A beginning is worth one limit: robust unlearnable laws in noisy cellular automata*

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Abstract

We compare nested expectation laws of begun and stationary probabilistic cellular automata. At equal scale depth d, a computable begun law has ceiling Δ^0_{d+2}, whereas consensus on a nonempty effectively Σ^0_2 class of stationary translation-invariant states has ceiling Δ^0_{d+1}. We attain the begun ceiling at every finite depth using positive-rate three-dimensional rules, an open kernel region and a uniform renewal-gap readout schema. The canonical seed clones a uniform Bernoulli row on the physical block alphabet. Seed universality refers to blocked full-support, one-cell-shift-invariant ergodic laws on the pre-blocking alphabet. The seed has no distinguished origin or program but retains infinite transverse redundancy. At depth one, local error certificates and a continuous plateau quantizer give every Δ^0_3 value in [0,1] with a clock-free readout and genuine inner time limits. Every ordinary and Cesaro cluster state carries the same law; at computable kernels its stationary Cesaro cluster class is Π^0_2 and has no low member when the value is outside Δ^0_2. An exact conditional-cone identity transfers any computable-seed begun depth-d law to a depth-(d+1) law in every stationary state of the same positive-rate rule. The extra limit is sufficient and sharp across the arithmetic ranges, but need not be minimal for an individual law. In the normalized four-dimensional Gibbs formulation its lag is a slab separation. At computable begun data, fixed targets outside Δ^0_2 cannot be learned with positive probability by effective finite-history inquiries. Removing cloned preparation from the realization remains open.

1 Definitions and scope

We compare two ways of assigning a number to an infinite noisy system. A begun law takes a time limit from prescribed initial data, followed by scale limits. An eternal law takes scale limits of stationary ensemble statistics. Stationarity supplies no extra time limit. At equal scale depth this distinction changes an arithmetic ceiling by one level. The comparison concerns expectation functionals and effective consensus classes. It makes no assertion that a finite experiment completes a limit.

1.1 Dynamics and preparations

Let Σ be a finite labelled alphabet, $N \Subset \mathbb{Z}^D$ a finite neighbourhood, and $K(a \mid r)$ a stochastic vector for every $r \in \Sigma^N$. The probabilistic cellular automaton (PCA) P_K updates sites independently conditional on the previous configuration. It commutes with translations. Positive rates mean

$$\kappa(K) = \min_{a,r} K(a \mid r) > 0.$$

*The authors developed this work with AI models from more than one developer and take responsibility for every claim. How the work was done, and what the machine checks cover, is described at the end of the paper.

The finite product of row simplices has its relative topology. Open kernel sets below are relatively open in this space. Computability means computability of all finitely many entries. The ceiling results also apply to kernels with zero entries; the realization and fluctuation results explicitly require positive rates.

Write $\mathcal{M}$ for probability measures on $\Sigma^{\mathbb{Z}^D}$ with the weak topology and $\mathcal{M}_{\mathrm{TI}}$ for the translation-invariant ones. The stationary TI class is

$$\mathcal{S}_{\mathrm{TI}}(K) = \{\mu \in \mathcal{M}_{\mathrm{TI}} : \mu P_K = \mu\}.$$

A phase means an extreme point of this compact convex class. This definition does not identify extremality with shift ergodicity. An eternal world is a stationary ensemble, with its stationary path law when histories are observed. A begun world has law λP_K^t at age t, where λ is a specified computable TI measure unless another hypothesis is stated.

For the constructions, coordinates are $(a, b, i) \in \mathbb{Z}^2 \times \mathbb{Z}$. The noiseless line rule uses an enlarged pre-blocking alphabet A. Grouping B consecutive cells gives the physical alphabet $\Sigma = A^B$, as specified in Section 4. An admissible row law is the blocked image of a full-support, one-cell-shift-invariant ergodic law on $A^{\mathbb{Z}}$. This universality class concerns laws before blocking; it does not include arbitrary shift-ergodic laws on $\Sigma^{\mathbb{Z}}$. A cloned row has $x(a, b, i) = y(i)$. Its law is TI in all three coordinates. Our canonical seed clones a uniform Bernoulli row on Σ; it is the blocked image of the uniform Bernoulli row on A and is therefore included. There is no distinguished site or finite program in this computable seed. There is a time-zero preparation and exact infinite transverse redundancy. This seed is not a product measure on $\mathbb{Z}^3$.

1.2 Readouts and the counting of limits

Definition 1.1 (Effective finite statistics)**.** A scale-depth-d family has indices $\mathbf{k} \in \mathbb{N}^d$ and

$$s_{\mathbf{k}}(\mu) = R_{\mathbf{k}}\big(\mu(F_{\mathbf{k},1}), \ldots, \mu(F_{\mathbf{k},j_{\mathbf{k}}})\big).$$

One algorithm, uniformly in every index, returns all finite supports, rational cylinder functions F, and a real-computable evaluator R on an effectively open domain. It computes the value to any requested rational accuracy wherever defined. A stationary-history statistic may use finitely many space-time cylinders instead. The family is bounded linear if $s_{\mathbf{k}}(\mu) = \mu(F_{\mathbf{k}})$ with $|F_{\mathbf{k}}| \le 1$.

The eternal law and begun law are respectively

$$\Phi(\mu) = \lim_{k_1} \cdots \lim_{k_d} s_{\mathbf{k}}(\mu), \tag{1}$$

$$\Phi_{\mathrm{beg}}(\lambda, K) = \lim_{k_1} \cdots \lim_{k_d} \lim_{t\to\infty} s_{\mathbf{k},t}(\lambda P_K^t). \tag{2}$$

All limits are finite, and every intermediate limit must exist for each fixed tuple of remaining indices. These are partial functionals; their admissible indices need not be decidable. At $d = 0$, the eternal law has no limit, while the begun law still has its time limit. Ratios are used only on a positive-denominator domain. A clock-free begun family has $s_{\mathbf{k},t} = s_{\mathbf{k}}$: its statistic uses scale but not age. A lag in a stationary history is an ordinary index; if sent to infinity it counts toward depth.

Consensus on $C \subseteq \mathcal{S}_{\mathrm{TI}}(K)$ means the entire law is defined and has the same value on every member of C. Robust begun realization means one value for all K in a nonempty open set U. Robust all-state eternal consensus means one value for every stationary state at every $K \in U$. A selected state is not a consensus class. All our origin-free statements mean that translating the finite reference window changes no ensemble value; they do not remove the window's coordinate or block presentation.

1.3 Arithmetic and effective classes of states

Let $0^{(n)}$ denote the nth Turing jump, with $0^{(0)}$ computable. Put

$$\mathsf{D}_n = \{x \in \mathbb{R} : x \text{ has a fast Cauchy name computable in } 0^{(n-1)}\}, \qquad n \geq 1.$$

Thus D_1 is the computable reals and D_2 the limits of computable rational sequences. The finite jump hierarchy is strict. One existing limit of a uniformly X-computable real sequence is computable in X'.

We fix a compact representation compatible with cylinder Cauchy names. For each probability p, code $2p-1 = \sum_{n\geq 1} d_n 2^{-n}$ with $d_n \in \{-1, 0, 1\}$, and interleave all these signed digits. Three symbols are encoded by two bits, excluding the fourth pair by an effective closed condition. Decoding is computable and continuous. Conversely, given Cauchy data for a residual $y \in [-1, 1]$, choose $d = -1, 0, 1$ by certifying, respectively, $y < 0$, $|y| < 1/2$, or $y > 0$; these overlapping alternatives cover the interval. Replace y by $2y - d$ and continue. This uniformly computes a code without testing equality to a dyadic rational. Normalization and marginal consistency impose further effective closed conditions on the coordinate arrays. A state is low when it has such a name X with $X' \equiv_T 0'$. An effective closed class means a Π^0_1 class of these valid codes; an effective Σ^0_2 class is a computable union of such closed pieces. This is stronger than an arbitrary noneffective F_σ description. All state-class hypotheses use this fixed representation, including validity constraints.

Uniform finite statistics are crucial: a noncomputable site map would be advice. Even a computable decoder can carry a target program when it varies with the target. The upper bounds quantify over all effective readouts. For attainment we instead fix a uniform readout schema, independent of the target, parameterized only by the finite labelled alphabet, block length and the prescribed depth. The target program goes into the finite dynamics. We do not claim one literal observation table on one common alphabet for all targets.

2 The main results

The following statements give the comparison before the constructions. Proofs occupy Sections 3–8. The two substantial realization inputs are the noiseless self-organization of Hellouin de Menibus–Sablik and the reliable stacking of Toom and Gács–Reif, in the forms stated below.

Theorem 2.1 (Consensus and begun ceilings)**.** *For computable K, an eternal depth-d law constant on a nonempty effective Σ^0_2 class of stationary TI states belongs to D_{d+1}. At depth zero its common value is computable. For bounded linear families, agreement and existence of every intermediate limit on all extreme states suffice for agreement on all stationary TI states. For computable λ and K, a begun depth-d law belongs to D_{d+2}. These ceilings transfer to common robust values on open kernel sets. They also hold for finite stationary-history statistics, counting each additional limiting lag as one limit.*

Theorem 2.2 (Realization without a distinguished origin)**.** *For each $\beta \in \mathsf{D}_3 \cap [0, 1]$ there is a TI positive-rate PCA on $\mathbb{Z}^3$ and a nonempty open kernel region U_β with the following property. From the canonical computable cloned Bernoulli seed, a fixed uniform gap-readout schema has begun scale-depth-one law β for every $K \in U_\beta$. Its initial version uses the world's age for copy spacing. The same value holds for the clone of the blocked image of every full-support, one-cell-shift-invariant ergodic row law on the pre-blocking alphabet A. This quantifier concerns A-row laws before blocking, rather than arbitrary shift-ergodic laws on the physical alphabet $\Sigma = A^B$. The uniform Bernoulli row on Σ is included. The program is in the finite rule, not at a site. At every finite scale depth $d \geq 0$,*

a corresponding uniform schema attains exactly $\mathsf{D}_{d+2} \cap [0,1]$ *over computable begun constructions of this type. The time limit is innermost.*

Theorem 2.3 (Clock-free realization and selected late states)**.** *The depth-one realization can be chosen with a statistic depending on scale only and with every required inner time limit existing. It has value* β *at every kernel in its open region. The same clock-free depth-one law has value* β *at every ordinary and every Cesàro cluster state of the begun orbit. At computable* K*, the Cesàro cluster class is a nonempty* Π^0_2 *class of stationary TI states. If* $\beta \notin \mathsf{D}_2$*, none of its members has a low name, and no nonempty effective* Σ^0_2 *stationary class has this consensus value at depth one.*

Clock removal here uses dyadic renewal parameters and a continuous plateau quantizer. A uniform small error for an unmodified ratio would not establish its inner time limit.

Theorem 2.4 (Fluctuation transfer)**.** *Let* K *have positive rates, and let a begun depth-*d *law from a computable seed exist. There is an effective transformation of its seed, statistic family and cone geometry into an eternal depth-*$(d+1)$ *space-time statistic having exactly that value in every stationary state of the same PCA, whether TI or not. It preserves robustness on any open set on which the begun law is robust. It does not query the target real. The added index is a time lag in the eternal path law.*

Corollary 2.5 (Sharp arithmetic comparison)**.** *Among laws with values in* $[0,1]$*, at every* $d \geq 0$*, robust all-state eternal laws of depth* $d+1$ *and the begun constructions of depth* d *have exact image* $\mathsf{D}_{d+2} \cap [0,1]$ *when dynamics and admissible readouts are varied as specified above. In particular every* D_3 *value has a robust all-state eternal realization at depth two. A value outside* D_{d+1} *cannot have eternal consensus at depth* d *on a nonempty effective* Σ^0_2 *stationary class at computable parameters. Transfer supplies one sufficient extra limit; it does not prove a minimal cost of one extra limit for each individual law. A computable constant already has eternal depth zero.*

Corollary 2.6 (Static formulation)**.** *For the normalized PCA interaction* $\varphi_K = -\log K$ *on* $\mathbb{Z}^3 \times \mathbb{Z}$*, the depth-two fluctuation law is a two-scale spatial law with value* β *in every fully translation-invariant Gibbs state. The lag is the fourth-direction slab separation. Robustness is relative to this PCA interaction family, not to arbitrary four-dimensional interaction perturbations.*

For fixed targets outside D_2, the begun values at computable data cannot be learned with positive probability by the effective finite-history inquiries described in Section 9.1. This is a fixed-target assertion, not a restriction on every realized random limit. The selected stationary class in Theorem 2.3 is not an effective closed consensus class. There is consequently no conflict with Theorem 2.1, nor with computability of a unique stationary PCA measure.

3 Effective consensus and arithmetic ceilings

3.1 Finite probabilities and compact stationary classes

For any computable initial measure, the probability of a fixed time-t cylinder is uniformly computable. Enumerate assignments on its finite backward cone and sum the initial cylinder weights times products of the corresponding transition probabilities. The number of terms is immaterial to computability. The same formula shows that $\mu \mapsto \mu P_K$ is computable and weakly continuous.

Lemma 3.1 (Effective stationary classes)**.** *For computable* K*, the valid Cantor codes of* $\mathcal{S}_{\mathrm{TI}}(K)$ *form a nonempty* Π^0_1 *class. Adding uniformly computable non-strict cylinder inequalities keeps the class effectively closed. Every nonempty such class has a low member.*

Proof. For a cylinder indicator f, stationarity is the equation $\mu(f) = \mu(P_K f)$. The latter function is a finite computable linear combination of cylinder indicators. Normalization, marginal consistency, TI equations and stationarity are a computable list of equalities of computable continuous coordinate functions. Failure has a finite positive-precision witness. The same holds for failure of a non-strict inequality.

Take any TI seed and let $C_T = T^{-1} \sum_{t<T} \lambda P_K^t$. Compactness supplies cluster states, TI passes to limits, and for $|f| \leq 1$,

$$|C_T(P_K f) - C_T(f)| \leq 2/T. \tag{3}$$

Every such cluster state is stationary, proving nonemptiness.

Apply the low basis theorem of Jockusch–Soare [12]. Here is the particular compact argument. Starting with a nonempty effective closed code class P_e, let V_e be the effective open set of oracles on which machine e halts on input e. Emptiness of $P_e \setminus V_e$ is semidecidable by finite-cover compactness, and hence decidable by $0'$. Retain that difference if nonempty; otherwise retain P_e. The previous finite choices can be hard-coded into an ordinary index for the next closed class. The nested compact intersection is nonempty. Every code in it has exactly the $0'$-computable jump answers just recorded. Its jump is therefore equivalent to $0'$. This is an existence argument for a low state, not a computable point selector. □

3.2 Consensus, including the zero-depth case

Proof of Theorem 2.1. On a low name X in an effective closed class, all finite statistics are uniformly X-computable. Replace each real term of a converging sequence by a rational within 2^{-s}. With X', decide whether a candidate rational Cauchy-tail bound fails: failure is an existential statement about rational terms. Search for a tail satisfying a slightly stronger bound than the requested tolerance. Its limit is X'-computable. Relativize uniformly in all remaining indices and iterate. At depth $d \geq 1$ the value is computable from $X^{(d)} \equiv_T 0^{(d)}$, hence belongs to D_{d+1}.

At $d = 0$, lowness alone is insufficient. Instead let P be the nonempty effective compact code class and s the computable continuous statistic, defined on it and constant with value v. Enumerate rational open intervals I of width at most 2^{-n}. The condition $P \subset s^{-1}(I)$ is semidecidable: the effective open complement of P together with the effective open preimage must cover Cantor space, and a finite basic cover witnesses this. An interval containing v in its interior eventually succeeds. Its midpoint computes v to the required precision. Partial definedness causes no difficulty because the compact class is contained in the statistic's open domain.

For a nonempty effective Σ^0_2 code class, choose a nonempty closed piece. The common value on that piece has the preceding bound. The choice is a finite nonuniform constant; no algorithm selecting the piece is claimed. Pieces need not be saturated under alternative names, since their decoded images remain effectively compact and the value agrees on all their codes.

For the pure-state assertion, $\mathcal{S}_{\mathrm{TI}}(K)$ is metrizable compact convex. Choquet representation [3] expresses each member as a barycentre of a probability measure on its extreme boundary. For bounded linear statistics, $\mu(F_{\mathbf{k}}) = \int e(F_{\mathbf{k}})\, dw_\mu(e)$. At each fixed outer tuple the innermost pointwise limit on extreme states is bounded Borel. Dominated convergence moves that limit through the integral. Repeat through every remaining limit. All intermediate mixture limits exist, and agreement at the extremes forces agreement everywhere. No effective decomposition or effective presentation of the extreme boundary is required.

At computable λ, K, finite-time statistics in (2) are uniformly computable. One time limit and d scale limits give a value computable in $0^{(d+1)}$. For a robust value, evaluate at a rational kernel in the nonempty open region. Equality transfers that absolute ceiling to all kernels there; there is no uniform procedure locating this witness from an arbitrary presentation of the open set.

Finally, finite stationary-history cylinders are computable from the state name and K by finite-cone sums. Thus the same argument works for these readouts. Each limiting lag is an index among the d limits, not a free observation-time operation. □

The pure-state extension is specifically linear and bounded. The nonlinear map $p \mapsto p(1-p)$, where $p = \mu(x_0 = 1)$, vanishes on the two constant-state phases of the noiseless AND automaton but equals $1/4$ at their equal mixture. This already shows why the Choquet step cannot be applied to arbitrary ratios. To check the example, TI stationarity for AND gives $\mu(1) = \mu(11)$, hence $\mu(10) = \mu(01) = 0$, so the stationary TI class consists exactly of those constant-state mixtures.

Computability of a unique invariant measure for a computable PCA was proved by Marcovici–Sablik–Taati [13, Proposition 2.2]. The compact-consensus mechanism also belongs to the effective invariant-measure framework of Galatolo–Hoyrup–Rojas [9]. Our comparison uses these classical principles for nested laws, rather than claiming a new computability mechanism for unique stationary measures.

4 Renewal targets and reliable realization

4.1 Encoding a real in gap probabilities

Fix $\beta \in \mathsf{D}_3 \cap [0,1]$ and uniformly $0'$-computable rationals $b_k \in [0,1]$ converging to it. Let the number G of zeros between successive ones have distribution

$$p_{2k} = 2^{-k} b_k, \qquad p_{2k+1} = 2^{-k}(1 - b_k), \qquad k \geq 1. \tag{4}$$

The pair weights sum to one. There is a stationary renewal measure ν_b with these independent gaps: choose the block containing the origin by the length-biased distribution of $G+1$, place the origin uniformly inside that block, and extend by independent gaps on both sides. This construction is valid even for a periodic renewal process.

Lemma 4.1 (Renewal coordinates)**.** *Writing $\rho = (1 + \mathbb{E}G)^{-1}$, we have $4 \leq \mathbb{E}G \leq 5$ and, for $m \geq 0$,*

$$\nu_b([0^a 1\, 0^{g_1} 1 \cdots 0^{g_m} 1\, 0^b]) = \rho \Pr(G \geq a) \prod_{j=1}^{m} p_{g_j} \Pr(G \geq b), \tag{5}$$

$$\nu_b([0^n]) = \rho \sum_{j \geq n} \Pr(G \geq j). \tag{6}$$

In particular the denominator in

$$R_k(\nu) = \frac{\nu([1\, 0^{2k} 1])}{\nu([1\, 0^{2k} 1]) + \nu([1\, 0^{2k+1} 1])} \tag{7}$$

is $\rho 2^{-k} \geq 2^{-k}/6$ at ν_b, and $R_k(\nu_b) = b_k$. The measure ν_b is limit-computable, uniformly from the approximation program.

Proof. Each renewal block has one one and G zeros. Its mean length is $1 + \mathbb{E}G$, so renewal starts have intensity ρ. Conditional on a start, forward and backward gaps have their original independent laws. This proves (5). There are $(G - n + 1)_+$ possible starts of an all-zero word of length n within a gap, giving (6). Also

$$\mathbb{E}G = \sum_{k \geq 1} 2^{-k}(2k + 1 - b_k) \in [4, 5].$$

The even and odd specified gaps have total mass 2^{-k}, which proves the ratio assertions, including $b_k = 0, 1$.

For effective tails use

$$\sum_{k>K} 2^{-k}(2k+1) = (2K+5)2^{-K}. \tag{8}$$

Thus every cylinder coordinate is uniformly $0'$-computable. More explicitly, approximate each rational code b_k by a total computable array of rational guesses, clamped to $[0, 1]$, eventually correct for each fixed k. At stage s use the first s gap pairs and normalize their total $1 - 2^{-s}$. Their stationary finite-state renewal measures are computable. For every fixed finite prefix of pairs the guesses settle, while (8) uniformly controls the remaining mean and cylinder errors. These computable measures converge to ν_b. The finite algorithm contains no oracle for $0'$. □

Mixing is not needed and need not hold. If every $b_k = 0$, all interarrival lengths $G + 1$ are even. The intensity is $1/6$, but $\nu_b(x_0 = x_{2n+1} = 1) = 0$ at every odd separation. The relevant period is that of interarrival lengths, not zero-run lengths.

4.2 The two realization inputs

Hellouin de Menibus–Sablik [11, Corollary 1] prove that every limit-computable shift-invariant measure on a finite output alphabet is the limit of a CA F on an enlarged alphabet A, from every full-support shift-ergodic measure on $A^{\mathbb{Z}}$. The target is supported on the output subalphabet; auxiliary symbols vanish in the limit. This is the singleton target-measure result; no stronger seed-mixing hypothesis for general limit sets is used. This seed quantifier concerns the pre-blocking alphabet A and its one-cell shift. Write μ_{row} for such an A-row law. The physical preparation clones its blocked image on $\Sigma = A^B$. The canonical uniform Bernoulli row on Σ arises by blocking the uniform Bernoulli row on A. No claim is made here for arbitrary ergodic physical block-row laws. The self-organizer formats segments, computes approximating words, copies them and merges segments that are too short. We import this finite-rule construction rather than reprove its segment estimates.

Pad the radius of F to an integer $r \geq 1$, let $B = 2r$, and write $(Sx)_i = x_{i+1}$. For $H = S^r F$,

$$H^t(x)_i = F^t(x)_{i+rt}. \tag{9}$$

Grouping B original cells turns H into a two-block rule h with neighbourhood $\{0, 1\}$ on $\Sigma = A^B$. Before applying h, take full-symbol NEC majority in every redundant plane, with a fixed tie rule for three distinct symbols. The stacked rule G_β has six parents

$$\{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,0,1), (0,1,1)\}.$$

It is an ordinary finite-alphabet TI rule; the finite target approximation program is part of F.

The arbitrary-trajectory reliability theorem of Gács–Reif [8], in the explicit form of Taati–Miękisz [16, Theorem 3.1], applies to every cloned initial row and all logical times. For quantitative choices we also use Gács [7, Theorem 1], whose infinite-plane site-error bound is $4D_*\varepsilon$ for $D_* = 24$ and $\varepsilon < 1/(32D_*^8)$. The factor D_* is part of that bound. These results descend from Toom's stability construction [17].

Fix the rational number

$$\varepsilon_* = \frac{1}{64D_*^8}, \qquad U_\beta = \{K : K(a \mid r) > 0, \quad K(G_\beta(r) \mid r) > 1 - \varepsilon_* \text{ for all } a, r\}. \tag{10}$$

Uniform positive rational wrong-symbol noise of total mass $\varepsilon_*/2$ gives a computable member. The set is nonempty and relatively open. Conditional probabilities of actual faults are at most ε_*; conditioning backwards through time layers gives the finite joint-fault bound required by reliability. Actual fault indicators need not be unconditionally independent. The one-site error is at most $96\varepsilon_* < 1/20$.

4.3 Age-dependent amplification

At integer $t \geq 2$ use the odd integer schedule

$$m_t = 2\lceil \log_2 t \rceil^2 + 1$$

and copies spaced by $2t + 1$ in the a direction. This integer rounded log-squared schedule is computed by bit length and has the required growth and summability. Decode a blocked symbol by strict majority, using a fixed default if no symbol has a majority, and then unpack the B original coordinates. These operations are one schema on labelled block alphabets.

Conditional on the whole seed row $Y = y$, the initial data are deterministic. Separated endpoint cones use disjoint update coins, so their symbols are conditionally independent. Each has the same intended symbol and error at most $1/20 \leq 1/4$. Hoeffding gives

$$\Pr(\text{wrong decoded block} \mid Y = y) \leq e^{-m_t/8} =: e_t. \tag{11}$$

For a fixed original word u, let W be the finitely many blocks covering it. A union bound and integration over y yield

$$\left| \Pr_{\lambda P_K^t} (\text{decoded } u) - (F_*^t \mu_{\text{row}})([u]) \right| \leq |W| e_t. \tag{12}$$

Here μ_{row} is the pre-blocking A-row law. Its one-cell shift invariance passes to $F_*^t \mu_{\text{row}}$, so (9) causes no moving-frame correction: that invariance absorbs the drift rt on the original cells. Block-shift invariance alone would not justify this step. There is still age in the copy geometry.

Proof of the depth-one part of Theorem 2.2. Apply (12) to the two gap words in (7). The noiseless probabilities converge to their renewal values by the self-organization input. Their limiting sum is $\rho 2^{-k} > 0$, so the inner time limit of the decoded ratio exists and equals b_k for every fixed k. Every finite-time denominator is positive: a finite assignment of the copies can realize either binary word, and positive-rate updates give it positive probability. The initial full-support cloned row also gives these events positive probability. Taking $k \to \infty$ gives β. The bound is uniform over U_β and over row contents. The noiseless theorem applies to every full-support, one-cell-shift-invariant ergodic law on $A^{\mathbb{Z}}$. Thus the same value holds for clones of their blocked images, including noncomputable admitted row laws. It is not asserted for every shift-ergodic law on $\Sigma^{\mathbb{Z}}$. For the arithmetic upper bound and later inquiry statement, use the canonical computable row. Theorem 2.1 gives the reverse inclusion. The image is over finite rules and their labelled block presentations, not one universal alphabet. □

4.4 Attainment at every finite depth

Proof of the remaining part of Theorem 2.2. For $d \geq 1$ and $\beta \leq_T 0^{(d+1)}$, iterated relativized real limit approximation gives rationals $b_{k_1,\ldots,k_d} \in [0,1]$, uniformly $0'$-computable, whose d prescribed nested limits exist and equal β. This uses real Cauchy names and uniform limit approximation, not uniformly decidable strict cuts of arbitrary oracle reals. Fix a computable bijection p_d from

positive d-tuples to positive integers. Put $b_{\mathbf{k}}$ into gap pair $p_d(\mathbf{k})$ in (4). The same geometric bound controls all tails, so the renewal measure is still limit-computable. Use the same uniform readout schema, now asking gap question $p_d(\mathbf{k})$. Its inner time limit is $b_{\mathbf{k}}$ and the d outer limits give β. Every intermediate limit is part of the hypothesis on that chosen array; arbitrary programs are not presumed to converge.

At $d = 0$, choose a limit-computable Bernoulli target with parameter $\beta \in \mathsf{D}_2 \cap [0, 1]$. Its cylinder probabilities are limits of computable probabilities by a computable rational approximation to β. Apply the same noiseless realization and reliable stack, then read the decoded one-site probability with (11). Its single time limit is β. Endpoints zero and one are allowed target measures even though the initial row has full support. The ceilings in Theorem 2.1 give exactness at each depth. The schema is fixed separately for each d. □

5 A clock-free law on late states

Age-dependent cone separation proves Theorem 2.2, but does not produce a fixed statistic of a late state. We localize error certificates with a seed-independent fault envelope. Independence belongs to local certificate events, not actual errors.

5.1 Independent local certificates

Implement updates with independent uniforms $U_{z,t}$, putting the deterministic output first in the sampling interval. Actual faults are contained in the independent envelope

$$\mathcal{F} = \{(z, t) : U_{z,t} \geq 1 - \varepsilon_*\}.$$

It is independent of the seed. Put

$$D_* = 24, \quad c = 16D_*^8\varepsilon_* = \tfrac{1}{4}, \quad \delta_* = (1 + 8D_*)\varepsilon_* \leq \tfrac{1}{16}.$$

Gács's explanation-tree and weighted-tree counting lemmas [7, Lemmas 2.3 and 2.6], in the PDF numbering, give: a nonfaulty error has an explanation with n fault vertices and at most $4(n-1)$ edges; the graph degree is D_*; each edge moves each space-time coordinate by at most one; and the number of weighted rooted trees of ℓ edges is at most $2D_*(2D_*^2)^\ell$.

Lemma 5.1 (Certificate domination)**.** *At m equal-time endpoints of pairwise sup-distance at least $L \geq 8$, for $1 \leq h \leq m$ and every cloned row,*

$$\Pr(\textit{at least } h \textit{ errors} \mid Y) \leq 8D_* m\varepsilon_* c^{\lceil L/8 \rceil - 1} + \binom{m}{h}\delta_*^h. \tag{13}$$

The estimate is uniform in time and $K \in U_\beta$.

Proof. With $A_* = 2D_*^2$, the number of allowed weighted trees with at most $4(n-1)$ edges is bounded by $2D_* \sum_{\ell=0}^{4(n-1)} A_*^\ell \leq 4D_* A_*^{4(n-1)}$. Requiring their n marked vertices to lie in the fault envelope has probability ε_*^n; extra faults are allowed. The union $S_n(u)$ of such certificates at a root therefore satisfies $\Pr(S_n(u)) \leq 4D_*\varepsilon_* c^{n-1}$. Each tree lies in a sup-ball of radius $4(n-1)$.

Let $n_0 = \lceil L/8 \rceil$. At endpoint j let A_j be the root-fault event or a certificate with $n < n_0$. Its supporting coins lie at radius strictly less than $L/2$. These supports are disjoint for distinct endpoints. The A_j are independent, including conditional on the row, and

$$\Pr(A_j) \leq \varepsilon_* + 4D_*\varepsilon_*/(1 - c) \leq \delta_*.$$

The remaining large-certificate event has probability at most $4D_*\varepsilon_* c^{n_0-1}/(1-c) \le 8D_*\varepsilon_* c^{n_0-1}$ per root. Every actual error belongs to a small or large event. Union-bound the large events and the choices of h independent small events. This gives (13). Intersecting supports with the available time half-space only reduces them. □

Actual errors can remain correlated at every transverse distance. For a fair cloned bit Y, take time-one error probabilities p_0, p_1 depending on its value. Distant errors are conditionally independent but have unconditional covariance $(p_1 - p_0)^2/4$. The lemma does not claim decay of this covariance.

5.2 Scale-only geometry and cluster states

At scale $k \ge 1$ choose

$$m_k = 8k + 9, \qquad L_k = 64(k+1), \qquad h_k = 2^{-k}. \tag{14}$$

Decode each blocked position by strict majority of m_k copies spaced by L_k, then unpack. Write r_k for the resulting gap ratio. The two words use at most $2k+3$ blocks. With e_k the right side of (13) at majority threshold,

$$e_k \le (8k+9)2^{-16k-14} + 2^{-8k-10} \le (8k+10)2^{-8k-10}, \tag{15}$$

$$(2k+3)e_k \le 90\,2^{-6k-12} \le 4^{-k}/96. \tag{16}$$

Indeed $8D_*\varepsilon_* \le 1$, $c = 1/4$ and $\delta_* \le 1/16$. Also $(2k+3)(8k+10) \le 90\,4^{k-1}$: equality holds at $k = 1$ and successive polynomial ratios are less than four. The last comparison holds at $k = 1$ and improves by a factor sixteen. These are all-scale bounds, not numerical fits.

Conditional on the row, apply the lemma at majority threshold, union-bound over the word's blocks, and average over the row. For either gap word u and every time,

$$\left| \Pr_{\lambda P_K^t} (\text{scale-}k \text{ decoded } u) - (F_*^t \mu_{\text{row}})([u]) \right| \le \eta_k := 4^{-k}/96. \tag{17}$$

The reference probabilities converge. Finite decoded probabilities are continuous affine functions of the physical state, so every ordinary cluster state inherits their η_k bounds against ν_b. Cesàro cluster states inherit them too, by averaging the convergent reference sequence.

For nonnegative pairs with $s = a + b > 0$ and individual perturbations at most $\eta \le s/4$, subtraction gives

$$\left| \frac{a'}{a'+b'} - \frac{a}{a+b} \right| \le \frac{\eta}{s - 2\eta} \le \frac{2\eta}{s}. \tag{18}$$

The numerator is $b(a'-a) - a(b'-b)$. Since $s \ge 2^{-k}/6$, every ordinary or Cesàro cluster state satisfies

$$|r_k(\bar{\mu}) - b_k| \le h_k/8. \tag{19}$$

The decoded denominator is positive and $\lim_k r_k(\bar{\mu}) = \beta$. This alone does not give fixed-k inner time limits.

5.3 Continuous quantization gives genuine inner limits

Choose the realization with $b_k = 2^{-k}\lfloor 2^k a_k \rfloor$, for uniformly $0'$-computable rationals $a_k \in [0,1]$ tending to β. This is exact rational flooring and $b_k \to \beta$. On $[0,1]$ define

$$Q_k(x) = h_k \sum_{j=0}^{2^k-1} \min\{1, \max\{0, 2x/h_k - 2j - \tfrac{1}{2}\}\}. \tag{20}$$

This target-independent continuous rational piecewise-linear map equals jh_k on $|x - jh_k| \le h_k/4$, intersected with $[0,1]$, including the endpoints. Use $s_k = Q_k \circ r_k$.

Proof of Theorem 2.3. Equation (19) gives $s_k(\bar{\mu}) = b_k$ exactly at every cluster state. At fixed k the same estimate gives $\limsup_t |r_k(\lambda P_K^t) - b_k| \le h_k/8$. There is a strict margin to the plateau boundary $h_k/4$. Hence $s_k(\lambda P_K^t) = b_k$ for all sufficiently large t. Every inner time limit exists and the outer limit is β, uniformly on the kernel region. Denominators are positive at finite times by positive rates, and at cluster states by the displayed bounds. The statistic uses scale but no age. This chooses a dyadic-parameter realization; it does not prove convergence of every earlier unquantized ratio.

At computable K and canonical seed, C_T is uniformly computable and its cluster class V' has the valid-code description

$$\mu \in V' \iff (\forall n, T)(\exists t \ge T)\ d(C_t, \mu) < 2^{-n}. \tag{21}$$

Strict inequality has a finite-precision positive witness, giving a Π^0_2 condition. Compactness gives nonemptiness; TI and (3) give stationarity. On V' the clock-free law is β. A low name X would compute that one-limit value in $X' \equiv_T 0'$, impossible if $\beta \notin \mathsf{D}_2$. Every nonempty effective Σ^0_2 class has a nonempty closed piece and a low code, so cannot have this depth-one consensus. Ordinary cluster states share the law but need not be stationary. □

6 Exact fluctuation transfer

6.1 Finite patterns and their future cones

Lemma 6.1 (Positive patterns and the conditional cone identity)**.** *If K has positive rates, every stationary state satisfies $\mu([w]_B) \ge \kappa(K)^{|B|}$ for every finite B and pattern w. For any initial law ν, output window W, lag t, and B containing the whole depth-t backward dependency region,*

$$\Pr_{\nu}(X_t|_W = u \mid X_0|_B = w) = g_{K,t}(w; u) \tag{22}$$

whenever the conditioning event has positive probability. The function $g_{K,t}$ is independent of the exterior distribution.

Proof. Stationarity and conditionally independent site updates give

$$\mu([w]_B) = \int \prod_{z \in B} K(w_z \mid x|_{z+N})\, d\mu(x) \ge \kappa(K)^{|B|}.$$

In the graphical construction, the output on W at time t is a fixed function of $X_0|_B$ and finitely many future uniforms, independent of the initial state. Conditioning on the whole pattern w fixes every relevant initial input. Every global extension therefore has the same conditional output law. A sufficient region for radius r is $B = W + [-rt, rt]^D$. Conditioning on only part of it is insufficient in general. □

6.2 Reweighting the fluctuations

Proof of Theorem 2.4. Express the source's finitely many cylinder functions as finite linear combinations of pattern indicators on one common output window $W_{\mathbf{k},t}$. Choose its whole backward region B. For each output pattern u and stationary μ, set

$$q_u(\mu) = \sum_{w \in \Sigma^B} \lambda([w]_B) \frac{\Pr_\mu(X_0|_B = w,\ X_t|_W = u)}{\mu([w]_B)}. \tag{23}$$

All denominators are positive by Lemma 6.1. Its conditional identity gives the exact finite equality

$$q_u(\mu) = \sum_w \lambda([w]_B) g_{K,t}(w; u) = (\lambda P_K^t)([u]). \tag{24}$$

Evaluate the source's $R_{\mathbf{k},t}$ on these probabilities. The transferred statistic equals the source statistic at every finite tuple and lag, in every stationary state. Its iterated limits exist whenever the source limits do, with the same order and value.

The sum is finite, its weights are computable, and division is computable on the positive-denominator domain without a supplied uniform lower bound. The source domain pulls back to an effectively open domain. Thus the new statistics are admissible finite space-time functionals. Counting t as one added innermost index gives depth $d + 1$. The transformation uses the specified seed, family and geometry and no target oracle. It preserves target independence where those inputs have it. Equality (24) also proves the robustness assertion. □

The weights are essential. Waiting for an unspecified fluctuation samples patterns with frequencies $\mu([w]_B)$, not $\lambda([w]_B)$. One interpretation first samples a desired pattern with the source's computable finite law, then waits for it. Conditional on the past its chance at each next update is at least $\kappa^{|B|}$; avoiding it for n updates has probability at most $(1 - \kappa^{|B|})^n$. The local Markov identity applies after that occurrence. This concerns repeated finite events, not one positive-probability event containing an entire infinite cloned beginning. No useful uniform sampling efficiency is claimed.

For the canonical cloned uniform row the weights have a simple schema: a finite pattern has zero weight unless transverse sites at each common line coordinate agree; otherwise its weight is $|\Sigma|^{-j}$, for j distinct specified line coordinates. Alphabet and block metadata determine these numbers. They contain neither β nor its approximants. Finite cloned patterns remain explicit in the readout, although cloning is not a required initial condition of the eternal world.

Proof of Corollary 2.5. Transfer each realization of Theorem 2.2, including the tuple-indexed ones. This attains every D_{d+2} value at eternal depth $d + 1$ in every stationary state. For the upper bound, restrict all-state consensus to the nonempty $\mathcal{S}_{\mathrm{TI}}(K)$ at a computable kernel in the robust region and apply Theorem 2.1. At depth d its ceiling is D_{d+1}, so a value outside that class cannot have such a consensus realization. Over all admissible readouts, lower-level values can also be attained by computable arrays of constants. This does not say that one fixed gap formula attains every value in one fixed automaton. □

The transfer is a use of the Markov property and positive rates. For fixed K, its finite values can also be computed directly from λ, K and are independent of μ. This breadth comes from the admitted class of probability functionals. The result is not a theorem that a bounded local observer efficiently estimates these ratios, nor a minimal-cost theorem for every source law. The arithmetic obstruction is sharp for values outside the lower consensus ceiling.

7 Stationary histories and static Gibbs states

For a positive-rate kernel, put one translated energy

$$\varphi_K(x) = -\log K\big(x_{0,0} \mid x|_{N\times\{-1\}}\big)$$

at each point of $\mathbb{Z}^D \times \mathbb{Z}$. Each interaction has finite range and bounded finite energy at fixed K. Row normalizations are retained. At computable K the interaction is computable.

Extend a stationary PCA state μ to its consistent bi-infinite path law Γ_μ. Changing finitely many space-time spins changes exactly the transition factors whose parent-child families meet that set. Conditional probability ratios are products of these K factors and hence exponentials of the corresponding energy differences. This is the finite-range DLR specification. Factors centred at future children of resampled spins are included.

The converse for fully space-time TI Gibbs states is the PCA/Gibbs correspondence of Goldstein–Kuik–Lebowitz–Maes [10], restated by Taati–Miękisz [16, Theorem 3.2]. Such a Gibbs state has the PCA's forward transition law and a stationary TI time-zero marginal. Full space-time invariance is part of this converse. It is not inferred for arbitrary non-TI Gibbs states, and a one-time marginal is not itself the four-dimensional Gibbs history.

Proof of Corollary 2.6. Take a fully TI Gibbs state Γ for φ_K. By the correspondence it is a stationary PCA history. The two-time probabilities in (23) are static cylinders on two slabs separated by t in the fourth direction. Equations (22) and (24) hold. Apply the depth-two transfer of the depth-one realization. Every such state has value β, with slab separation sent to infinity as the added limit.

For a computable finite-range finite-alphabet interaction, full TI Gibbs classes have effective closed code descriptions: impose all local DLR equations against finite exterior cylinder tests and all translation equations. Finite conditional weights are computable and positive, so failed equations have finite precision witnesses. Nonemptiness follows from finite-volume Gibbs measures, compactness and translation averaging, as boundary fractions vanish on growing boxes. The low-basis and compact-cover arguments of Section 3 give the same consensus ceilings. A depth-one consensus at computable parameters cannot equal a fixed value outside D_2. A computable kernel witness transfers the robust ceiling. □

The map $K \mapsto -\log K$ transports the open kernel region into the normalized PCA interaction family, not into an open set of all four-dimensional interactions. Uniform-fault weights can instead be parameterized by inverse temperature b: the total wrong-symbol rate is $(q-1)e^{-b}/(1+(q-1)e^{-b})$, for alphabet size q. A sufficiently large open interval of b stays in the reliable region with normalization preserved.

A begun half-space formulation fixes a cloned bottom row, uses the causal law above it, and then averages over the prescribed row distribution. Interior DLR ratios have the same transition-factor calculation, including future children. Its begun statistics are those already proved. When full-space time invariance is needed in a limiting history, use Cesàro time shifts or prove it separately. Ordinary accumulation alone does not give invariance. The static all-state result uses the correspondence and transfer, not an assumption of a unique full-space Gibbs phase.

8 Selection, separated phases and product seeds

8.1 What a beginning specifies

Proposition 8.1 (Selection bounds)**.** *If a computable begun orbit converges, its limit is a* $0'$*-computable stationary state, TI for a TI seed. In general both ordinary and Cesàro cluster classes*

have Π^0_2 valid-code descriptions; only the latter is automatically stationary. At computable K, a depth-d law at a state named by X is computable from $X^{(d)}$. For $d \geq 1$, a selected value outside D_{d+1} forces the state to have no low name.

Proof. Each limiting cylinder coordinate of a convergent computable orbit is a uniformly computable-sequence limit, hence uniformly $0'$-computable. Weak continuity gives stationarity. The cluster formula (21) applies to any computable measure sequence, including ordinary orbits. Equation (3) supplies the additional stationarity for Cesàro clusters. The relative law bound is the iterated limit argument of Section 3. At $d \geq 1$, a low name satisfies $X^{(d)} \equiv_T 0^{(d)}$, proving the last assertion. □

There is no unrestricted one-jump point-selection conclusion. A Π^0_2 description of a nonempty compact cluster class does not supply a $0'$-computable member. Computable bounded sequences can lack any limit-computable accumulation point [2, Corollary 11.8]. This does not by itself exhibit a positive-rate PCA with that property; it invalidates the inference from description complexity alone.

The restriction $d \geq 1$ in the non-lowness statement matters. Take a noncomputable low set A and its base-four Cantor real r, of the same degree. In the computable identity CA, $(1-r)\delta_{0^{\mathbb{Z}}} + r\delta_{1^{\mathbb{Z}}}$ is stationary, has a low name, and its depth-zero local expectation is the noncomputable r. No effective class pins that expectation. This example concerns the general computable-PCA selection statement; the identity rule is not a positive-rate realization.

Ordinary accumulation need not mean stationarity, even with positive rates. Let T be symmetric low-noise binary Toom dynamics and C global complement. They commute, and $T^t\delta_0 \to \mu_-$, $T^t\delta_1 \to \mu_+$ with $\mu_- \neq \mu_+$. For $P = CT$, the even and odd subsequences from δ_0 tend to μ_- and μ_+, which P interchanges. Neither is P-stationary. The Cesàro orbit converges to their stationary equal mixture.

8.2 Sharpness and unrestricted mixtures

The Toom phases in this example follow from the classical stability theorem [17, 7]. At computable symmetric noise $p < 1/2$, the kernel is attractive. Starting from the two constant configurations gives monotone measure sequences and stationary limits μ_+, μ_- with local one probabilities $m > 1/2$ and $1 - m$. Stability supplies a uniform separation. Their increasing-event probabilities are monotone limits of computable values; arbitrary cylinder coordinates follow by finite inclusion-exclusion. Both phase measures are uniformly $0'$-computable.

For any real $r \in [0, 1]$ their mixture has

$$\mu_r = r\mu_+ + (1-r)\mu_-, \qquad \mu_r(x_0 = 1) = 1 - m + r(2m-1). \tag{25}$$

Free selection therefore gives a nondegenerate interval of depth-zero eternal values, including nonarithmetical ones. No arithmetic ceiling applies to arbitrary selected states. No robust growing-majority formula for recovering r is needed.

This also gives the depth-zero specification boundary. If m is noncomputable choose $r = 0$. Otherwise choose a noncomputable $r \in \mathsf{D}_2 \cap [0, 1]$; the local expectation in (25) then computes r and is noncomputable. In both cases μ_r is a limit of computable genuine measures: approximate the weight rationally and use finite Toom evolutions from the constant seeds. Its singleton consequently has the Π^0_2 cluster description, but cannot be a nonempty effective Σ^0_2 consensus class for that depth-zero statistic. Theorem 2.3 supplies the robust depth-one boundary with an entire stationary Cesàro cluster class.

8.3 Separated stationary phases in the realization

Proposition 8.2 (Scoped divergence in one automaton)**.** *For every $K \in U_\beta$ in the realization there are separated extreme stationary TI states. At computable K, a nonempty effective Σ^0_2 stationary class cannot have any depth-one consensus outside D_2, whereas the canonical begun law is β. The statement concerns the same dynamics.*

Proof. The finite diagonal symbol map of F has a periodic orbit. Start the noisy stack from its spatially constant cloned row. Let E mean that original offsets zero and one in a block agree; $B = 2r \geq 2$ makes this a one-block event. Full-symbol reliability gives $\Pr(E) \geq 19/20$ at all times. Every Cesàro cluster state μ_c therefore has $\mu_c(E) \geq 19/20$.

From the canonical cloned Bernoulli seed, the noiseless pair-disagreement probability tends to $2/(1 + \mathbb{E}G) \geq 1/3$, since every one in the renewal measure is flanked by zeros. Reliability gives $\mu_b(E) \leq 2/3 + 1/20 = 43/60$ in every begun Cesàro cluster state. Both classes are stationary TI by Cesàro averaging. Maximize and minimize the continuous affine functional $\mu(E)$ over the compact convex stationary class. The corresponding extremizing faces contain extreme points of the full class, so separated extreme phases exist.

At computable K the nonempty region $R = \{\mu \in \mathcal{S}_{\mathrm{TI}}(K) : \mu(E) \leq 3/4\}$ is effectively closed. It contains all the begun Cesàro cluster states and excludes the constant-trajectory class. The consensus exclusion follows from Theorem 2.1; the begun value follows from Theorem 2.2. When $\beta \notin \mathsf{D}_2$ the clock-free selected cluster class is a strictly more restrictive specification than any such effective closed consensus class. □

8.4 Two limited product-seed obstructions

Proposition 8.3 (Finite-time cones and invariant decoders)**.** *From an i.i.d. seed, time-t restrictions to sets separated by more than $2rt$ are independent for a radius-r PCA. At fixed finite t, a measurable symbol decoder invariant under both transverse shifts is almost surely constant. A countable row of such decoded symbols is deterministic; it is homogeneous if the decoder is also equivariant under longitudinal shifts.*

Proof. Restrictions are functions of the initial symbols and update coins in their backward cones. At the stated separation these input sets are disjoint. A fixed-time plane field is a finite-range factor of the i.i.d. input slab, hence mixing and ergodic under transverse shifts. An invariant finite-valued function is therefore almost surely constant. Countable intersection gives a deterministic row. Longitudinal equivariance and TI force its coordinates to agree. Without that equivariance a decoder could output a predetermined alternating row. □

Independent odd majorities need not converge to a deterministic word: a reset PCA outputting fair bits has fair majority at every copy count. Strict full-symbol majority also differs from plurality when, for example, probabilities are $(.4, .35, .25)$. We do not use a universal modal-word conclusion.

Proposition 8.4 (Direct comparison to a cloned reference)**.** *An i.i.d. seed with marginal q can disagree with a cloned row on a set of plane density less than one half in every plane only when that row is homogeneous.*

Proof. Simultaneous strong laws for the finitely many symbols and countably many planes give disagreement density $1 - q(y(i))$. Thus $q(y(i)) > 1/2$ in every plane. At most one symbol can satisfy this inequality, so y is constant, even if selected after observing the configuration. A deterministic finite-alphabet line rule takes homogeneous rows through an eventually periodic computable symbol orbit. □

This excludes only direct initialization of a nonhomogeneous cloned reference by treating the raw product seed as its sparse perturbation. It excludes neither preprocessing, changing local references, hierarchical encoding nor self-organizing correction. Density less than one half is not the full hypothesis of every fault theorem. No universal impossibility or clock-necessity claim follows.

9 Inquiry, prior work and remaining questions

9.1 Unlearnability is a fixed-target statement

An effective inquiry has finitely encoded histories, a computable adaptive controller choosing each finite observation and rational estimate, and uniformly computable finite conditional outcome probabilities on positive-probability histories. Each successful stage finishes in finite time; no computable cost or convergence bound is required. The target is fixed independently of the outcome tape. Auxiliary finite experiments, including Born-random outcomes, are admitted only with the same effective conditional description.

For computable K and the canonical computable seed λ, adaptively chosen finite space-time cylinders of the begun history satisfy these hypotheses. Finite-cone sums compute every finite joint observation probability uniformly. On a positive-probability history, division by its computable positive probability computes the next conditional law. The controller chooses finite windows, observation times and outputs computably. Thus these observations meet the hypotheses of the released fixed-target randomized reach theorem [5, Theorem 21.1]. It follows that convergence to one prescribed real with positive probability forces that real into D_2. Conversely a program can print a chosen computable rational approximation to any D_2 target while ignoring observations. This existential converse is not one common learner for an unknown system.

The companion proof combines effective sampling, almost-everywhere generalized lowness and a relativized positive-measure cone argument. Its conclusion concerns a fixed target, not every realized random limit. Accordingly, our values outside D_2 cannot be learned with positive probability by these inquiries at computable begun data. Noncomputable kernels or preparations can supply advice; no such claim is made for arbitrary noncomputable parameters or arbitrarily selected noncomputable stationary laws. The realization, consensus and transfer proofs do not depend on the companion's inquiry theorem; only this inquiry conclusion uses it.

The anchored precursor [5, Theorem 1.5] uses one fixed reliable rule and a distinguished program origin. Here the finite program moves into the rule, and self-organization removes that origin in the seed. Both cited theorems are in the public version 2.1.0 of *Uncomputable critical exponents*, with version DOI `10.5281/zenodo.23780285`.

9.2 Prior mechanisms and the scope of the synthesis

Noiseless realization of limit-computable target measures from full-support one-cell-shift-ergodic A-row laws is due to Hellouin de Menibus and Sablik [11]. Boyer–Delacourt–Poupet–Sablik–Theyssier [1] obtain high arithmetic complexity for persistent-word languages of noiseless μ-limit sets. These are substantive antecedents of the begun construction. Our use is at fixed positive noise on an open kernel region, for a uniform real-valued readout schema.

Positive-rate storage and reliable computation come from Toom, Gács–Reif and Gács [17, 8, 7, 6]. The present paper does not introduce a new reliability paradigm. The conditional certificate argument composes their local-tree bounds with renewal denominators and continuous quantization. Gács's self-organizing construction uses manufactured synchronization and treats memory and

computation initialization separately [6]; those features are evidence for a route, not proof that every possible construction must contain a clock.

The eternal ceiling uses low basis, compact consensus and iterated limits, with Choquet representation for the bounded linear extension. The unique-stationary-measure result is already Marcovici–Sablik–Taati [13]. Taati [15] proves rapid forgetting for surjective CA with positive additive noise. Such results concern particular dynamics and noise models; they do not exclude the reliable stacks used here. Marsan–Sablik [14] study invariant measures in the zero-noise limit, including effective constraints and realization of connected compact limit sets. Their noise parameter tends to zero, whereas ours stays in a fixed positive-rate open region. The history/Gibbs correspondence is prior work [4, 10, 16].

The bounded primary-source comparisons available to us found no earlier statement of the full robust renewal realization with its seed quantifiers, nor of its all-state fluctuation-transfer and effective-consensus comparison. This is a bounded-search outcome, not an exhaustive priority claim. Most of the mechanisms above are classical; the new statements are the specified compositions and quantifiers.

9.3 What is required, and what is still open

At fixed depth, effective stationary consensus is the hypothesis that imposes the ceiling. A selected eternal state may carry a non-low value. Our beginning supplies one Π^0_2 stationary cluster-class specification, but another noneffective selection could supply a similar class. There is no theorem that a physical or cosmological beginning is its only possible description.

The world's age is absent from the clock-free cluster readout and, after quantization, from the strict begun statistic. Infinite cloned redundancy remains in those preparations. Eternal transfer needs no cloned initial state, but uses finite cloned-pattern weights and an elapsed-time lag; the static form uses a reference direction and slab separation. These resources have explicit places in the definitions. Neither the eternal observer nor a finite simulation completes any of the limits.

Open problem 9.1 (Product-seed realization)**.** A TI positive-rate PCA admits a robust begun law outside D_2 under a target-independent effective readout schema from a full product seed, with a suitably specified seed-universality class.

Removing the two cloned directions requires a new construction. A possible route is a self-organized synchronization layer carrying the noiseless segment computation as payload. A self-built clock may help; its universal necessity is not proved. The direct comparison obstruction in Proposition 8.4 does not settle this problem.

Open problem 9.2 (Individual transfer cost and restricted observations)**.** Characterize the least eternal depth of a given begun law when the physical observation schema is fixed, and determine when the sufficient additional limit of Theorem 2.4 is necessary.

Constants already show that it is not necessary for every law. The arithmetic ceiling proves necessity for values outside the lower consensus class, but does not classify all individual readouts. A further question is a quantum version with precisely specified effective local channels, stationary-state classes and conditional observation operations. The classical finite-cone identity cannot simply be replaced by conditioning quantum states without defining the measurement and disturbance model.

How this work was done

The authors developed the definitions and proofs, checked the specified primary statements, and take responsibility for the mathematics and attribution. The substantial imported self-organization and reliability constructions are identified where used; a complete local table for those large automata is not generated or exhaustively enumerated here.

The ancillary programs check exact renewal-cylinder identities, block-coordinate formulas, finite cones, conditional transfer, certificate constants and continuous-quantizer plateaus. Independent implementations and explicit counterexamples test the limitations of unconditional independence, modal decoding and guarded updates. Seeded finite-torus simulations illustrate preparation dependence, metastability and failure of particular guarded designs. Observations are kept separate from mathematical pass conditions; no preregistration claim is made from final files. All intended numerical criteria raise on failure.

In particular, high sitewise agreement in a guarded rule-30 experiment can come from a spatially alternating fixed row rather than active computation. Clean-clone tests expose the same design's large tracking errors. These are finite-size diagnostics, not universal obstructions. The chosen simulation noise values are not certified members of the tiny analytic reliability region. Every finite positive-rate torus has a unique stationary law; its observed branches are finite-time metastability, not distinct stationary phases. Exact infinite-volume claims, noncomputability, seed universality and every-scale error bounds rest on the proofs. The sequential runner records sampled process-tree RSS, not an operating-system historical peak. Per-script memory fields may be absent where a platform API is unavailable.

References

[1] L. Boyer, M. Delacourt, V. Poupet, M. Sablik, and G. Theyssier. μ-limit sets of cellular automata from a computational complexity perspective. *Journal of Computer and System Sciences*, 81, 2015. https://arxiv.org/abs/1309.6730v2.

[2] V. Brattka, G. Gherardi, and A. Marcone. The Bolzano–Weierstrass theorem is the jump of weak König's lemma, 2017. Version 3. https://arxiv.org/abs/1101.0792v3.

[3] G. Choquet. Le théorème de représentation intégrale dans les ensembles convexes compacts. *Annales de l'Institut Fourier*, 10:333–344, 1960. https://doi.org/10.5802/aif.105.

[4] E. Domany and W. Kinzel. Equivalence of cellular automata to Ising models and directed percolation. *Physical Review Letters*, 53:311–314, 1984. https://doi.org/10.1103/PhysRevLett.53.311.

[5] S. Douglas and N. Mghirbi. Uncomputable critical exponents, 2026. Version 2.1.0. https://doi.org/10.5281/zenodo.23780285.

[6] P. Gács. Reliable cellular automata with self-organization. *Journal of Statistical Physics*, 103(1–2):45–267, 2001. Corrected and strengthened manuscript, version 2 (2024): https://arxiv.org/abs/math/0003117v2.

[7] P. Gács. A new version of Toom's proof, 2021. Version 1; theorem and lemma numbers refer to the PDF. https://arxiv.org/abs/2105.05968v1.

[8] P. Gács and J. Reif. A simple three-dimensional real-time reliable cellular array. *Journal of Computer and System Sciences*, 36(2):125–147, 1988. https://doi.org/10.1016/0022-0000(88)90024-4.

[9] S. Galatolo, M. Hoyrup, and C. Rojas. Dynamics and abstract computability: computing invariant measures. *Discrete and Continuous Dynamical Systems*, 29(1):193–212, 2011. https://doi.org/10.3934/dcds.2011.29.193.

[10] S. Goldstein, R. Kuik, J. L. Lebowitz, and C. Maes. From PCA's to equilibrium systems and back. *Communications in Mathematical Physics*, 125:71–79, 1989. https://doi.org/10.1007/BF01217769.

[11] B. Hellouin de Menibus and M. Sablik. Characterisation of sets of limit measures of a cellular automaton iterated on a random configuration, 2016. Version 3. https://arxiv.org/abs/1301.1998v3.

[12] C. G. Jockusch, Jr. and R. I. Soare. π_1^0 classes and degrees of theories. *Transactions of the American Mathematical Society*, 173:33–56, 1972. https://doi.org/10.1090/S0002-9947-1972-0316227-0.

[13] I. Marcovici, M. Sablik, and S. Taati. Ergodicity of some classes of cellular automata subject to noise. *Electronic Journal of Probability*, 24(41):1–44, 2019. https://doi.org/10.1214/19-EJP297.

[14] H. Marsan and M. Sablik. Characterization of the set of zero-noise limits measures of perturbed cellular automata, 2024. Version 2. https://arxiv.org/abs/2412.04672v2.

[15] S. Taati. Reversible cellular automata in presence of noise rapidly forget everything. In *AUTOMATA 2021*, volume 90 of *OASIcs*, pages 3:1–3:15, 2021. https://doi.org/10.4230/OASIcs.AUTOMATA.2021.3.

[16] S. Taati and J. Miękisz. Quasicrystalline Gibbs states in 4-dimensional lattice-gas models with finite-range interactions, 2025. Version 1. https://arxiv.org/abs/2512.24436v1.

[17] A. L. Toom. Stable and attractive trajectories in multicomponent systems. In R. L. Dobrushin and Y. G. Sinai, editors, *Multicomponent Random Systems*, pages 549–575. Marcel Dekker, 1980.