4 the set of computable couplings for which the exponent has degree $\mathbf{0}''$ is exactly the set of computable $\lambda \in (1, 2^s)$, where $s \in [23/12, 2]$ is the number of Theorem D(3). Indeed, for computable $\lambda \geq 2^s$ the exponent is $\max\{0, 1 - \log \lambda / \beta\} = 0$; and $\lambda = 2^s$ is impossible for computable λ, since s is not computable. So the dynamical transition in the coupling occurs at a threshold 2^s of degree $\mathbf{0}''$. Replacing 2 by an integer $M \geq 2$ in the definition of s (that is, $s_M = M - \sum_{e \in K_2} 4^{-(e+2)}$; the digit argument of Theorem D(3) is unchanged) gives, for each M, one computable frequency for which every computable $\lambda \in (1, 2^{M-1/12})$ yields exponents of degree $\mathbf{0}''$. No single frequency can serve every $\lambda > 1$, since the exponent vanishes for $\lambda \geq e^{\beta(\alpha)}$ when $\beta(\alpha) < \infty$ and equals 1 when $\beta(\alpha) = \infty$.

The numerical exponent varies with λ. The corollary concerns its computability degree, and only computable coupling changes.

11.2 Every finite frequency prefix admits every allowed exponent

Corollary 11.6. *Fix $\lambda = 2$ and $\theta = 1/4$. For each real $\gamma \in [0, 1]$ that is right-c.e. relative to $\mathbf{0}'$, and each finite continued-fraction prefix, there is a computable irrational continuation with eventually odd denominators whose upper transport exponents equal γ. In particular, every such exponent is realized densely among computable frequencies.*

Proof. Consecutive convergent denominators are coprime and hence not both even. Starting with their parities, append at most two positive partial quotients to obtain two consecutive odd denominators. Subsequent even partial quotients keep them odd.

For $0 < \gamma < 1$, the proof of Theorem D supplies a bounded computable rational sequence $v_n \geq 1$ with $\limsup v_n = (1-\gamma)^{-1}$. Continue with $a_{n+1} = 2^{\lceil v_n q_n \rceil}$. Then

$$\frac{\log q_{n+1}}{q_n} = (\log 2) v_n + o(1).$$

A finite prefix does not affect the limsup, so Theorem A gives γ. For $\gamma = 0$ use an eventually constant even tail, and for $\gamma = 1$ use $a_{n+1} = 2^{q_n^2}$. The convergents provide a computable approximation to each continuation. Every open interval contains a continued-fraction cylinder, proving density. □

Combined with Proposition 12.5, this gives a precise limitation on inference from finite-time dynamics. Frequencies can be chosen arbitrarily close, with uniformly close predictions on any prescribed finite time interval, while their asymptotic upper exponents differ. Even the extreme values zero and one already have dense realizations by the known endpoint theorems. The present classification gives the full allowed range of computability-theoretic values.

12 Individual times and finite-volume approximation

The transport theorems concern Abel averages. Two elementary deductions clarify what those averages imply at individual times and how their finite-time predictions can be computed. The first uses positivity of probabilities; the second uses only nearest-neighbor hopping.

12.1 A positive duration of instantaneous escape

Lemma 12.1. *Let $f : [0,\infty) \to [0,1]$ be measurable and suppose*

$$\frac{2}{T}\int_0^\infty e^{-2t/T} f(t)\, dt \geq c, \qquad 0 < c \leq 1.$$

Set $\tau = (T/2)\log(4/c)$. Then

$$|\{t \in [0,\tau] : f(t) \geq c/2\}| \geq cT/8.$$

Here $|\cdot|$ denotes Lebesgue measure.

Proof. The weight of $[\tau,\infty)$ is $c/4$, so the weighted integral over $[0,\tau]$ is at least $3c/4$. Its part where $f < c/2$ is at most $c/2$. Thus the remaining set has weighted measure at least $c/4$. The weight is at most $2/T$, giving the claimed duration. □

Corollary 12.2. *Use the hypotheses of Theorem B, fix $c_\lambda \in (0,1)$, and write $N = Te^{-hq_n}$. For every sufficiently large permitted scale and every $e^{hq_n} \leq T \leq |\delta_n|^{-1}$, the set*

$$\left\{ t \in \left[0, \frac{T}{2}\log\frac{4}{c_\lambda}\right] : \sum_{|j|>N} |\langle \delta_j, e^{-itH}\delta_0\rangle|^2 \geq c_\lambda/2 \right\}$$

has measure at least $c_\lambda T/8$.

Proof. Apply Lemma 12.1 to the instantaneous outside probability and use Theorem B. □

Corollary 12.3 (Escape at almost all times of the window)**.** *Use the hypotheses of Theorem B, put* $N = Te^{-hq_n}$*, and fix* $b > 0$ *and* $s \in (0,1)$*. Then*

$$\lim_{\substack{n\to\infty \\ b_n \geq c}} \sup_{\theta\in\mathbb{T}} \sup_{e^{hq_n} \leq T \leq |\delta_n|^{-1}} \frac{1}{T} \Big| \Big\{ t \in [0, bT] : \sum_{|j|\leq N} |\langle \delta_j, e^{-itH_{\lambda,\alpha,\theta}} \delta_0 \rangle|^2 > s \Big\} \Big| = 0.$$

If $\theta \in G_\alpha^*$ *and* $\beta(\alpha) < \infty$*, the same holds, for this fixed* θ*, with* $\sum_{|j|\leq N}$ *replaced by the probability outside the annulus* $N_- < |j| \leq N_+$ *of Corollary 12.4.*

Proof. Let $f(t) = \sum_{|j|\leq N} |\langle \delta_j, e^{-itH} \delta_0 \rangle|^2 \in [0,1]$. Then $\frac{2}{T} \int_0^\infty e^{-2t/T} f(t)\, dt = 1 - P(N,T) =: \tau_n$, and $\tau_n \to 0$ uniformly in θ and T by Theorem B. On $[0, bT]$ the weight $\frac{2}{T} e^{-2t/T}$ is at least $\frac{2}{T} e^{-2b}$, so Chebyshev's inequality gives $|\{t \in [0,bT] : f(t) > s\}| \leq \frac{T}{2} e^{2b} \tau_n / s$. For the annulus, the Abel average of the probability outside $N_- < |j| \leq N_+$ is $1 - P(N_-, T) + P(N_+, T)$, which tends to 0 by Remark 9.10; argue in the same way. □

For good phases the upper estimate also places a positive fraction in a specified annulus at individual times.

Corollary 12.4. *Let* $\theta \in G_\alpha^*$ *and* $\beta(\alpha) < \infty$*. Fix* $h > L$*,* $c > h$*,* $\zeta > 0$ *and* $c_\lambda \in (0,1)$*. At every sufficiently large scale with* $b_n \geq c$*, and every* $T \in [e^{hq_n}, |\delta_n|^{-1}]$*, set*

$$N_- = Te^{-hq_n}, \qquad N_+ = e^{\zeta q_n}(1 + T\lambda^{-q_n}).$$

There is a measurable set of times of measure at least $c_\lambda T/16$ *inside* $[0, (T/2)\log(8/c_\lambda)]$ *on which the probability in* $N_- < |j| \leq N_+$ *is at least* $c_\lambda/4$*.*

Proof. Remark 9.10, with $\varepsilon = \zeta$, bounds the Abel-averaged annulus probability below by $1 - o(1)$, hence by $c_\lambda/2$ for large n. Use Lemma 12.1 with $c = c_\lambda/2$. □

The annulus is fixed by the selected averaging scale T. These statements do not say that its radius follows an instantaneous power law, that the indicated times are close to T, or that such times have positive asymptotic density over the whole half-line.

12.2 Dependence on frequency and phase

Proposition 12.5. *Let* v *be a real Lipschitz sampling function with constant* K*, and let* $H_{\alpha,\theta}$ *be its nearest-neighbor Schrödinger operator. For representatives of the frequencies and phases,*

$$\|(e^{-itH_{\alpha,\theta}} - e^{-itH_{\alpha',\theta'}})\delta_0\| \leq K(|\theta - \theta'|\,|t| + |\alpha - \alpha'|\,t^2).$$

Consequently the Abel-averaged position distributions satisfy

$$\mathrm{TV}(a_{\alpha,\theta}(\cdot,T), a_{\alpha',\theta'}(\cdot,T)) \leq \min\left\{1, \frac{K}{2}(T|\theta - \theta'| + T^2|\alpha - \alpha'|)\right\}.$$

One may minimize phase differences over integer translates.

Proof. The bounded commutator $[X, H] = S^{-1} - S$ has norm at most two. Duhamel's formula, first with bounded position cutoffs and then by a strong limit, gives $\|Xe^{-isH}\delta_0\| \leq 2|s|$. The difference of the two potentials at site j has modulus at most $K(|\theta - \theta'| + |j|\,|\alpha - \alpha'|)$. Apply the propagator Duhamel identity to δ_0 and integrate from zero to $|t|$. For unit vectors, the total variation distance

of the outcome distributions of a measurement is at most their trace distance, which is at most their Hilbert-space distance. Convexity and

$$\frac{2}{T}\int_0^\infty te^{-2t/T}\,dt = T/2, \qquad \frac{2}{T}\int_0^\infty t^2e^{-2t/T}\,dt = T^2/2$$

prove the second assertion. □

For almost Mathieu, $K = 4\pi\lambda$. At fixed frequency and phase, changing the coupling adds the propagator bound $2|\lambda - \lambda'||t|$ and the Abel-distribution bound $|\lambda - \lambda'|T$. These estimates concern evolution from a localized vector. They do not assert operator-norm continuity of the infinite-volume Hamiltonian in its frequency.

Corollary 12.6. *Fix $\lambda = 2$, $\theta = 1/4$, a reference frequency α_0, a finite time horizon $T_* > 0$, and an accuracy $\eta > 0$. For every $\gamma \in [0,1]$ that is right-c.e. relative to $\mathbf{0}'$, there is a computable irrational α with eventually odd denominators and upper transport exponent γ such that*

$$\sup_{|t|\le T_*} \|(e^{-itH_{2,\alpha,1/4}} - e^{-itH_{2,\alpha_0,1/4}})\delta_0\| < \eta.$$

The outcome distributions of any measurement of these evolved states then differ in total variation by less than η throughout that interval.

Proof. By Corollary 11.6, choose a frequency realizing γ within $\eta/(8\pi T_*^2)$ of α_0. Proposition 12.5 gives the uniform norm estimate. Measurement cannot increase trace distance, which is at most the norm distance of the two unit vectors. □

The proof uses $\lambda = 2$ only through Corollary 11.6. The frequencies built there with $\beta(\alpha) = 0$ and with $\beta(\alpha) = \infty$ have eventually odd denominators, so by Theorem A their exponents at the phase $1/4$ are 0 and 1 for every $\lambda > 1$. With 8π replaced by $4\pi\lambda$, finite-horizon data at nonzero tolerance therefore do not determine the exponent at any coupling $\lambda > 1$.

Thus finite-time data with nonzero error tolerance cannot give a uniform restriction on the asymptotic exponent without an additional arithmetic promise. This is a statement about finite precision. It does not identify distinct frequencies with exactly identical evolution, and it does not rule out inference within a more restricted frequency class.

12.3 An explicit finite-volume certificate

Let H have an arbitrary bounded real potential. Let H_N be its restriction to $[-N, N]$, with the two boundary bonds removed, and extend its evolved vector by zero outside that interval. Write $a_N(\cdot, T)$ for its Abel-averaged distribution. Define

$$R_N(\tau) = 2\sum_{k=N+1}^{\infty} \frac{(2\tau)^k}{k!}.$$

Proposition 12.7. *For $N \ge 0$ and $t \ge 0$,*

$$\|e^{-itH}\delta_0 - e^{-itH_N}\delta_0\| \le \min\{2, R_N(t)\}.$$

For every $\tau > 0$,

$$\mathrm{TV}(a(\cdot,T), a_N(\cdot,T)) \le \min\{1, R_N(\tau) + e^{-2\tau/T}\}.$$

The same bound controls the error in the probability of any prescribed set of sites.

Proof. Write $H = V + K$, where V is diagonal and $K = S + S^{-1}$. Remove the boundary bonds to obtain an operator on all of $\ell^2(\mathbb{Z})$ whose evolution from δ_0 is the zero extension of the finite-volume evolution. In the interaction picture for V, both hopping operators have norm at most two. Their Dyson terms of orders $k \le N$ agree on δ_0: a path starting at zero cannot cross a removed bond with fewer than $N+1$ nearest-neighbor steps. Each term of order k has norm at most $(2t)^k/k!$. The difference of the remaining terms is therefore at most $R_N(t)$. The norm difference of two unit vectors is at most two. For the averaged distributions, use this estimate up to time τ and the trivial total variation bound one afterwards. □

The series is explicit; for example $R_N(\tau) \le 2e^{2\tau}(2\tau)^{N+1}/(N+1)!$. Given an error tolerance $0 < \eta < 1$, take $\tau = (T/2)\log(2/\eta)$ and then choose N so that $R_N(\tau) \le \eta/2$. Finite-matrix exponentiation and integration then give a certified approximation to the infinite-volume distribution, with any additional numerical error accounted for separately.

For computable $\lambda, \alpha, \theta, T$, all entries of the finite matrix are computable. Matrix exponentials on the compact interval $[0, \tau]$ can be approximated with explicit Taylor remainders, and their integrals by uniform approximation. The proposition thus proves effective computability of the finite-time Abel probabilities without extracting a rate from the asymptotic transport theorem. It does not supply an effective onset scale for Theorem B.

13 Discussion and open problems

The dynamical picture. For comparison with the upper exponents, define

$$\beta^-(p) = \liminf_{T\to\infty} \frac{\log M_p(T)}{p \log T}, \qquad \alpha_u^- = \sup_{p>0} \beta^-(p).$$

Damanik and Tcheremchantsev proved that these lower Abel-averaged transport exponents vanish for every $\lambda > 1$, every irrational frequency and every phase [11, Theorem 4]. For $0 < \log\lambda < \beta(\alpha) < \infty$ and $\theta \in G_\alpha^*$, Theorem A gives $\beta^+(p) = \alpha_u^+ = 1 - \log\lambda/\beta(\alpha)$ for every $p > 0$. When $\beta(\alpha) \le \log\lambda$, the upper exponents also vanish at phases in G_α^*. The positive-exponent regime is therefore intermittent: upper exponents record resonant bursts, whereas lower exponents record sequences of confinement times. More precisely, fix $\zeta > 0$ and $c > \log\lambda + \zeta$. For $\theta \in G_\alpha^*$ and $\beta(\alpha) < \infty$, at sufficiently large scales with $b_n \ge c$, and with $T = |\delta_n|^{-1}$, Theorem B and Remark 9.10, with $h = \log\lambda + \zeta$, put Abel-averaged mass $1 - o(1)$ between $T\lambda^{-q_n}e^{-\zeta q_n}$ and $2T\lambda^{-q_n}e^{\zeta q_n}$, with exponentially small mass beyond the outer radius. Corollary 12.3 shows that, for each permitted averaging scale T and fixed $b > 0$ and $s \in (0,1)$, the instantaneous mass outside the corresponding annulus exceeds s at only $o(T)$ times in $[0, bT]$. This does not determine the asymptotic density of burst times across all scales.

Universality. For $0 < \beta(\alpha) < \infty$, Theorem 8.1 gives the fastest-exponent lower bound at every phase for Lipschitz sampling functions whose Lyapunov exponent is continuous near a spectral minimizer. The rest of this paper uses the almost Mathieu structure: Chambers' formula makes the phase dependence of the rational discriminant a single cosine with energy-independent amplitude. The released companion [12, Theorem A] proves the exact fastest-exponent law $\max\{0, 1 - \ell_*/\beta(\alpha)\}$ for almost every phase of every real-analytic operator that is supercritical and of type I on its spectrum, where $\ell_* = \min_\Sigma L$. Its lower bound holds at every phase; at $\beta = 0$ and $\beta = \infty$ its equality holds at every phase, with the same endpoint conventions as here. If $v(x) = w(kx)$, the operator has reduced sampling function w, frequency $k\alpha$ and phase $k\theta$, so the relevant arithmetic

quantity is $\beta(k\alpha)$. Beyond type I, the companion proves the almost-every-phase fastest-exponent law, with $\beta(s\alpha)$ in place of $\beta(\alpha)$, for $v(x) = w(sx)$ whose primitive part w is supercritical and of type I [12, Theorem C], and, for every real-analytic v with positive Lyapunov exponent on the spectrum, conditionally on a separation hypothesis for the periodic approximants of the primitive sampling function [12, Theorem D].

Hausdorff versus packing dimension. Jitomirskaya and Last proved that for $\lambda > 1$ the spectral measures of the almost Mathieu operator have zero Hausdorff dimension [19]. For $0 < \log\lambda < \beta(\alpha) < \infty$ and $\theta \in G_\alpha^*$, Theorem C gives packing dimension $1 - \log\lambda/\beta(\alpha) > 0$ for every nonzero spectral measure, whereas its Hausdorff dimension is zero. Thus these measures have different Hausdorff and packing dimensions at the same phases where lower transport exponents vanish and upper ones are positive. Theorem 10.2 also determines the upper and lower generalized Rényi dimensions of the canonical spectral measure for orders between zero and one. Exact values at other orders, and analogous statements for arbitrary initial vectors with uncontrolled spatial tails, are not determined here.

Exceptional phases. The upper bounds are proved for $\theta \in G_\alpha^*$, a full-measure set invariant under $\theta \mapsto \theta + \alpha$ that tolerates any subexponential decay of $|\sin(2\pi q_n\theta)|$ along scales with $b_n \geq c$. The transport and packing lower bounds of Theorems A–C hold for every phase. Exceptional phases can nevertheless have larger fastest exponents, as the released companion proves [13, Theorems B(1) and C]. Let $\beta_e(\alpha)$ be the limsup of b_n along even denominators q_n, with value zero if there are only finitely many. If $\beta_e(\alpha) = \beta(\alpha) \in [\log\lambda, \infty)$, then

$$\sup_{\theta\in\mathbb{T}} \alpha_u^+(H_{\lambda,\alpha,\theta}) = \alpha_u^+(H_{\lambda,\alpha,1/4}) = 1 - \frac{\log\lambda}{2\beta(\alpha)}.$$

The same value occurs on a dense G_δ set of phases and is strictly larger than the almost-every-phase value. This is a theorem about the fastest exponent, not equality of all moment exponents at the exceptional phases. Without any condition on even denominators, the same companion bounds every phase uniformly: for $\log\lambda \leq \beta(\alpha) < \infty$, $\alpha_u^+(H_{\lambda,\alpha,\theta}) \leq 1 - \log\lambda/(2\beta(\alpha))$ at every phase, and for $0 < \beta(\alpha) < \log\lambda$ the corresponding bound is $\beta(\alpha)/(\beta(\alpha) + \log\lambda)$ [13, Theorem A and Corollary 5.6].

Turning phases. The barrier of Proposition 4.4 holds at every phase. At an exact turning phase of the drifting cosine, where $y_0 = 2\cos(2\pi q_n\theta) = \pm 2$, such as $\theta = 0$ at every scale, the confinement comes from the quadratic part of the cosine: for $b_n > L$ the radius $e^{(b_n - L + \varepsilon)q_n}$ becomes $e^{(b_n - L/2 + \varepsilon)q_n}$, and for $b_n < L$ the proof gives only $e^{(b_n/2 + O(\varepsilon))q_n}$. The reflection-symmetric phases $\theta = (m + k\alpha)/2$ with $k \neq 0$ are not exact turning phases: since $|\sin(2\pi q_n\theta)| = |\sin(\pi k\delta_n)|$, the value y_0 only approaches ± 2, exponentially fast along the resonant scales, and this is what excludes them from G_α^* (Lemma 2.2). At turning phases the non-turning inverse-branch argument of Section 5 no longer supplies the upper velocity scale λ^{-q}. The filters are centred at y_0, and the sine-based lower bound (13) degenerates when y_0 equals or is exponentially close to ± 2. A filter of width ρ requires the relevant real inverse branches e_b to extend smoothly across $[y_0 - 4\rho, y_0 + 4\rho]$, with $\rho \geq e^{-o(q)}$. Requiring every critical value of P_q to have modulus greater than $(2 + 4\rho)\lambda^q$ is a sufficient stronger condition; the actual obstruction is a critical value in the required interval after division by λ^q. Whether the branches extend depends on the relevant gaps of the periodic approximant. At $\theta = 0$ and scales with $4 \mid q_n$ it fails. For even q the conserved polynomial is even and satisfies $P_q(0) = 2(-1)^{q/2}(\lambda^q + 1)$ [13, Corollary 7.2]; so for $4 \mid q$ the critical value at $E = 0$ is $(2 + 2\lambda^{-q})\lambda^q$, and the central gap of the approximant $H_{\lambda,p/q,0}$ is closed. Next to this

closed gap, the companion's argument constructs, for every fixed $h > L$ and $0 < \eta \le 1/4$, an elliptic energy window of width at least $c_{\eta,h} q^{-1} e^{-hq/2}$. This gives the polynomial-tail lower bound $\alpha_u^+(H_{\lambda,\alpha,0}) \ge \max\{0, 1 - \log\lambda/(2\beta_4(\alpha))\}$ derived in Remark 13.1, where $\beta_4(\alpha)$ is the limsup of b_n along denominators divisible by four, and the example there has exponent $3/4$ at $\theta = 0$ and $1/2$ at almost every phase. The exponent at a turning phase is therefore not determined by the radius of the quadratic barrier alone. We do not know it for other arithmetic patterns of the resonant denominators, where the gap at the turning point may stay open, and we do not know whether the packing dimension depends on the phase. The single-scale finite-volume observations in Remark 4.5 are consistent with the radius $\lambda^{-q/2}/|\delta|$, both for $b_n > \log\lambda$ and for $\frac{1}{2}\log\lambda < b_n < \log\lambda$; in the latter examples the non-turning phases had much smaller observed radii. They concern one observable at small denominators and say nothing about exponents.

Remark 13.1 (The phase 0 along denominators divisible by four)**.** This remark records a consequence of the argument of [13, Theorem 7.5], applied at $\theta = 0$ along the denominators divisible by four. It is not a result printed in [13], whose exponent conclusion is stated only at the phase $1/4$. Let $\lambda > 1$, $L = \log\lambda$, let α be irrational, and put

$$\beta_4(\alpha) = \limsup_{n:\ 4|q_n} b_n,$$

with $\beta_4(\alpha) = 0$ if only finitely many q_n are divisible by four. Then

$$\alpha_u^+(H_{\lambda,\alpha,0}) \ge \max\Big\{0,\ 1 - \frac{L}{2\beta_4(\alpha)}\Big\}.$$

The right side is read as 0 if $\beta_4(\alpha) = 0$ and as 1 if $\beta_4(\alpha) = \infty$. The argument has four steps.

(i) *Closed gap.* For even q, [13, Corollary 7.2] gives $P_q(0) = 2(-1)^{q/2}(\lambda^q+1)$ and $P_q(-E) = P_q(E)$. If $4 \mid q$, the discriminant $\Delta_t = P_q - 2\lambda^q$ of the approximant $H_{\lambda,p/q,0}$ therefore satisfies $\Delta_t(0) = 2$ and $\Delta_t'(0) = 0$: its central gap is closed.

(ii) *Hypotheses and phase condition.* [13, Theorem 7.5] assumes $\lambda > 1$, α irrational, $h > L$ and $\eta \in (0, 1/4]$, and works at scales with q_n even and $b_n \ge h/2 + (h - L)$, at averaging times $e^{(h/2+(h-L))q_n} \le T \le |\delta_n|^{-1}$, and, besides the phase $1/4$, at every phase φ with $|\cos(2\pi q_n\varphi) - (-1)^{q_n/2}| \le \eta\lambda^{-q_n}/16$. At $\varphi = 0$ and $4 \mid q_n$ the left side is $|1 - 1| = 0$. For such a phase the argument compares the first q_n-block of $H_{\lambda,\alpha,\varphi}$ to the right of the origin with the rational block $A_{q_n}^{(p_n/q_n)}(E, \varphi + p_n/q_n)$; at $\varphi = 0$ its trace is Δ_t, which by (i) has a closed gap at $E = 0$, and the potentials of the two blocks differ by at most $4\pi\lambda|\delta_n|$ at each site.

(iii) *Tail estimate.* There are then $c_\eta > 0$ and n_0 such that, for every $n \ge n_0$ with $4 \mid q_n$ and $b_n \ge h/2 + (h - L)$, and every T with $e^{(h/2+(h-L))q_n} \le T \le |\delta_n|^{-1}$,

$$P_0\big(\lfloor c_\eta T e^{-hq_n/2}\rfloor, T\big) \ge T^{-2} e^{-2hq_n},$$

where P_0 is the outside probability for $H_{\lambda,\alpha,0}$ and the initial state δ_0. The ingredients are an energy $E^{(1)}$ next to the closed gap with $|\Delta_t(E^{(1)})| = 2 - \eta$ and $|P_{q_n}'(E^{(1)})| \le q_n\sqrt{2C_h\eta}\, e^{hq_n/2}$ [13, Proposition 7.4], the insertion lemma (Lemma 9.3) and an elliptic run of the Weyl solution (Lemma 9.6). The statement of [13, Theorem 7.5] includes this tail estimate, with modified constants, for every phase satisfying the condition in (ii).

(iv) *Exponent.* At $T_n = |\delta_n|^{-1} = e^{b_n q_n}$ the tail in (iii) is at least T_n^{-2-2h/b_n} beyond a radius at least $c_\eta T_n^{1-h/(2b_n)} - 1$. Since $M_p(T) \ge N^p P(N, T)$, along scales with $4 \mid q_n$ and $b_n \to \beta_4(\alpha)$ we get $\beta^+(p) \ge 1 - h/(2\beta_4(\alpha)) - (2 + 2h/\beta_4(\alpha))/p$ whenever $h/2 + (h - L) < \beta_4(\alpha) < \infty$. Let $p \to \infty$ and

then $h \downarrow L$. If $\beta_4(\alpha) \le L/2$ there is nothing to prove, and if $\beta_4(\alpha) = \infty$ the same computation with $b_n \to \infty$ gives 1.

If $\beta_4(\alpha) = \beta(\alpha) \in [L, \infty)$, the all-phase bound $\alpha_u^+ \le 1 - L/(2\beta(\alpha))$ of [13, Theorem A and Corollary 5.6] gives equality: $\alpha_u^+(H_{\lambda,\alpha,0}) = 1 - L/(2\beta(\alpha))$. For example, take $\lambda = 2$ and the computable frequency with partial quotients $a_1 = 4$, $a_{2k} = 4^{q_{2k-1}}$ and $a_{2k+1} = 4$ for $k \ge 1$. Since $q_0 = 1$, $q_1 = 4$ and $4^q \equiv 0 \pmod 4$ for $q \ge 1$, induction gives $q_{2k-1} \equiv 0$ and $q_{2k} \equiv 1 \pmod 4$. Along $n = 2k - 1$ we have $q_n^{-1} \log q_{n+1} = \log 4 + O(q_n^{-1} \log q_n)$, and along $n = 2k$ this ratio tends to 0; by (7) the same holds for b_n. Hence $\beta_4(\alpha) = \beta(\alpha) = 2 \log 2 = 2L$, and $\alpha_u^+(H_{2,\alpha,0}) = 3/4$, whereas Theorem A gives $1/2$ at almost every phase.

The size of the fraction. Theorem B gives the fraction $1 - o(1)$. More precisely, for each fixed $0 < \eta \le 1/4$, throughout its permitted resonant windows,

$$1 - P(Te^{-hq_n}, T) \le 1 - \frac{2}{\pi} \arcsin\Big(1 - \frac{3\eta}{2}\Big) + C_\eta q_n e^{-(h - \log\lambda)q_n/2} + o_\eta(1),$$

uniformly in the phase and in T. The last term comes from convergence in the arcsine law; no explicit rate for it is given here. First taking the scale limit and then $\eta \downarrow 0$ makes the deficit vanish. The behavior well after the resonant time remains open: for $T \gg |q_n\alpha - p_n|^{-1}$ the elliptic propagation estimates are limited by the drift $|q_n\alpha - p_n|$ rather than the damping $1/T$, so the near-block count used here no longer gives a fraction tending to one. This does not affect the exponent proof, which uses resonant averaging times.

Critical coupling. At $\lambda = 1$ the Lyapunov exponent vanishes. For $\beta(\alpha) > 0$, Jitomirskaya and Zhang proved quasiballistic transport at every phase [18, Corollary 3]. For $\beta(\alpha) = 0$ both halves of our method fail: the velocity λ^{-q} of the upper bound becomes 1, and the elliptic propagation estimates need scales with $|q_n\alpha - p_n| \le e^{-cq_n}$ for some $c > 0$, which then do not exist. Transport at critical coupling with $\beta(\alpha) = 0$ is a separate problem.

Physical meaning. The golden mean has $\beta(\alpha) = 0$, and the upper power-law transport exponents vanish for $\lambda > 1$ at every phase [11, Theorem 4]. Positive upper exponents can occur for other irrational frequencies at the same coupling. The finite-scale statement is Theorem B: for fixed $c > h > \log\lambda$, along sufficiently large denominators with $b_n \ge c$, the Abel-averaged mass beyond Te^{-hq_n} tends to one uniformly in phase and in $e^{hq_n} \le T \le |\delta_n|^{-1}$. Such scales require correspondingly large next partial quotients; the theorem does not assert this behavior for an arbitrary single large partial quotient or from time λ^{q_n} without an exponential margin. The exponent is an infinite-time quantity of an infinite system. Its noncomputability for the explicit operator of Theorem D(3) does not prevent effective approximation of finite-time probabilities for computable data and computable times. Figure 3 gives finite-volume, finite-time observations of radii comparable to $1/(|\delta|\lambda^q)$ and initial growth comparable to the reference speed λ^{-q}; it does not measure a limsup exponent.

How this work was done

The authors developed the results and proofs with assistance from AI models from more than one developer. They take responsibility for every claim. The authors chose the question, directed the work and decided which lines to pursue; the models did most of the exploration, derivation and drafting.

The proofs of version 1.0.0 were developed over several long working sessions, each building on the previous ones and recording its proofs, failed routes and self-checks. Each step was then checked

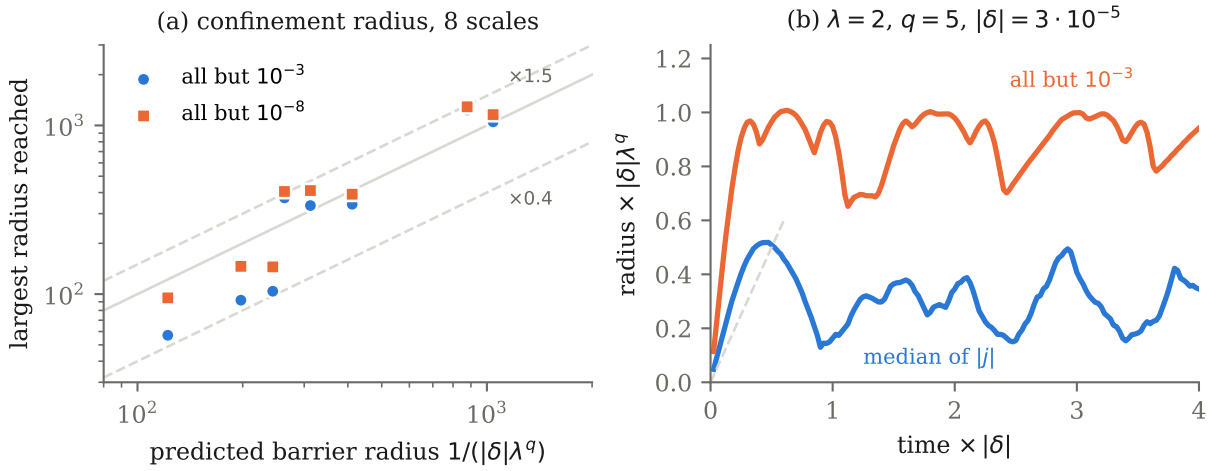


Figure 3: Finite-volume numerical illustration at one scale per frequency; not used in any proof. We propagated the initial state δ_0 at phase $\theta = 0.2137$ by a Chebyshev expansion, with $\alpha = p/q + \eta_\alpha$, $\delta = q\alpha - p$ and $N_b = 1/(|\delta|\lambda^q)$. The box is $[-M, M]$ with Dirichlet boundary conditions and $M = \lfloor 5N_b \rfloor + 200$. We sampled the instantaneous distribution at 160 equally spaced times up to $4/|\delta|$; the mass within 50 sites of a box edge was below 10^{-80} at these sampled times. (a) Maximum sampled radii enclosing all but 10^{-3} (circles) or all but 10^{-8} (squares) of the instantaneous mass, compared with N_b. The eight choices $(\lambda, p/q, \eta_\alpha)$ are $(1.5, 2/5, 10^{-4})$, $(1.5, 2/5, 3 \cdot 10^{-5})$, $(2, 2/5, 2 \cdot 10^{-5})$, $(2, 2/5, 6 \cdot 10^{-6})$, $(3, 2/5, 2 \cdot 10^{-6})$, $(1.5, 3/8, 2 \cdot 10^{-5})$, $(2, 3/8, 4 \cdot 10^{-6})$ and $(1.5, 5/13, 2 \cdot 10^{-6})$. The radius ratios lie between 0.4 and 1.6; decreasing $|\delta|$ in the repeated (λ, q) pairs keeps them of the same order, rather than exactly unchanged. (b) For $\lambda = 2$, $p/q = 2/5$ and $\eta_\alpha = 6 \cdot 10^{-6}$, the median and the radius enclosing all but 10^{-3} of the instantaneous mass are shown in units of N_b, against $t|\delta|$. The dashed line is the reference growth $R(t) = t\lambda^{-q}$, not a fitted or proved velocity with that prefactor. The later oscillation and the sampled edge-mass diagnostic are numerical observations, not a certified infinite-volume error bound.

by a separate session that had not seen the material, with instructions to refute it and to read the primary sources it relies on, and the repairs found were incorporated. The mathematics new in this version was checked by one blind, refutation-first session that had not seen the paper; its findings, including two false statements that are now repaired, are incorporated here. The text then had an editorial review and a reconciliation of the references and citation locators. Priority and literature searches were AI-assisted. The proofs are analytic. Numerical computations appear only in remarks and figures labelled as illustrations, and are not used in any proof. The finite-volume estimate in Section 12 provides an explicit error bound for numerical evaluation; finite simulations are not used to infer the asymptotic transport law.

References

[1] Serge Aubry and Gilles André. Analyticity breaking and Anderson localization in incommensurate lattices. *Ann. Israel Phys. Soc.*, 3:133–164, 1980.

[2] Artur Avila, Jiangong You, and Qi Zhou. Sharp phase transitions for the almost Mathieu operator. *Duke Math. J.*, 166(14):2697–2718, 2017. https://doi.org/10.1215/00127094-

2017-0013.

[3] Jean-Marie Barbaroux, François Germinet, and Serguei Tcheremchantsev. Quantum diffusion and generalized Rényi dimensions of spectral measures. In *Journées Équations aux Dérivées Partielles (Nantes, 2000)*, pages 1–16. Université de Nantes, 2000. Exp. No. I; https://doi.org/10.5802/jedp.565.

[4] Verónica Becher, Yann Bugeaud, and Theodore A. Slaman. The irrationality exponents of computable numbers. *Proc. Amer. Math. Soc.*, 144(4):1509–1521, 2016. https://doi.org/10.1090/proc/12841.

[5] Patrick Billingsley. *Probability and Measure*. Wiley, New York, third edition, 1995.

[6] Jean Bourgain and Svetlana Jitomirskaya. Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential. *J. Statist. Phys.*, 108(5-6):1203–1218, 2002. Theorem 1 and Corollary 2 cited in arXiv:math-ph/0110040v1; https://doi.org/10.1023/A:1019751801035.

[7] Matthew Bradshaw, Titus de Jong, Wencai Liu, Audrey Wang, Xueyin Wang, and Bingheng Yang. Quantum dynamical bounds for quasi-periodic operators with Liouville frequencies. Preprint, arXiv:2510.25603v1, 2025. https://doi.org/10.48550/arXiv.2510.25603.

[8] W. G. Chambers. Linear-network model for magnetic breakdown in two dimensions. *Phys. Rev.*, 140(1A):A135–A143, 1965. https://doi.org/10.1103/PhysRev.140.A135.

[9] Toby S. Cubitt, David Perez-Garcia, and Michael M. Wolf. Undecidability of the spectral gap. *Nature*, 528(7581):207–211, 2015. https://doi.org/10.1038/nature16059.

[10] David Damanik, Rowan Killip, and Barry Simon. Perturbations of orthogonal polynomials with periodic recursion coefficients. *Ann. of Math.*, 171(3):1931–2010, 2010. https://doi.org/10.4007/annals.2010.171.1931.

[11] David Damanik and Serguei Tcheremchantsev. Upper bounds in quantum dynamics. *J. Amer. Math. Soc.*, 20(3):799–827, 2007. Theorems 4 and 7 and Section 5 cited in arXiv:math-ph/0502044v1; https://doi.org/10.1090/S0894-0347-06-00554-6.

[12] Seth Douglas and Nidhal Mghirbi. Universality of the transport law for supercritical quasi-periodic operators. Zenodo, version 1.0.0, 2026. https://doi.org/10.5281/zenodo.23830333.

[13] Seth Douglas and Nidhal Mghirbi. Worst-case transport and light cones for quasi-periodic operators. Zenodo, version 1.0.0, 2026. https://doi.org/10.5281/zenodo.23845228.

[14] Italo Guarneri and Hermann Schulz-Baldes. Lower bounds on wave packet propagation by packing dimensions of spectral measures. *Math. Phys. Electron. J.*, 5(1):1–16, 1999. Paper 1; https://web.ma.utexas.edu/mpej/Vol/5/1.abs.

[15] Lian Haeming. Quasiballistic transport for discrete one-dimensional quasiperiodic Schrödinger operators. *J. Spectr. Theory*, 15(4):1477–1502, 2025. https://doi.org/10.4171/JST/566.

[16] Svetlana Jitomirskaya and Helge Krüger. Exponential dynamical localization for the almost Mathieu operator. *Comm. Math. Phys.*, 322(3):877–882, 2013. https://doi.org/10.1007/s00220-013-1743-9.

[17] Svetlana Jitomirskaya, Wencai Liu, and Serguei Tcheremchantsev. Lower bounds on concentration through Borel transforms and quantitative singularity of spectral measures near the arithmetic transition. Preprint, arXiv:2501.12153v1, 2025. https://doi.org/10.48550/arXiv.2501.12153.

[18] Svetlana Jitomirskaya and Shiwen Zhang. Quantitative continuity of singular continuous spectral measures and arithmetic criteria for quasiperiodic Schrödinger operators. *J. Eur. Math. Soc.*, 24(5):1723–1767, 2022. https://doi.org/10.4171/JEMS/1139.

[19] Svetlana Ya. Jitomirskaya and Yoram Last. Power law subordinacy and singular spectra. II. Line operators. *Comm. Math. Phys.*, 211(3):643–658, 2000. https://doi.org/10.1007/s002200050830.

[20] Yoav Lahini, Rami Pugatch, Francesca Pozzi, Marc Sorel, Roberto Morandotti, Nir Davidson, and Yaron Silberberg. Observation of a localization transition in quasiperiodic photonic lattices. *Phys. Rev. Lett.*, 103(1):013901, 2009. https://doi.org/10.1103/PhysRevLett.103.013901.

[21] Pertti Mattila. *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. https://doi.org/10.1017/CBO9780511623813.

[22] Giacomo Roati, Chiara D'Errico, Leonardo Fallani, Marco Fattori, Chiara Fort, Matteo Zaccanti, Giovanni Modugno, Michele Modugno, and Massimo Inguscio. Anderson localization of a non-interacting Bose–Einstein condensate. *Nature*, 453(7197):895–898, 2008. https://doi.org/10.1038/nature07071.