Where the quantum capacity sits in the arithmetical hierarchy

Seth Douglas Nidhal Mghirbi

Version 1.0.0, November 2026

Abstract

Quantum capacity can be approximated uniformly from below. For effectively described finite-dimensional channels, we give certified algorithms for every finite-block coherent-information maximum and an increasing rational sequence converging to the capacity Q. The closed threshold Q at least one half therefore lies in the second universal level of the arithmetical hierarchy. This contradicts the third-level hardness conjectured by Bhattacharyya, Mehta and Zhao, under standard computable reductions. Positivity of Q is semidecidable. On exact algebraic inputs it is decidable precisely when zero capacity has a complete effective certificate system; an activation-bound converse is conditional on decidability of the real exponential field. A measure-and-erase wrapper gives an undecidable all-length predicate for precisely specified MPO-generated families. For zero-error communication we distinguish known one-shot results from an unproved OR-repetition step. On the synchronous magic-square graph, all evaluated spectral families give six, including fractional Haemers bounds over every field, while a certified computation gives strong-square independence number 29. Either its Shannon capacity is six or a spectral point outside those families lies below six.

AI disclosure. This paper was developed with AI assistance. The authors take responsibility for its claims; methods and verification limits appear in Section 8.

Contents

1 Introduction **2**

2 Uniform lower approximation and closed thresholds **3**

3 Zero certificates and activation **5**

4 An external index: MPO-generated channel families **7**

5 Zero error: finite witnesses and the missing repetition step **8**

6 The magic-square certificates **9**

7 Discussion **11**

8 Methods and provenance **12**

A Two effective finite-block derivations **12**

B Certificate conventions **13**

1 Introduction

Can a finite description of a quantum channel tell us whether its quantum capacity exceeds a threshold? There are two different questions here. An algorithm may approximate a capacity from below without ever being able to decide equality with the threshold. The distinction determines where the decision problem belongs in the arithmetical hierarchy.

The picture is small. Write $\Sigma_1^0 = \mathrm{RE}$ for the computably enumerable sets, and $\Pi_1^0 = \mathrm{coRE}$ for their complements. A predicate in Π_2^0 has the form $\forall k\, \exists s\, R(k, s)$ with R decidable. The following table gives upper bounds, except where completeness is expressly stated. The ordinary capacity Q is measured in qubits per channel use.

Question	Class or status
$Q(\Phi) > 0$; $Q(\Phi) = 0$	Σ_1^0; Π_1^0
$Q(\Phi) \geq \frac{1}{2}$; $Q(\Phi) < \frac{1}{2}$	Π_2^0; Σ_2^0
$Q(\Phi) \geq \frac{3}{4}$ versus $Q(\Phi) \leq \frac{1}{4}$ (promise)	RE [5]
Closed thresholds for $C_{0E}(G)$ and $\Theta_q(G)$	Π_2^0
One-shot game-graph ceiling, q / qc models	RE-complete / coRE-complete
$Q(\Phi_L) = 0$ for every MPO-family index L	coRE, undecidable

Input and reduction conventions. All dimensions are finite and supplied. For ordinary capacity we allow *effective channel names*: computable Kraus entries with programs giving certified approximations at requested precision, under the promise that they describe a CPTP map. Computable Choi names are equivalent for this purpose. Our default discrete domain $\mathcal{E}$ consists of explicit algebraic Choi matrices, encoded by integer polynomials and isolating data; validity is decidable. Circuit descriptions over a fixed computable gate set are also covered; Clifford+T gives algebraic data. Zero-error results use explicit finite graphs or exact algebraic channel data. Hardness means computable many-one hardness unless stated otherwise. Promise reductions must produce valid instances; Turing reductions must make valid queries. These conventions apply throughout.

Main contribution: the channel application and its hierarchy consequence. We apply standard effective finite-dimensional optimization and the effective Fekete principle to quantum capacity. This gives increasing rational lower bounds uniformly from a channel description and yields the second-level closed-threshold bound below. In particular, the problem called (QCap) by Bhattacharyya, Mehta and Zhao (BMZ) lies in Π_2^0. Their Conjecture 1.4 reads

> "(QCap) is Π_3^0-hard."

For their effective circuit setting, this is false as printed, under computable many-one or Turing reductions. The correction concerns the hierarchy discussion, rather than their hardness constructions.

Remark 1.1 (Known ingredients and closest prior approximation result)**.** The coherent-information formula is due to Lloyd, Shor and Devetak [21, 30, 12]. Our finite-dimensional optimization uses Winter's entropy continuity bound [33]. Wolf–Cubitt–Pérez-García [34] distinguish exact decisions from approximation. For quantum capacity, Khabbazi Oskouei–Mancini [16, Theorem 4.1] establish recursive approximability as convergence of a computable rational sequence without a modulus or monotonicity. The generic effective transformation from computable superadditive representations to lower approximations is explicit in Boche, Böck and Deppe [7, Lemmas 17–19 and Theorem 21], building on Zheng and Weihrauch. Our contribution here is its channel-specific implementation and the resulting correction to the QCap hierarchy discussion.

New synthesis: what an undecidability proof would require. Positivity is semidecidable. On $\mathcal{E}$, it is decidable exactly when zero capacity has a sound and complete effective system of finite certificates. A separate activation-bound characterization has one unconditional direction and one direction conditional on decidability of the real exponential field. An undecidable all-length predicate already exists for precisely specified MPO-indexed channel families. Its external index is essential to the statement.

Remark 1.2 (Known in special settings; recoveries)**.** BMZ prove QMA-hardness of a gapped ordinary-capacity problem and RE-hardness of a restricted one-shot zero-error problem [5]. Cubitt–Elkouss–Matthews–Ozols–Pérez-García–Strelchuk [9] prove that no fixed block length suffices across all channels; this does not give one channel that fails to attain capacity at every length. The one-shot game-graph equivalences and model separations below recover Harris [14], Kim–Paulsen–Schafhauser [17], and Paddock–Slofstra [27]. Our role in that part is the communication phrasing.

The third takeaway is a concrete obstruction to extending one-shot hardness to asymptotic zero-error coding. A sufficient OR-repetition conjecture remains unproved. Its classical shadow reaches a specific 24-vertex magic-square graph: all the evaluated spectral families give the upper bound 6, while a certified computation gives strong-square independence number 29. The resulting dichotomy isolates the missing spectral information without deciding the Shannon capacity.

2 Uniform lower approximation and closed thresholds

For a channel $\Phi : A \to B$, choose a complementary channel Φ^c and put

$$b_n(\Phi) = \max_{\rho} \{S(\Phi^{\otimes n}(\rho)) - S((\Phi^c)^{\otimes n}(\rho))\}, \qquad a_n(\Phi) = b_n(\Phi)/n.$$

All logarithms are base two unless ln is written. Pure inputs give $b_n \geq 0$. Product inputs give $b_{n+m} \geq b_n + b_m$, so b_n is nondecreasing, although a_n need not be. The capacity formula and Fekete's lemma give

$$Q(\Phi) = \lim_{n\to\infty} a_n(\Phi) = \sup_{n\geq 1} a_n(\Phi). \tag{1}$$

Theorem 2.1 (Uniform finite-block and capacity approximation)**.** *There is a uniform algorithm returning rationals $c(\Phi, n, k)$ such that*

$$|c(\Phi, n, k) - a_n(\Phi)| \leq 2^{-k}. \tag{2}$$

Consequently Q is uniformly left-c.e.: a computable increasing rational sequence $r_s(\Phi)$ converges to $Q(\Phi)$.

Proof. Here is the finite-block algorithm; Appendix A supplies two explicit effective-net derivations. A reference system of dimension $D = d_A^n$ purifies every input. Thus b_n is the maximum of $-S(R|B)$ over a unit sphere in $\mathbb{C}^D \otimes \mathbb{C}^D$. For unit vectors at Euclidean distance at most $\eta \leq 1$, their output states have trace distance at most η. Winter's bound gives the computable modulus

$$|I_c(\psi) - I_c(\varphi)| \leq 2\eta n \log d_A + g(\eta), \quad g(\eta) = (1+\eta)h_2\left(\frac{\eta}{1+\eta}\right), \tag{3}$$

where $g(0) = 0$. Optimize certified entropy evaluations over a sufficiently fine finite net. Both the mesh error and evaluation error are controlled uniformly. Divide by n to obtain (2).

The following lower-approximation step is the effective Fekete principle [7, Lemmas 17–19 and Theorem 21], applied uniformly to the finite-block channel quantities. Explicitly, define

$$r_s = \max\big(\{0\} \cup \{c(\Phi, n, k) - 2^{-k} : 1 \le n, k \le s\}\big). \tag{4}$$

These are increasing rational lower bounds. Given any a_n, taking k arbitrarily large shows $\sup_s r_s \ge a_n$. Equation (1) proves convergence to Q. □

Corollary 2.2 (Threshold bookkeeping)**.** *For every fixed computable threshold t, the predicate $Q(\Phi) > t$ is c.e., $Q(\Phi) \ge t$ is in Π^0_2, and $Q(\Phi) < t$ is in Σ^0_2. In particular, (QCap) is in Π^0_2, (GapQcap) is in RE, and $Q = 0$ is co-RE.*

Proof. For rational t the decisive formulas have decidable matrices:

$$Q > t \iff \exists s\ [r_s > t], \tag{5}$$

$$Q \ge t \iff \forall k \ge 1\ \exists s\ [r_s > t - 2^{-k}], \tag{6}$$

$$Q < t \iff \exists k \ge 1\ \forall s\ [r_s \le t - 2^{-k}]. \tag{7}$$

For computable t, strict comparison with it is c.e.; its precision witness merges with the existential stage in (6). Complementation gives the last threshold class. Search for $r_s > 1/2$ accepts every promised $Q \ge 3/4$ instance and no promised $Q \le 1/4$ instance. Finally $Q \ge 0$, so (5) at zero proves the positivity and zero-capacity claims. □

The RE upper bound for (GapQcap) is already stated by BMZ [5, Section 1.3]; we include it to compare the gapped and closed thresholds.

Equality at $Q = t$ is included in (6): every threshold $t - 2^{-k}$ is eventually crossed. This is why numerical approximation does not turn the closed threshold into an existential test. For general program names one can dovetail completed lower evaluations, including partial computations off the valid-input promise. This gives a Π^0_2 extension agreeing on valid names. No validity test for such names is required.

BMZ's conjecture. Computable preimages preserve Π^0_2. Thus Π^0_3 many-one hardness of (QCap) would collapse the strict arithmetical hierarchy. Under Turing reductions, a Π^0_2 language is computable in $\mathbf{0''}$, whereas a Π^0_3-complete language has degree $\mathbf{0'''}$. The same contradiction holds for valid-query promise reductions, using the extension just described. BMZ's stated Π^0_4 membership remains a valid loose bound. Their separate gap-hardness conjecture is unaffected.

Conjecture 2.3 (A corrected form)**.** *On the effective algebraic circuit domain, (QCap) is Π^0_2-complete.*

This is a proposed replacement, not a lower bound proved here.

Question 2 and finite attainment. BMZ ask whether a channel exists with $a_n(\Phi) < Q(\Phi)$ for every integer $n \ge 1$. The following implication uses those integers, rather than only powers of two.

Proposition 2.4. *If every channel in the encoded family attains Q at some finite block length, then (QCap) is in Δ^0_2. Hence Π^0_2 hardness, under many-one or Turing reductions, would force a channel in that family with $a_n < Q$ for all n.*

Proof. Under finite attainment, $Q \geq t$ is equivalent to $\exists n\ [a_n \geq t]$. For rational t,

$$a_n \geq t \iff \forall k\ [c(\Phi, n, k) + 2^{-k} \geq t].$$

Thus the threshold is Σ_2^0 and, independently by Corollary 2.2, Π_2^0. Their intersection is Δ_2^0, computable in $\mathbf{0}'$. This excludes Π_2^0 Turing hardness as well as many-one hardness. □

No all-length nonattaining channel follows without the hardness hypothesis. Nor does undecidability of an exact threshold alone imply noncomputability of the capacity function: equality tests for computable reals can already be undecidable. By contrast, many-one RE-hardness of this closed threshold would exclude a uniformly computable Q, since its closed threshold would then be co-c.e.; RE-hardness of the gapped promise would also exclude it, since a computable Q decides that promise.

3 Zero certificates and activation

For the rest of this section the discrete domain is $\mathcal{E}$. Define the activation length

$$n_*(\Phi) = \min\{n \geq 1 : b_n(\Phi) > 0\},$$

with value ∞ when $Q = 0$. Positivity has a finite witness, although the witness length is not supplied by the channel description.

Proposition 3.1 (Zero-certificate equivalence; unconditional)**.** *The predicate $Q > 0$ is decidable if and only if $Z = \{\Phi : Q(\Phi) = 0\}$ is c.e. Equivalently, zero capacity has a sound and complete system of finite certificates with a computable verifier.*

Proof. The complement of Z is c.e. by Corollary 2.2. If Z is c.e., dovetail its enumeration with the positive-capacity search. Exactly one terminates. Conversely a decision algorithm enumerates Z. A c.e. enumeration yields finite certificates by recording the enumeration stage, and a computable verifier yields an enumeration by searching all finite certificates. □

Remark 3.2 (Known structural certificates and a recent claim)**.** PPT and antidegradability give effective zero-capacity certificates. For algebraic data, PPT is an exact positive-semidefiniteness test on the partially transposed Choi matrix. Antidegradability is an exact semialgebraic feasibility test for a symmetric two-output extension of the normalized Choi state. Both are decidable, by real quantifier elimination rather than approximate SDP termination [34].

Zhu–Wang [35, Theorems 1.1 and 4.3; Section 5] claim that the rational qutrit channel

$$\Lambda(X) = \frac{1}{2}X + \frac{1}{4}(\mathrm{tr}(X)I - X^{\mathsf{T}})$$

has zero private and quantum capacities while being NPT and non-antidegradable. Their proposed certificate uses a signed lift, a completely positive defect factorization, and a complete information comparison that tensorizes. We cite this as a preprint claim; its full signed-lift/BKM argument is not independently verified here and is not an input to our proofs. It changes the appropriate structural question: the issue for Proposition 3.1 is an effective complete enumeration of all zero-capacity channels. Additional examples or certificate classes can enlarge that enumeration without making it complete.

The obstruction can be stated more sharply without assuming any particular structural classification.

Proposition 3.3 (A fixed certificate family)**.** *Suppose $e \mapsto \Phi_e$ is a computable many-one reduction from HALT to $\{Q > 0\}$. For every fixed sound c.e. set $C \subseteq Z$, the set*

$$D_C = \{e : e \notin \text{HALT},\ \Phi_e \notin C\}$$

is not c.e., and is infinite. If C includes the PPT and antidegradable channels, infinitely many distinct channel outputs evade both classes.

Proof. The preimage of C is c.e. and consists of nonhalting indices. Its union with D_C is the complement of HALT, which is not c.e. If only finitely many distinct outputs occurred in D_C, exact equality of algebraic channel matrices with those finitely many outputs would enumerate D_C, a contradiction. □

This is a statement about uniform completeness. It does not prohibit a proof of zero capacity for an individual channel. The elementary certificate principle is analogous to the distillability discussion in [34, Section 3.1] and the tensor-stable-positivity discussion in [31, Section 4].

Let $|\Phi|$ denote description length and set

$$A(s) = \max\{n_*(\Phi) : |\Phi| \leq s,\ Q(\Phi) > 0\},$$

with maximum zero when the set is empty. Finitely many descriptions have length at most s, so $A(s)$ is a finite number.

Proposition 3.4 (Activation bounds; conditional reverse direction)**.** *If positivity is decidable, A is bounded by a computable function. Conversely, assume that* $\mathrm{Th}(\mathbb{R}, +, \cdot, <, \exp)$ *is decidable. Then a computable bound on A decides positivity. In particular the equivalence holds under Schanuel's conjecture, by the Macintyre–Wilkie theorem.*

Proof. With a positivity decider, list all valid descriptions of length at most s and identify the positive ones. Run the lower-bound searches for those finitely many channels until each has a witness $b_n > 0$. The largest witnessed length, or zero for an empty list, is a computable upper bound on $A(s)$.

For the converse, fixed-block positivity is an existential sentence over the real exponential field. Purify the input in dimension $D \times D$. Its output matrices are polynomial in the real and imaginary input coordinates, with algebraic coefficients. Quantify a unitary diagonalization and nonnegative eigenvalues λ_i for each matrix. Represent $\lambda \ln \lambda$ by $t = \lambda y$, $\exp(y) = \lambda$ when $\lambda > 0$, and by $t = 0$ when $\lambda = 0$. Algebraic coefficients have finite isolating definitions. The resulting sentence expresses positive coherent information; changing the logarithm base does not change its sign. The assumed theory decider therefore decides $b_n > 0$. If $A \leq f$ computably, test $b_{\max(1, f(|\Phi|))} > 0$. Monotonicity of b_n proves correctness. □

Remark 3.5 (Conditionality and attribution)**.** Decidability of the real exponential field is not supplied by semialgebraic feasibility. Macintyre–Wilkie's implication from Schanuel is conditional [22]; see also [4, version 3, Theorem 14.4]. The proposition is a bounded-witness consequence with a channel-specific entropy encoding. Under that hypothesis, undecidability of positivity would mean that A is not eventually bounded by any computable function: finitely many exceptional sizes could be hardcoded. It does not say that every positive channel requires long activation.

4 An external index: MPO-generated channel families

An all-length question about a family can be undecidable even when every zero-capacity member has a structural certificate. The following wrapper makes the distinction explicit.

Proposition 4.1 (Measure-and-erase wrapper)**.** *Let W be a finite Hermitian matrix and let $c > 0$ be rational with $c \geq \|W\|$. There is a channel $\Phi[W]$ with algebraic data when W is algebraic, and rational Choi data when W is rational, such that*

$$Q(\Phi[W]) = 0 \iff W \geq 0.$$

Proof. Put $A = (I - W/c)/2$ and $R = (I + W/c)/2$. On an input $X \otimes q$, with q a message qubit, measure the effects A, R on X. Bob receives the qubit and an accept flag in the first branch, and an erasure and reject flag in the second. An isometry sends the measured X to the environment, with $\sqrt{A}$ or $\sqrt{R}$ applied; the environment gets the qubit exactly in the reject branch. Orthogonal flags in both outputs remove cross terms. Bob's map is

$$\rho \longmapsto |\mathrm{acc}\rangle\langle\mathrm{acc}| \otimes \mathrm{tr}_X[(A \otimes I)\rho] + \mathrm{tr}[(R \otimes I)\rho]\, |\mathrm{rej}, e\rangle\langle\mathrm{rej}, e|.$$

It is linear in A, R, which proves the rational Choi assertion.

If $W \geq 0$, then $A \leq R$ and $R \geq I/2$. On its reject branch the environment uses the effect $G = R^{-1/2} A R^{-1/2} \leq I$ on X, forwards its qubit on success, and erases otherwise. On its accept branch it erases. The identity $\sqrt{R} G \sqrt{R} = A$ recovers Bob's accept map, including correlations with the message; the remaining weight is his reject map. This is an antidegrading channel.

If $\langle v, Wv\rangle = -\mu < 0$ for a unit vector v, fix $X = v$. The message sees a qubit erasure channel with acceptance probability $(1 + \mu/c)/2$. The maximally mixed message input has coherent information $\mu/c > 0$. □

Corollary 4.2 (Indexed-family undecidability)**.** *Given a rational TI MPO description generating diagonal Hermitian operators W_L, set $\Phi_L = \Phi[W_L]$ with effective positive rational norm bounds c_L. The predicate $\forall L \geq 1\ Q(\Phi_L) = 0$ is co-c.e. and undecidable. For rational open-boundary descriptions the complementary existential predicate is RE-complete. Periodic fixed-size descriptions also give undecidability, without a many-one-completeness claim here.*

Proof. Kliesch–Gross–Eisert [18, Theorem 5 and Lemmas 11–12] give a many-one reduction to existence of a negative eigenvalue at some length for rational open-boundary diagonal MPOs, even at physical dimension two and fixed bond bound. Finite-length positivity is decidable exactly, so existence of a negative eigenvalue is c.e. and RE-complete. The wrapper preserves this predicate length by length.

De las Cuevas et al. [11, Problem 2 and Lemma 3] give a periodic version with seven rational 7×7 matrices. Their promise separates positivity at every length from a negative eigenvalue at every sufficiently large length. Their reduction framework permits Turing reductions; this supports the periodic undecidability assertion. If the matrices are $C_i \in \mathbb{Q}^{s \times s}$, take $K = \max(1, \max_i \|C_i\|_\infty)$ and $c_L = sK^L$ for trace contraction. For open boundaries take $c_L = \max(1, \|\ell\|_1 \|r\|_\infty) K^L$. These bound every diagonal coefficient. Complementation proves co-c.e. membership of the universal predicate. □

The translation invariance is that of the generating MPO tensor. The channel has a distinguished message qubit and terminal flags; its Choi matrix has fixed-bond MPO form on the X chain with terminal auxiliary systems. This specifies an effectively generated family, rather than a spatially

translation-invariant local channel with identical input and output sites. In particular Φ_L is not asserted to be $\Phi_1^{\otimes L}$. Every universal-YES length is antidegradable; undecidability remains in the external quantifier over L.

Remark 4.3 (Attribution)**.** The undecidability is imported from MPO positivity. Transport of that problem to tensor positivity already appears in [31]. The wrapper adds a CPTP capacity interpretation. The memory-channel capacity theorem of Elkouss–Pérez-García [13] is an operational analogue, for classical capacity with memory.

5 Zero error: finite witnesses and the missing repetition step

For a finite confusability graph G, write $\alpha_*(G)$ for its one-use message maximum with arbitrary finite-dimensional entanglement assistance. Write $\alpha_q(G)$ for the restricted maximally entangled, projective code parameter. These are different resources. Beigi's entanglement-assisted theta bound [3] also bounds the restricted parameter. For recent resource-sensitive game-to-classical-channel correspondences and an arbitrary-versus-maximally-entangled separation, see Mančinska [23, Theorem 6 and Corollary 8]; we do not use that result in the proofs below. Set

$$C_{0E}(G) = \sup_m \frac{\log \alpha_*(G^{\boxtimes m})}{m}, \qquad \Theta_q(G) = \sup_m \alpha_q(G^{\boxtimes m})^{1/m}.$$

The strong product has distinct tuples adjacent when every coordinate is equal or adjacent. Product protocols give the required supermultiplicativity.

Proposition 5.1 (Uniform zero-error lower approximation)**.** *On explicit finite graphs, C_{0E} and Θ_q are uniformly left-c.e. Their closed computable-threshold predicates lie in Π_2^0. The same lower-approximation conclusion for C_{0E} holds with exact algebraic channel data.*

Proof. For fixed message number M and dimension d, entanglement-assisted graph coding is finite real-algebraic feasibility. One may use positive semidefinite matrices ρ, ρ_i^v with

$$\operatorname{tr}\rho = 1, \quad \sum_v \rho_i^v = \rho, \quad \rho_i^v \rho_j^w = 0 \quad (i \neq j,\ v = w \text{ or } v \sim w).$$

For α_q, replace these by projections $E_{i,v}$ with $\sum_v E_{i,v} = I$ and the same cross-message zero products. Real quantifier elimination decides each fixed-dimensional system. Dovetail over m, M, d and enumerate rational lower estimates for $\log M/m$, or $M^{1/m}$. For algebraic channel data the fixed-dimensional protocol constraints likewise have exact algebraic coefficients. The supremum argument of (6) completes the proof. □

Remark 5.2 (Known criterion; representation restriction)**.** These code descriptions are standard [10, 24, 20]. An approximation program for arbitrary computable-real entries does not decide exact support or zero products. The zero-error statement therefore uses the exact inputs specified in Section 1. No such restriction was needed for ordinary capacity. Spectral duality in [20] is a minimum over an abstract spectrum; it supplies no effective convergent family of upper bounds. Li–Li's multiplicative SDP bound [19] concerns no-signalling assistance and supplies a single bound. Neither it nor the spectral families in our related paper [26] supply the missing convergence theorem.

One-shot ceilings: a recovery. Normalize a finite synchronous game by retaining self-legal answers, rejecting unequal answers at equal questions even off the sampling support, and making only unsupported off-diagonal tests vacuous. If a question loses every self-answer, detect it and use a fixed NO instance. Its graph X has vertices (q, a) and edges at distinct pairs for which either orientation is forbidden. There are $N = |Q|$ question cliques. Here q means finite-dimensional strategies, qa their closure, and qc commuting-operator strategies. Harris's Theorem 5.5 [14] identifies a perfect strategy with $\alpha_t(X) = N$ in the relevant q, qa, qc models.

In communication terms, aggregate code projections into $P_{q,a} = \sum_i E_{i,(q,a)}$ and $F_q = \sum_a P_{q,a} \leq I$. At the ceiling, $\sum_q \tau(F_q) = N$ forces $F_q = I$ for a faithful trace, giving perfect game measurements. Conversely the code is $E_{i,(q,a)} = \mathbf{1}_{i=q} P_{q,a}$, also in the tracial commuting model.

The finite-dimensional ceiling is RE-complete; the commuting ceiling is co-RE-complete, by the known satisfiability results [15, 27]; for the latter see [27, Theorem 5.11(c)]. KPS [17, Theorem 5.4 and Corollary 5.5] give earlier q versus qa graph separations. Harris's Corollary 5.6 gives a qa versus qc separation. These are recoveries, not new separation or hardness theorems. Nonattainment of a finite-dimensional perfect strategy at supremum value one does not meet the soundness hypothesis below.

The precise missing step (conjectural). A projective packing consists of projections P_v with $P_v P_w = 0$ on graph edges, of weight $\sum_v \tau(P_v)$. Its supremum is α_p. Aggregating a projective code gives a packing, so $\alpha_q \leq \alpha_p$. A packing does not supply the message-indexed PVM decomposition needed for the converse.

Conjecture 5.3 (Packing OR-repetition)**.** *For every normalized finite synchronous game with an effective rational question distribution having uniform marginals and entangled value $\omega^* \leq 1/2$, there is $c_G \in (0, 1)$ such that, for every $m \geq 1$,*

$$\alpha_p(X^{\boxtimes m}) \leq (1 - c_G)^m N^m.$$

The constant may depend on the game; no computable modulus is asserted. The version with α_q would suffice for the restricted projective capacity and may be weaker. The strong power corresponds to the *special OR predicate*: identical tuple vertices are accepted, as are pairs with a coordinate that is distinct and nonadjacent. It is not Boolean OR of coordinate acceptance. AND repetition instead gives the disjunctive product, adjacent when some coordinate loses.

The verified line extraction gives success at least $2M/N^m - 1$ from a packing of weight M, after completing abstentions by fixed answers. It uses $\tau(PQ) \geq \tau(P) + \tau(Q) - 1$, without commutativity. This is only a constant density loss. Anchored AND parallel repetition [2] does not give exponential OR loss. The attempted AND-enforcing route fails on product inputs because tensor-product output overlaps multiply and hence give the strong product.

JNVWY's Theorem 12.6 [15] provides attained finite-dimensional perfect completeness and soundness at most $1/2$ for the halting reduction. Diagonal normalization preserves that witness. After symmetrizing the rational distribution, a common marginal $\nu(q) = k_q/K$ becomes uniform by replacing q with k_q clones. Averaging clone-indexed measurements preserves the soundness bound. Thus Conjecture 5.3 would separate the asymptotic Θ_q ceiling on this family. This is a conditional route, with no conclusion here for unrestricted entanglement-assisted codes.

6 The magic-square certificates

Let $X = X_{\mathrm{MS}}$. Its vertices are the four legal bit assignments to each of $R_1, R_2, R_3, C_1, C_2, C_3$ in a 3×3 square; all context parities are even except C_3. Distinct vertices are adjacent if they belong to

the same context or disagree on a shared cell. This fixes the 24-vertex synchronous graph, rather than another magic-square event graph. Ordinary independence is $\alpha(X) = 5$.

Proposition 6.1 (All-field certificate)**.** *For every field $\mathbb{F}$, the fractional Haemers bound satisfies $\mathcal{H}_f^{\mathbb{F}}(X) = 6$.*

Proof. A d-representation is a block fitting matrix M with diagonal I_d and zero blocks at distinct nonneighbors; its cost is $\operatorname{rank} M/d$. Factor $M = BA$ and put $V_v = \operatorname{im} A_v$. Then $B_v A_v = I_d$ and $B_v A_w = 0$ at nonneighbors. Consequently

$$r(S) = d^{-1} \dim \sum_{v \in S} V_v$$

is normalized, monotone and submodular, with

$$r(\{v\}) = 1, \qquad r(\overline{N}(v) \cup \{v\}) - r(\overline{N}(v)) = 1, \tag{8}$$

where $\overline{N}(v)$ denotes distinct nonneighbors.

The induced subgraph on

$$R_1 : 011; \quad R_2 : 000, 011; \quad R_3 : 000, 011; \quad C_1 : 000;$$
$$C_2 : 101, 110; \quad C_3 : 100, 111$$

is $2K_1 \sqcup W_8$, where W_8 is the eight-cycle with opposite chords. The dimension inequalities force cost at least 6 on this core. This is also the $n = 8$ Möbius-ladder Shannon-inequality argument of Blasiak–Kleinberg–Lubetzky [6, Theorem 5.2 and proof], applied to dimensions over any field. For an explicit finite verification, the accompanying certificate gives the identity

$$r(T) - 6 = \sum_i \lambda_i g_i(r) + \sum_j \mu_j h_j(r), \qquad \lambda_i \geq 0, \tag{9}$$

using 33 monotonicity/submodularity rows $g_i \geq 0$ and 10 rows $h_j = 0$ from (8) and normalization. Every coefficient is rational and the identity is checked exactly. Restriction to the core gives the lower bound on X; its six question cliques give the upper bound 6. □

Remark 6.2 (Recovery and exact quantum ceiling)**.** The subspace framework is standard [8, Section 2.1]; the Möbius-ladder lower bound is a recovery of [6]. The specific induced-core application supplies a useful certificate, not a new LP method. The Mermin–Peres observables [25, 28], displayed in Appendix B, give an exact four-dimensional perfect strategy. Therefore $\alpha_q(X) = \vartheta(X) = 6$. Beigi's entanglement-assisted theta bound [3] also applies to this restricted code parameter. Vertex transitivity and clique number four give $\overline{\chi}_f(X) = 24/4 = 6$. These equalities use exact projections and combinatorial certificates, rather than floating SDP estimates.

Certified computation. The exact strong-square value is

$$\alpha(X^{\boxtimes 2}) = 29. \tag{10}$$

A 29-word independent set was independently constructed in the first blind check and verified pairwise. For the upper bound, slicing by the first question gives at most $6\alpha(X) = 30$ words. A question-pair block contains at most one word. A 30-word code therefore has five words in each row and column of question blocks, so its empty blocks form a permutation. Checked graph symmetries reduce the 720 permutations to two double cosets of sizes 72 and 648. The earlier independent encodings were UNSAT with CaDiCaL and Glucose. A further independent encoding was UNSAT with Lingeling for both representatives, and drat-trim separately verified both saved proof traces. The programs, formulas, traces, checker source, symmetry verification and lower witness are included in the computational supplement.

Which spectral bounds are exhausted? Every evaluated known spectral family gives 6 on X: ϑ, $\overline{\chi}_f$, all fractional Haemers bounds, and every point in the quantum spectrum, the last squeezed between $\alpha_q(X) = 6$ and the question-clique ceiling. This also covers entanglement-assisted spectral points, including those reported in [26], provided their spectral membership is accepted. No proof in this note depends on that preprint's new membership claim.

Vrana's refinements [32, Theorems 1.1–1.2] do not improve these evaluations. For a vertex-transitive graph, concavity and averaging over automorphisms show $F_f(X, U) = \log f(X)$ at the uniform distribution U. Thus interpolation of refinements gives $f_F(X) = \prod_i f_i(X)^{\lambda_i} = 6$, also for pointwise limits of these interpolations. This specifies the evaluated closure, without claiming an exhaustive classification of all spectral constructions.

Corollary 6.3 (Spectral dichotomy)**.** *Either $\Theta(X) = 6$, or a classical spectral point outside the evaluated families has value below 6 on X. In the second case $\Theta(X) < \Theta_q(X) = 6$. Presently the certified interval is $\sqrt{29} \le \Theta(X) \le 6$.*

Proof. Zuiddam's duality [36] says $\Theta(X) = \min_f f(X)$. Equation (10) gives the lower endpoint; the preceding evaluations give the assertions about a minimizing point. □

Vertex transitivity gives $\vartheta(\overline{X}) = 24/\vartheta(X) = 4$; the clique of size four gives $\Theta(\overline{X}) = 4$. The diagonal independent set in $X \boxtimes \overline{X}$ has size 24, so the second branch would also give $\Theta(X \boxtimes \overline{X}) > \Theta(X)\Theta(\overline{X})$. Supermultiplicativity of this kind is known in general from Alon [1]. The direct evaluated Haemers bounds fail to certify a strict gap here; structural arguments using auxiliary graphs remain possible.

The classical shadow (conjectural). Qualitative classical OR-repetition for all full-support synchronous games is equivalent to

$$\alpha(G) < \overline{\chi}(G) \quad \Longrightarrow \quad \Theta(G) < \overline{\chi}(G).$$

A minimum clique cover can be partitioned into question cliques; a perfect strategy is exactly an independent transversal. The only live case is $\alpha < \overline{\chi}_f = \overline{\chi}$. This is the integral-cover case of Shannon's C_0 versus C_{0F} question [29], using the classical channel whose output supports are all maximal cliques. Then feedback fractional packing equals $\overline{\chi}_f$, and Shannon's feedback formula applies when $C_0 > 0$. The realization matters: an edge-symbol channel with the same graph can have a different feedback packing value [10]. For X, the two realizations give packing values 6 and 12, respectively. This broad classical conjecture is not equivalent to the narrower $\omega^* \le 1/2$ quantum conjecture; the magic square itself is quantum-perfect. The first branch of Corollary 6.3 would refute the classical conjecture, since $\alpha(X) = 5 < 6$.

7 Discussion

The proved picture separates four issues. Ordinary capacity is computably accessible from below; equality with a positive threshold adds a universal precision quantifier. Positivity has finite witnesses, but complete zero certificates need not be enumerable. Under the real exponential hypothesis, computable bounds on activation would settle positivity. In the MPO family, undecidability comes from an external all-length quantifier. For restricted asymptotic zero-error coding, the unproved issue is exponential packing loss in strong powers.

Outlook (conjectural and heuristic). An asymptotic projective zero-error ceiling hardness result remains open. Three developments would justify reopening the broader search: a spectral point below six on X_{MS}; a mechanism for coherent amplification within tensor powers; or effective zero-capacity certificates beyond PPT and antidegradability. The third condition must now take the Zhu–Wang claim into account. An independently validated new certificate could enlarge the structural toolkit; it would not by itself establish completeness of that toolkit. The first condition would settle the concrete classical test, not prove the quantum repetition conjecture. The second is a missing mechanism, not an existence theorem.

Heuristic. Distillability, tensor-stable positivity, $Q > 0$ and quantum $C_0 > 0$ may be manifestations of a common problem about witnesses in tensor powers. Their analogies suggest looking simultaneously for positive finite witnesses and effective zero certificates. No reductions or equivalence between these four questions are claimed. In particular, incompleteness of a named structural certificate class does not alone imply undecidability.

8 Methods and provenance

How this work was done

The authors developed the results and proofs with assistance from AI models from more than one developer. They take responsibility for every claim. The authors chose the question, directed the work and decided which lines to pursue; the models did most of the exploration, derivation and drafting.

Three rounds of proof development recorded arguments, unsuccessful approaches and self-checks. Separate sessions that had not seen the material were instructed to refute it and check primary sources. This version incorporates their repairs. Literature and priority searches were AI-assisted and bounded to the checked sources; recent preprint claims were not accepted as independently verified proofs. A subsequent independent whole-paper audit verified the finite certificates and both strong-square UNSAT traces with a separate checker, and requested the attribution and reference repairs recorded with that audit.

Computational evidence. The all-field dimension certificate uses only rational arithmetic after LP discovery; its semantic replay reconstructs the graph and checks every inequality row and coefficient of (9). This replay and the exact non-diagonal wrapper check were run again during assembly. The square upper bound has independent CaDiCaL, Glucose and Lingeling encodings; a separate drat-trim check verifies both saved Lingeling UNSAT traces. The supplement contains the finite certificate data, checker programs and source, formulas, traces, symmetry verification, lower witness and replay instructions. The graph reduction, encodings and checker implementation remain part of the trust base; this is not a proof-assistant formalization. These computations certify the stated finite quantities; none extrapolates a Shannon-capacity gap from finitely many powers.

A Two effective finite-block derivations

In the purification proof, grid the real unit sphere in dimension $2D^2$. Choose rational mesh small enough that rounding a unit vector has Euclidean error at most $\eta/2$, discard zero vectors, and normalize. Normalization changes the error by at most another $\eta/2$. The resulting finite algebraic net is effective. To evaluate entropy, approximate each computable Hermitian output by a rational Hermitian matrix in operator norm. Isolate its algebraic eigenvalues and use Weyl's inequality to

bound the ordered true eigenvalues. Intersect the eigenvalue intervals with $[0,1]$. The continuous extension of $-x\log x$ at zero has an effective modulus on $[0,1]$; zero eigenvalues cause no rank decision. Choose the net and evaluation errors to sum to the requested tolerance in (3).

The independent complementary-channel derivation starts with a computable positive square root of the unnormalized Choi matrix $J_\Phi = (\mathrm{id}\otimes\Phi)(|\Omega\rangle\langle\Omega|)$, where $|\Omega\rangle = \sum_{i=1}^{d_A} |i\rangle|i\rangle$. Reshape all $d_A d_B$ square-root columns into Kraus operators, including zero columns, to obtain a fixed-size environment $d_E = d_A d_B$. If starting from the normalized Choi state J_Φ/d_A, multiply its square-root columns by $\sqrt{d_A}$ before reshaping. No minimal-rank decomposition is needed. For input states at trace distance $\delta \le 1$, separate entropy continuity bounds on the B and E outputs give

$$|I_c(\rho) - I_c(\sigma)| \le 2n\delta\log(d_B d_E) + 2g(\delta).$$

Density matrices have effective nets YY^* with Y drawn from normalizations of nonzero rational complex matrices in Frobenius norm: every state is YY^* for some unit-Frobenius Y, and $\frac{1}{2}\|YY^* - ZZ^*\|_1 \le \|Y-Z\|_F$. Finite maximization with this modulus again proves (2). An even shorter lower-only proof enumerates dense algebraic inputs and certified coherent-information lower evaluations over every length. Continuity identifies their supremum with Q.

B Certificate conventions

The exact magic-square observables, with Pauli matrices X, Y, Z, are

$$\begin{pmatrix} X\otimes I & I\otimes X & X\otimes X \\ I\otimes Z & Z\otimes I & Z\otimes Z \\ X\otimes Z & Z\otimes X & Y\otimes Y \end{pmatrix}.$$

They commute within contexts and multiply to I except in the last column, where the product is $-I$. Their legal joint eigenspace projections have rank one and give the asserted perfect code.

For the all-field certificate, subsets of the ten core vertices are encoded by bit masks in the order printed in Section 6. The accompanying `dimension-dual.json` records all nonzero multipliers and rows. Monotonicity rows are $r(S\cup\{i\}) - r(S)$; submodularity rows are $r(S\cup\{i\}) + r(S\cup\{j\}) - r(S) - r(S\cup\{i,j\})$. The replay verifies that the equality rows correspond to the graph's actual nonneighbors before summing them. This gives a finite exact proof of the numerical LP optimum, independent of floating solver accuracy. The computational supplement includes the SAT formulas, symmetry builder, pairwise lower-witness check, both saved Lingeling UNSAT traces and drat-trim checker source. The two traces were separately verified by drat-trim. This reduces reliance on solver outcomes, while the graph reduction, Boolean encodings and checker implementation remain in the trust base; no proof-assistant verification is claimed.

References

[1] Noga Alon. The Shannon capacity of a union. *Combinatorica*, 18:301–310, 1998. URL: `https://www.cs.tau.ac.il/~nogaa/PDFS/shann3.pdf`.

[2] Mohammad Bavarian, Thomas Vidick, and Henry Yuen. Anchored parallel repetition for nonlocal games, 2015. URL: `https://arxiv.org/abs/1509.07466`.

[3] Salman Beigi. Entanglement-assisted zero-error capacity is upper bounded by the Lovász theta function, 2010. arXiv:1002.2488. URL: `https://arxiv.org/abs/1002.2488`.

[4] Alessandro Berarducci and Francesco Gallinaro. On the elementary theory of the real exponential field, 2026. URL: `https://arxiv.org/abs/2603.08365v3`.

[5] Archishna Bhattacharyya, Arthur Mehta, and Yuming Zhao. On the undecidability of quantum channel capacities, 2026. arXiv:2601.22471v3; FOCS 2026. URL: https://arxiv.org/abs/2601.22471v3.

[6] Anna Blasiak, Robert Kleinberg, and Eyal Lubetzky. Index coding via linear programming, 2011. URL: https://arxiv.org/abs/1004.1379v2.

[7] Holger Boche, Yannik Böck, and Christian Deppe. On effective convergence in Fekete's lemma and related combinatorial problems in information theory, 2024. arXiv:2010.09896v5. URL: https://arxiv.org/abs/2010.09896v5.

[8] Boris Bukh and Christopher Cox. On a fractional version of Haemers' bound, 2018. URL: https://arxiv.org/abs/1802.00476v2.

[9] Toby S. Cubitt, David Elkouss, William Matthews, Maris Ozols, David Pérez-García, and Sergii Strelchuk. Unbounded number of channel uses may be required to detect quantum capacity. *Nature Communications*, 6:6739, 2015. URL: https://arxiv.org/abs/1408.5115.

[10] Toby S. Cubitt, Debbie Leung, William Matthews, and Andreas Winter. Improving zero-error classical communication with entanglement. *Physical Review Letters*, 104:230503, 2010. URL: https://arxiv.org/abs/0911.5300.

[11] Gemma De las Cuevas, Toby S. Cubitt, J. Ignacio Cirac, Michael M. Wolf, and David Pérez-García. Fundamental limitations in the purifications of tensor networks. *Journal of Mathematical Physics*, 57:071902, 2016. URL: https://arxiv.org/abs/1512.05709v1.

[12] Igor Devetak. The private classical capacity and quantum capacity of a quantum channel. *IEEE Transactions on Information Theory*, 51:44–55, 2005. URL: https://arxiv.org/abs/quant-ph/0304127.

[13] David Elkouss and David Pérez-García. Memory effects can make the transmission capability of a communication channel uncomputable. *Nature Communications*, 9:1149, 2018. URL: https://www.nature.com/articles/s41467-018-03428-0.

[14] Samuel J. Harris. Universality of graph homomorphism games and the quantum coloring problem, 2023. URL: https://arxiv.org/abs/2305.18116v2.

[15] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. MIP* = RE, 2020. Theorem 12.6 in version 3. URL: https://arxiv.org/abs/2001.04383v3.

[16] Samad Khabbazi Oskouei and Stefano Mancini. Algorithmic complexity of quantum capacity. *Quantum Information Processing*, 17:94, 2018. URL: https://arxiv.org/abs/1608.07730v3.

[17] Se-Jin Kim, Vern I. Paulsen, and Christopher Schafhauser. A synchronous game for binary constraint systems, 2017. URL: https://arxiv.org/abs/1707.01016.

[18] Martin Kliesch, David Gross, and Jens Eisert. Matrix product operators and states: NP-hardness and undecidability. *Physical Review Letters*, 113:160503, 2014. URL: https://arxiv.org/abs/1404.4466v2.

[19] Xin Li and Yinan Li. Multiplicative semidefinite programming upper bounds on quantum no-signalling-assisted zero-error capacity, 2026. URL: https://arxiv.org/abs/2610.21071.

[20] Yinan Li and Jeroen Zuiddam. Quantum asymptotic spectra of graphs and non-commutative graphs, and quantum Shannon capacities, 2020. URL: https://arxiv.org/abs/1810.00744v3.

[21] Seth Lloyd. Capacity of the noisy quantum channel. *Physical Review A*, 55:1613–1622, 1997. URL: https://arxiv.org/abs/quant-ph/9604015.

[22] Angus Macintyre and Alex J. Wilkie. On the decidability of the real exponential field. In Piergiorgio Odifreddi, editor, *Kreiseliana: About and Around Georg Kreisel*, pages 441–467. A K Peters, 1996.

[23] Laura Mančinska. The right entanglement for perfect nonlocality and zero-error communication, 2026. arXiv:2610.06512v1. URL: https://arxiv.org/abs/2610.06512v1.

[24] Laura Mančinska and David E. Roberson. Quantum homomorphisms. *Journal of Combinatorial Theory, Series B*, 118:228–267, 2016. URL: https://arxiv.org/abs/1212.1724.

[25] N. David Mermin. Simple unified form for the major no-hidden-variables theorems. *Physical Review Letters*, 65:3373–3376, 1990. URL: https://doi.org/10.1103/PhysRevLett.65.3373.

[26] Nidhal Mghirbi and Seth Douglas. The Lovász number is not alone: a continuum of Rényi points in the entanglement-assisted spectrum of graphs, 2026. URL: https://arxiv.org/abs/2610.20775v1.

[27] Connor Paddock and William Slofstra. Satisfiability problems and algebras of boolean constraint system games, 2025. URL: https://arxiv.org/abs/2310.07901v2.

[28] Asher Peres. Incompatible results of quantum measurements. *Physics Letters A*, 151:107–108, 1990. URL: https://doi.org/10.1016/0375-9601(90)90172-K.

[29] Claude E. Shannon. The zero error capacity of a noisy channel. *IRE Transactions on Information Theory*, 2:8–19, 1956. URL: https://doi.org/10.1109/TIT.1956.1056798.

[30] Peter W. Shor. The quantum channel capacity and coherent information, 2002. MSRI lecture notes. URL: https://legacy.slmath.org/workshops/203/schedules/1181/documents/3863/assets/34150.

[31] Mirte van der Eyden, Tim Netzer, and Gemma De las Cuevas. Halos and undecidability of tensor stable positive maps, 2022. URL: https://arxiv.org/abs/2110.02113v2.

[32] Péter Vrana. Probabilistic refinement of the asymptotic spectrum of graphs, 2019. URL: https://arxiv.org/abs/1903.01857.

[33] Andreas Winter. Tight uniform continuity bounds for quantum entropies: conditional entropy, relative entropy distance and energy constraints. *Communications in Mathematical Physics*, 347:291–313, 2016. URL: https://arxiv.org/abs/1507.07775.

[34] Michael M. Wolf, Toby S. Cubitt, and David Pérez-García. Are problems in quantum information theory (un)decidable?, 2011. URL: https://arxiv.org/abs/1111.5425.

[35] Chengkai Zhu and Xin Wang. Quantum incapacity beyond no-cloning and PPT mechanisms, 2026. URL: https://arxiv.org/abs/2607.24693v1.

[36] Jeroen Zuiddam. The asymptotic spectrum of graphs and the Shannon capacity, 2018. URL: https://arxiv.org/abs/1807.00169.