The Resource Theory of Complexness References in Real Quantum Networks

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Abstract

Real quantum theory with independent sources cannot reproduce certain complex network correlations; the known remedy is the shared two-rebit state $\Omega = \frac{1}{4}(I - J \otimes J)$, the reference frame of complexness of Weilenmann, Gisin, and Sekatski. We develop the resource theory in which Ω is the unit. We show that two exact edge self-tests at one source do not fix a single frame; for commuting induced complex structures the missing datum is one binary observable, of whose $+1$ sector Ω is the normalized projector. The partial references $\eta_t = \frac{1}{4}(I - tJ \otimes J)$, $0 < t < 1$, are useful for $t > t_* \approx 0.708$ yet admit no exact finite-copy conversion to Ω under stochastic real separable operations with charge-free ancillas, while distilling Ω under rLOCC at unbounded rate: one exact copy of Ω converts into any finite number of copies between the same two holders. Every resource on m separated holders carries a transpose-charge code $H(\sigma) \leq \mathbb{F}_2^m$, non-increasing under real separable operations with charge-free ancillas; frame-code states realize every even-weight code, with a maximal reference one-shot obtainable between holders a, b iff $e_a + e_b$ is a codeword, and rate zero (in that class) otherwise. Charges can influence network statistics only if a parity condition holds on the source–party incidence graph, and, for frame-code resources, used charges are reducible to the weight-2 unit by two mechanisms we keep separate — partition coarsening, and activation consuming declared pairwise auxiliaries — the frame-sector transcription of the unlocking and activation of multipartite bound entanglement. All statements are relative to a declared separable-source-independence axiom.

1 Introduction

Renou et al. [1] showed that quantum theory over the real numbers, composed with independent sources, is experimentally falsifiable: certain network correlations of complex quantum theory are unreachable by any real model whose sources are uncorrelated (classically correlated sources do not help, by linearity of their witness), a separation realized experimentally [2, 3]. Weilenmann, Gisin, and Sekatski [4] identified the exact remedy: a shared two-rebit state

$$\Omega = \tfrac{1}{4}\,(I - J \otimes J), \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

locally invisible (we abbreviate the authors WGS), whose m-party extension restores the full simulating power of complex theory via the realification of McKague, Mosca, and Gisin [5]; one rebit of inter-source entanglement is necessary and sufficient at the ideal point. They name the

resource a *reference frame of complexness*; we adopt the contraction *complexness reference*. A debate has developed around the composition axiom: Hoffreumon and Woods [7, 8] argue the gap closes under an operational notion of independence, and Barrios Hita et al. [9] give a consistent real description in a related spirit, while Feng, Ren, and Vedral [10] and Moradi Kalarde, Xu, and Renou [12] defend the separation, and Yīng et al. [11] reframe the question through foil theories. Multipartite aspects of the gap are studied in [13]; the two-rebit skew sector is isolated as "tomographically nonlocal" entanglement in [14], whose multipartite structure is, to our knowledge, open. There is no tension between our results and the single-scenario uselessness theorems of [14]: the resourcefulness studied here is a network phenomenon, relative to a declared source-independence axiom.

This paper develops that multipartite structure as a resource theory with Ω as its unit. We state the honest relation to prior work at the outset. Once the transpose-sector decomposition is written down, our objects are stabilizer-like states in the σ_y sector: the family we study generalizes states of Wootters [15], is contained in the unlockable stabilizer class of Wang and Ying [32], and our four-source activation construction is the frame-sector transcription of the Smolin-state unlocking [26, 31] and of multiparticle activation [25, 27, 28, 29]. What this paper adds is the resource theory built on that transfer: a subgroup-valued monotone (the transpose-charge code) with a weight obstruction; an exact per-pair distillability dichotomy graded by the code, where the entanglement-theoretic ancestors give binary classifications; a graded hierarchy of partial references with a finite-copy/asymptotic separation; a parity law for which charges can influence network statistics at all; and the complexness (not entanglement) reading of all of it — the states in question are complex-separable, so no statement here is about ordinary entanglement. One feature has no entanglement-theoretic counterpart: the unit replicates exactly under the free operations (Lemma 4.5), so every rate is 0 or ∞ and the quantitative theory lives at finite copy number. The proof mechanisms also have ancestors, used gratefully: sector preservation is a modes-of-asymmetry argument [17] for an antiunitary symmetry, whose reference-frame treatment is [18]; bipartite transpose covariance appears in [19] within the imaginarity resource theory [20, 21, 22, 23], from whose monotones we differentiate ours in §5.1; the structural roots are bilocal tomography [24] and real entanglement geometry [16].

One terminological caution. The WGS reference state is real-QT entangled across every bipartition, and that real entanglement is undistillable by real LOCC precisely because its complexification is separable [4]; it carries no ordinary entanglement at all. Bound *real entanglement* and bound *complexness* (zero distillability of the reference itself) are different notions, and this paper concerns the latter.

2 Setting

Notation. A *rebit* is $\mathbb{R}^2$ with real density matrices (PSD, symmetric, unit trace) and real symmetric effects. Real CP maps are $\Phi(X) = \sum_k A_k X A_k^{\mathsf{T}}$; channels satisfy $\sum_k A_k^{\mathsf{T}} A_k = I$, and single terms without the normalization are stochastic branches. X, Y, Z are the Paulis; on a real carrier $Y = iJ$ is not a real operator while J is. $V := \mathrm{diag}(1, -1)$ denotes the reflection, $VJV^{\mathsf{T}} = -J$. Bell states are $|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$, $|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$. $e_a \in \mathbb{F}_2^m$ is the indicator vector of holder a; $|c|$ is Hamming weight, $\mathrm{supp}(c) = \{i : c_i = 1\}$, $|c \wedge c'|$ counts common support, and $d(C) = \min\{|c| : c \in C, c \neq 0\}$ for a code $C \neq \{0\}$. ∂ denotes the $\mathbb{F}_2$ boundary map of a graph (edges to vertex parities). For an operator ρ, $\mathrm{ran}\,\rho$ is its range; charge supports $\mathrm{supp}_{\mathbb{F}}$ are defined in §5.

Free scenario. Sources $S_1, S_2, \ldots$ each distribute subsystems to subsets of parties; parties measure jointly on what they receive, in a *single round*: sources emit, settings are drawn, parties measure and output. There is no communication within a round and no preparation stage. *Free models* satisfy **separable source independence**: conditioned on shared classical randomness λ, the joint source state is a product $\rho^{\lambda}_{S_1} \otimes \rho^{\lambda}_{S_2} \otimes \cdots$ (arbitrary internal source entanglement); all states, effects, and processing are real. This class is the convex hull of the strictly-independent class of [1]; any witness linear in the observed correlation retains its strictly-independent bound over it (we use this twice below). A correlation is *nonfree* if it is realizable in complex quantum theory with independent sources but by no free real model.

Resource-assisted realizations. A *resource* is an additional real state η whose subsystems (*shares*) are assigned to holders (sources or parties) — the assignment, the *partition* $\mathcal{P}$, is part of the embedding data. When the network routes several shares to one location, that location is their holder *ab initio*: partition coarsening is part of the declared embedding, never an in-protocol quantum transmission. An assisted realization may use a *preparation phase* before settings are drawn, in which holders may measure *η-descendants only* (resource shares and their images under local processing, together with local charge-free ancillas), broadcast classical outcomes — broadcasting is a free operation (the record is classical, though correlated with the surviving resource) — and apply local corrections to resource shares. Systems distributed by free sources exist only in-round and are never touched in preparation; within the round there is no communication.

Lemma 2.1 (No in-round manufacturing)**.** *Free models gain nothing from the structure above: with no resource, the preparation phase carries only classical randomness, which the free class already has; and free sources cannot convert their in-round payloads into shared references.*

Proof. The first claim is immediate from the restriction of preparation measurements to η-descendants (local charge-free ancillas yield only local randomness; a preparation phase granted to free sources would likewise only refine λ, since measurements on λ-conditioned product states generate classical data already expressible as shared randomness). The second claim is a closure statement: the free class is, by definition, closed under arbitrary local real instruments applied in-round by parties — parties only produce outputs, and there is no in-round channel through which a measurement outcome could steer any other location's operations. Hence no in-round action leaves the class. □

Usefulness. A resource η is *useful* if some assisted realization exactly produces a nonfree correlation P. No safeguard clause is needed: substituting any free state for η in the realization would exhibit a free realization of P — impossible. Usefulness is defined without reference to a partition; boundness (Definition 7.1) is the partition-relative notion. For *frame-code resources* (Definition 5.6), whose states are determined by their frame characters, the frame twirl of Definition 5.2 gives a certification handle: the twirled resource is free, so usefulness certifies that some nontrivial frame character is doing work (Theorem 6.6).

Conversion classes and rates. For a fixed partition of η's shares among permanently separated holders: **rLOCC** is finite-round local real operations and classical communication; **rSEP** is the larger real-separable class $\Lambda(X) = \sum_k (A_k^{(1)} \otimes \cdots \otimes A_k^{(m)})\, X\, (A_k^{(1)} \otimes \cdots \otimes A_k^{(m)})^{\mathsf{T}}$. In both,

ancillas are *charge-free*: local real states, product across holders. The distillation rate between holders a, b under class $\mathcal{C}$ is

$$D_\Omega^{a:b,\mathcal{C}}(\sigma) := \sup\Big\{ r : \exists\, \Lambda_n \in \mathcal{C},\ \tfrac{1}{2}\big\|\Lambda_n(\sigma^{\otimes n}) - \Omega^{\otimes \lfloor rn \rfloor}\big\|_1 \to 0 \Big\},$$

a joint (not per-copy) trace-distance criterion; $D_\Omega^{a:b}$ abbreviates the rLOCC rate, and D_Ω the two-holder case. Classical communication includes the shared randomness it generates; in rSEP, mixtures of product Kraus terms play the same role. A stochastic exact conversion with success probability $p > 0$, i.i.d.-repeated and heralded, gives a positive deterministic rate, hence an infinite one (Lemma 4.5); we use this silently. *Auxiliary-assisted* conversions, where declared additional resource states are consumed, appear in §6 with their accounting stated in each theorem.

Why Ω is not shared randomness. In complex quantum theory $\Omega = \frac{1}{2}(|y_+y_+\rangle\langle y_+y_+| + |y_-y_-\rangle\langle y_-y_-|)$ is a shared classical bit in the y basis. That ensemble does not exist for the free class: the states $|y_\pm\rangle\langle y_\pm| = \frac{1}{2}(I \pm iJ)$ are not real states, and Ω is real-*entangled* — by the real concurrence criterion (§4) it admits no decomposition into real product states conditioned on any λ. The bit is classical only in a description the free theory cannot express; that representational deficit is exactly what the resource theory quantifies, and it is operationally meaningful because it shows up in observable correlations [1]. Real local operations cannot write the bit down, but they can copy it (Lemma 4.5). Everything below is relative to the declared axiom; under operational-independence axioms the free class itself changes [8].

3 What the reference certifies: frame locking

An orthogonal complex structure on a real carrier is $\mathcal{J}$ with $\mathcal{J}^\mathsf{T} = -\mathcal{J}$, $\mathcal{J}^2 = -I$; its *frame line* is $[\mathcal{J}] = \{\mathcal{J}, -\mathcal{J}\}$. Exact two-source self-tests extract, at the ideal score, the state Ω across the tested edge and an effective complex structure on each end's support [4]; all real projective measurements are self-testable [33].

Theorem 3.1 (Edgewise exactness does not lock the source)**.** *There is a resource-assisted configuration — a four-dimensional source carrier $S = A \otimes B$ sharing $\Omega_{AC} \otimes \Omega_{BD}$ with two neighbor systems C, D — in which each edge block runs the exact ideal WGS realization on its own tensor factor, yet the two induced complex structures $\mathcal{J}_1 = J_A \otimes I_B$ and $\mathcal{J}_2 = I_A \otimes J_B$ on the same full support ($\rho_S = I_{AB}/4$) satisfy $[\mathcal{J}_1] \neq [\mathcal{J}_2]$.*

Proof. $\mathcal{J}_1\mathcal{J}_2 = J_A \otimes J_B$ has eigenvalues ± 1, so $\mathcal{J}_2 \neq \pm\mathcal{J}_1$, while both are orthogonal complex structures. Each edge block is the ideal WGS two-source realization on its own factor, extended by the identity on the spectator factor, which changes no probability; each edge's ideal score is that of the WGS construction [4]. □

Theorem 3.2 (Frame locking, commuting case)**.** *Let $\mathcal{J}_1, \mathcal{J}_2$ be commuting complex structures on a real carrier and $R := -\mathcal{J}_1\mathcal{J}_2$, which is real symmetric with $R^2 = I$. For any orthogonal projection Π commuting with $\mathcal{J}_1, \mathcal{J}_2$: $[\mathcal{J}_1|_\Pi] = [\mathcal{J}_2|_\Pi]$ iff $R\Pi = \nu\Pi$ for some $\nu \in \{\pm 1\}$, and explicitly $R\Pi = \nu\Pi \iff \mathcal{J}_2\Pi = \nu\mathcal{J}_1\Pi$.*

Proof. If $\mathcal{J}_2\Pi = \nu\mathcal{J}_1\Pi$ then $R\Pi = -\mathcal{J}_1\mathcal{J}_2\Pi = -\nu\mathcal{J}_1^2\Pi = \nu\Pi$; conversely multiply $R\Pi = \nu\Pi$ on the left by $\mathcal{J}_1$. □

Remark 3.3. In the tensor-factor setting of Theorem 3.1, where $\mathcal{J}_1\mathcal{J}_2 = J_A \otimes J_B$, one has $\frac{1}{4}(I - \mathcal{J}_1\mathcal{J}_2) = \frac{1}{2}\Pi_+$: the two-rebit reference coincides, as an operator on the source carrier S, with the normalized projector (equivalently, the maximally mixed state) on the $R = +1$ sector — the reference *is* the cross-port lock. The m-party WGS state $\rho_W^{(m)} = \frac{1}{2}(Q_+^{\otimes m} + Q_-^{\otimes m})$ with $Q_\pm = \frac{1}{2}(I \pm iJ)$ is real (the summands are complex conjugates), PSD and unit trace (the $Q_\pm$ are rank-one projectors), and has every pair marginal equal to Ω [4]: edge ideality and the cross-port lock are simultaneously satisfiable; there is no monogamy obstruction. For noncommuting $\mathcal{J}_1, \mathcal{J}_2$ the operator R need not be symmetric or involutive, and the criterion is simply $\mathcal{J}_2 = \pm\mathcal{J}_1$ on a common reducing subspace; a one-bit characterization there is open.

Remark 3.4 (Local invisibility forces a joint lock). J is skew, hence invisible to every local real symmetric observable (single-time real statistics determine only the symmetric part of an operator; see the companion manuscript [34]); $J \otimes J$ is symmetric and jointly measurable. The lock datum is intrinsically cross-port: real quantum theory is bilocally but not locally tomographic [24], and the lock is a bilocal datum. Local indistinguishability of exactly such state families underlies Wootters' information-hiding analysis [15].

Lemma 3.5 (Transpose-flag structure; restated from [6])**.** *In the exact self-testing normal form of Liu et al. [6], certified observables satisfy $\widehat{X} \mapsto X \otimes I$, $\widehat{Z} \mapsto Z \otimes I$, $\widehat{Y} \mapsto Y \otimes Z_f$ on a local flag register f: only Y sees the flag, and the flag is the sign of the extracted complex structure ($Y^{\mathsf{T}} = -Y$ corresponds to $\mathcal{J} \mapsto -\mathcal{J}$ under realification). With $K = XX - YY + ZZ$ measured using the literal same certified operators, $K_+ = 4|\Phi^+\rangle\langle\Phi^+| - I$ (spectrum $\{3, -1, -1, -1\}$) and $K_- = 2F_{\text{swap}} - I$ (spectrum $\{1, 1, 1, -3\}$, F_{swap} the two-qubit swap), the flag product commutes with K and $\Pr[f_1 \neq f_2] \leq \frac{3-\langle K\rangle}{2}$; Liu et al. prove this and chain it along a line of parties.*

Remark 3.6. Two observations are ours. Without literal operator reuse the lemma fails: two independently certified triples give ideal scores with uncorrelated flags. And K synchronizes *signs* between already-identified carriers; it cannot merge distinct tensor-factor complex structures, which requires the support condition of Theorem 3.2.

Conjecture 3.7 (Routed connected-graph rigidity)**.** *Define the* source-interaction graph*: one vertex per source, an edge when some party receives systems from both. Assume (A1) every edge supports an exact WGS block (three distinct parties; a source-interaction graph alone does not suffice — two sources feeding one party form a tree whose every correlation is trivially real-simulable); (A2) each edge self-test's effective complex structure on a source's support acts on a common logical carrier, with commuting induced structures across ports, and is the structure entering teleported K-tests supplied by source-internal Bell links between that source's receiving parties. Then for connected source-interaction graphs there is a single correlation, generated from complex product sources, whose exact real simulations force frame synchronization on every edge and reduce the assisting resource to $\rho_W^{(m)}$ under real local operations. Expected mechanism: port locks propagate flag equality along the line graph, which is connected whenever the graph is; acyclicity plays no role. No later section depends on this conjecture.*

4 The graded layer: partial references

Definition 4.1. $\eta_t := \frac{1}{4}(I - t\, J \otimes J)$ for $t \in [0, 1]$; $\eta_1 = \Omega$, $\eta_0 = I/4$, and $\operatorname{tr}[\eta_t(J \otimes J)] = -t$. By the real-concurrence formula $C_{\mathbb{R}}(\rho) = |\operatorname{tr}(\rho\, J \otimes J)|$ for two-rebit states, which vanishes iff ρ is real-separable [16], η_t is real-entangled iff $t > 0$.

Theorem 4.2 (Usefulness above threshold)**.** *Let $B_{\mathbb{R}} = 7.6605$ denote the numerical upper bound of [1] on the Renou functional over real models with independent (hence, by convexity, classically correlated) sources — a certified hierarchy bound, not an exact constant. Let P_t be the Renou two-source correlation with exactly one deformation: Alice's third input becomes the POVM $\frac{1}{2}(I \pm tY)$; both sources, Alice's other inputs, and Charlie's six observables are unchanged. Then the ideal complex value of the functional at P_t is $2\sqrt{2}\,(2+t)$, which exceeds $B_{\mathbb{R}}$ iff $t > t_* := \frac{B_{\mathbb{R}}}{2\sqrt{2}} - 2 \approx 0.7084$; and P_t is exactly realized by a real model assisted by one copy of η_t, whose two shares α (Alice) and γ (Charlie) realify the deformed Y's. Hence η_t is useful for $t > t_*$; whether some η_t-enabled correlation is nonfree for $t \le t_*$ is open.*

Proof sketch. Per Bob outcome, the Renou functional is a sum of one CHSH block in $(A_1, A_2; C_1, C_2)$ and two CHSH blocks pairing A_3 with A_1 and A_2 respectively (settings C_3, C_4 and C_5, C_6). P_t is defined by keeping the ideal Renou settings fixed and deforming only the A_3 POVM, so every correlator is linear in t: the first block keeps its ideal value $2\sqrt{2}$, and in each A_3 block the two A_3-correlators (jointly worth $\sqrt{2}$ at the ideal settings) scale by t while the two others (worth $\sqrt{2}$) are unchanged, giving $\sqrt{2}(1+t)$ per block and $2\sqrt{2}+2\cdot\sqrt{2}(1+t) = 2\sqrt{2}(2+t)$ in total (at $t = 1$, the complex maximum $6\sqrt{2}$). For the assisted realization: Alice realifies her deformed measurement as the real observable $J_{\mathrm{sys}} \otimes J_\alpha$ and Charlie realifies each Y in his observables through $J_{\mathrm{sys}} \otimes J_\gamma$; single-$Y$ correlators vanish on both sides (frame marginals maximally mixed; Bell states carry no single-Y moment), X- and Z-correlators are unchanged, and every YY correlator matches with the correct sign: $Y \otimes Y = -\,J \otimes J$ on the system pair, and the realified correlator carries the additional frame factor $\mathrm{tr}[\eta_t\, J_\alpha J_\gamma] = -t$, so $\langle Y \otimes Y \rangle_{\text{complex,deformed}} = t\,\langle Y \otimes Y \rangle_{\text{ideal}}$ is reproduced term by term across all Bob outcomes. □

Theorem 4.3 (No exact finite-copy conversion)**.** *For $0 < t < 1$ and every finite n, no stochastic real separable operation with charge-free ancillas maps $\eta_t^{\otimes n}$ to Ω exactly. Moreover every $n = 1$ separable branch has normalized output σ with $\frac{1}{2}\|\sigma - \Omega\|_1 \ge (1-t)/2$; the fixed-accuracy question for $n > 1$ is open (the rate statement below shows the gap closes as $n \to \infty$).*

Proof. Support half. $\mathrm{ran}\,\Omega = \mathrm{span}_{\mathbb{R}}\{|\Phi^-\rangle, |\Psi^+\rangle\}$; the vector $a|\Phi^-\rangle + b|\Psi^+\rangle$ has coefficient matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ of determinant $-(a^2+b^2)/2$, so the span contains no nonzero real product vector. Charge-free ancillas decompose into real pure product states and extend product vectors to product vectors; discarding a subsystem composes the branch with rank-one product effects on the discarded factor, re-expanding it into product-supported terms. Since $\eta_t^{\otimes n}$ is full rank, every nonzero separable branch $\sum_k (A_k \otimes B_k)(\cdot)(A_k \otimes B_k)^{\mathsf{T}}$ has each term supported on $\mathrm{ran}\,A_k \otimes \mathrm{ran}\,B_k$; exact output Ω forces these product subspaces into $\mathrm{ran}\,\Omega$ — contradiction. *Gap half ($n = 1$).* Absorb ancillas as above so each Kraus factor is an effective real 2×2 pair (A_k, B_k). From $A^{\mathsf{T}} J A = (\det A) J$, $\mathrm{tr}[\sigma\,(J \otimes J)] = -\frac{t}{p}\sum_k \det A_k \det B_k$ with $p = \frac{1}{4}\sum_k \|A_k\|_F^2 \|B_k\|_F^2$ (the cross term $\mathrm{tr}(J\,M)$ vanishes for symmetric M), and $\|A\|_F^2 \ge 2|\det A|$ gives $|\,\mathrm{tr}(\sigma\, J \otimes J)| \le t$ while $\mathrm{tr}(\Omega\, J \otimes J) = -1$; since $\|J \otimes J\| = 1$, $\frac{1}{2}\|\sigma - \Omega\|_1 \ge (1-t)/2$. □

Theorem 4.4 (Distillation from partial references)**.** *For every $t > 0$ there is an rLOCC protocol Λ_n producing $m_n = n$ output pairs with $\frac{1}{2}\|\Lambda_n(\eta_t^{\otimes n}) - \Omega^{\otimes n}\|_1 \to 0$; hence $D_\Omega(\eta_t) \ge 1$, and in fact $D_\Omega(\eta_t) = \infty$ by Lemma 4.5.*

Proof. Write $\eta_t = \frac{1+t}{2}\Omega + \frac{1-t}{2}\bar{\Omega}$ with $\bar{\Omega} = \frac{1}{4}(I + J \otimes J)$: copy i carries a hidden branch sign X_i, defined as the $J \otimes J$ eigenvalue of its branch ($X_i = -1$ for Ω), i.i.d. with $\Pr[X_i = -1] = \frac{1+t}{2}$. The

two holders measure the commuting real observables $J^{(1)} \otimes J^{(i)}$ on their local sides ($i = 2, \ldots, n$) and exchange outcomes; the product w_i of the two local parities equals $X_1 X_i$ deterministically given the branches, and the parity measurements commute with the branch decomposition, so conditioned on the branch vector the post-measurement *marginal* of pair i is its branch state. A majority vote over $\{w_i\}$ estimates X_1 with error $\epsilon_n \le \exp\lfloor -(n-1)t^2/2 \rfloor$; set $\hat{X}_i = \hat{X}_1 w_i$. One designated holder applies V to its side of every pair with $\hat{X}_i = +1$ (the $\bar{\Omega}$ branch), including pair 1 when $\hat{X}_1 = +1$. Conditioned on a correct vote, each pair's marginal is exactly Ω, but the joint conditional state carries the residual stabilizer group generated by the measured parities: $h_i = J_A^{(1)} J_A^{(i)}$ and their B-side partners, the latter products of the h_i with the pair stabilizers, which are twirl-invariant — so a group element's character below depends only on its h-content. To kill the residuals, both holders apply $V \otimes V$ on pair i keyed to an *independent* shared random bit b_i per pair: each pair's state is fixed ($V \otimes V$ commutes with $\Omega, \bar{\Omega}$) while $h_i \mapsto (-1)^{b_1 + b_i} h_i$, so every group element with nonempty h-content averages to zero once the b_i are discarded; the figure of merit is the b-averaged output, which on the correct-vote branch is exactly $\Omega^{\otimes n}$. The unconditioned output is then within joint trace distance ϵ_n of $\Omega^{\otimes n}$, with $m_n = n$ output pairs: rate ≥ 1 is achieved, and Lemma 4.5 below upgrades it to an infinite rate.[1] □

Lemma 4.5 (Exact replication)**.** *For every $m \ge 1$ there is a one-round rLOCC protocol with charge-free ancillas and $m-1$ bits of communication that maps Ω_{AB} exactly to $\Omega^{\otimes m}$ between the same two holders; it is also an rSEP operation, and $m-1$ bits are necessary. Consequently, for every resource σ, every pair (a, b) and $\mathcal{C} \in \{\mathrm{rLOCC}, \mathrm{rSEP}\}$, $D_\Omega^{a:b,\mathcal{C}}(\sigma) \in \{0, \infty\}$; in particular $D_\Omega(\eta_t) = \infty$ for every $t > 0$.*

Proof. Let $W : \mathbb{R}^2 \to \mathbb{R}^2 \otimes \mathbb{R}^2$ be the real isometry $W|0\rangle = |\Phi^-\rangle$, $W|1\rangle = |\Psi^+\rangle$, implemented by a local charge-free ancilla and a real orthogonal map. Complexified, $W|y_\pm\rangle = |y_\pm y_\pm\rangle$, so $W Q_s W^{\mathsf{T}} = Q_s \otimes Q_s$ with $Q_\pm = \frac{1}{2}(I \pm iJ)$, while $V Q_s V^{\mathsf{T}} = Q_{-s}$. Write $\Omega = \frac{1}{2}\sum_s Q_s \otimes Q_s$. Each holder applies W to its newest rebit $m-1$ times, giving $\frac{1}{2}\sum_s Q_s^{\otimes m} \otimes Q_s^{\otimes m}$; one holder draws uniform bits $r_2, \ldots, r_m$ and sends them; both apply V to copy j when $r_j = 1$; the bits are discarded. The sign of copy j becomes $s(-1)^{r_j}$, and since the r_j are uniform and independent of s, the signs are uniform on $\{\pm 1\}^m$: the output is $2^{-m}\sum_{s_1,\ldots,s_m} \bigotimes_j (Q_{s_j} \otimes Q_{s_j}) = \Omega^{\otimes m}$. As a map it is a uniform mixture of product Kraus terms, hence rSEP. The $m-1$ bits are necessary: the quantum mutual information of Ω is one bit and that of $\Omega^{\otimes m}$ is m bits; local operations do not increase it, and each communicated bit increases it by at most one. For the dichotomy: if some rate $r > 0$ is achievable from σ, keep one output pair, with error at most the joint error $\delta_n \to 0$, and replicate it; channels do not increase trace distance and replication maps Ω to $\Omega^{\otimes k}$ exactly, so every rate is achievable. □

The reference is graded only at finite copy number: one-shot and finite-copy exact inequivalence (Theorem 4.3) coexist with asymptotic equivalence at unbounded rate in both directions, since replication followed by a local V on each pair, independently with probability $\frac{1-t}{2}$, maps Ω exactly to $\eta_t^{\otimes k}$ for every k.

[1] Version 1.0.4 corrects this sentence. Versions 1.0.0–1.0.3 read "rate ≥ 1 is achieved (we expect equality but prove only the lower bound; the charge code is non-quantitative)". The expected equality is false: the unit replicates exactly (Lemma 4.5), so $D_\Omega(\eta_t) = \infty$ for every $t > 0$. No theorem of earlier versions is affected. This theorem is retitled (formerly "Distillation at unit rate"), and the rate and unit-counting phrasings the lemma touches are updated in the abstract, the Introduction, §2, Theorems 5.5, 5.7 and 6.6, §5.1, Corollary 7.3, the remark following it, Open Problem 7.5, and the Discussion. Found in internal review, confirmed by an independent check working from the paper alone, and verified by `verify-copy/t_copy.py`.

5 The transpose-charge code

Definition 5.1 (Charges)**.** Fix a partition of a resource into m holders with an orthogonal basis on each (the decomposition below is invariant under local orthogonal, not general invertible, basis changes). Let T_i be transpose on holder i; the commuting involutions T_i split operators into mutually Hilbert–Schmidt-orthogonal sectors $\rho = \sum_{c\in\mathbb{F}_2^m} \rho_c$ with $T_i(\rho_c) = (-1)^{c_i}\rho_c$. Global symmetry of real states forces even-weight c. The *charge support* is $\mathrm{supp}_{\mathbb{F}}(\rho) = \{c : \rho_c \neq 0\}$ and the *charge code* is $H_{\mathcal{P}}(\rho) = \mathrm{span}_{\mathbb{F}_2} \mathrm{supp}_{\mathbb{F}}(\rho)$ (we write $H(\rho)$ when the partition is clear; every statement below is relative to $\mathcal{P}$).

Definition 5.2 (Frame characters; twirl)**.** A *frame resource* carries one rebit per holder. On m frame rebits let $G_c := \bigotimes_i J_i^{c_i}$; for even c these are real symmetric involutions, mutually commuting, with the sign cocycle $G_cG_{c'} = (-1)^{|c\wedge c'|}G_{c+c'}$, and we write $\hat{\rho}(c) := \mathrm{tr}[\rho\, G_c]$ for the frame components of a resource ($\{G_c\}$ spans the commutative frame algebra, not the full operator space; for the frame-code states below the components determine the state). The *frame twirl* applies independent uniformly random reflections V per rebit; on a frame resource it kills every $c \neq 0$ component and leaves a diagonal, classically correlated — hence free — state. (The twirl frees *any* resource, but only for frame-code states does twirling coincide with zeroing all frame characters $\hat{\rho}(c)$ — for general states the twirl also removes non-frame components, so character bookkeeping certifies usefulness only in the frame-code setting; this is why Theorem 6.6 and Corollary 7.3 are stated for frame-code resources.) In a fixed assisted realization of a correlation P, whose outcome probabilities are linear functionals $\mathrm{tr}[E\,\rho]$ of the resource operator, a codeword c *contributes* if some such functional changes when the component $\hat{\rho}(c)$ is set to zero (linearity licenses evaluating the functionals off the state space).

Theorem 5.3 (Span monotonicity)**.** *For every rSEP protocol Λ with charge-free ancillas and every n,*

$$H\big(\Lambda(\sigma^{\otimes n})\big) \subseteq H(\sigma).$$

In particular $e_a + e_b \notin H(\sigma)$ implies $D_\Omega^{a:b,\mathrm{rSEP}}(\sigma) = 0$, and approximation cannot evade the obstruction.

Proof. Each holder-local real CP map commutes with the partial transpose on its own holder (a global real map commutes only with the global transpose), so every product Kraus term of an rSEP operation preserves each charge sector; charge-free ancillas occupy sector 0; charges add over tensor factors; and partial trace kills sectors charged on the discarded holder. Tensor powers therefore occupy sums of charges, whence the span is the invariant. For the approximation clause: a vanishing (a, b) weight-2 sector forces $\mathrm{tr}[\rho(-J_aJ_b)] = 0$ while Ω gives 1, so $\frac{1}{2}\|\rho - \Omega\|_1 \geq \frac{1}{2}$. □

Remark 5.4. Shared entangled ancillas are excluded by hypothesis; §6 shows they are exactly the doorway of activation.

Theorem 5.5 (Weight-2 sectors distill)**.** *(i) Every real-entangled two-rebit state has positive, hence (Lemma 4.5) infinite, rLOCC rate: independent local $SO(2)$ twirls (which fix J) project ρ onto $\frac{1}{4}(I + \zeta\, J\otimes J)$ with $\zeta = \mathrm{tr}(\rho\, J\otimes J)$; a single-side V when $\zeta > 0$ yields $\eta_{|\zeta|}$, and $|\zeta| = C_{\mathbb{R}}(\rho) > 0$ by the convex-roof identity of [16]; apply Theorem 4.4. (ii) If $\rho_{e_a+e_b} \neq 0$ (charge support, not merely span), rank-one real product effects on the other holders — which annihilate sectors charged there — followed by local two-dimensional projections at a, b (conjugations $\Pi J\Pi$ sweep all rank-two skew operators) leave, with nonzero probability, a two-rebit state with nonzero*

$J \otimes J$ moment — some choice succeeds, by the contrapositive: the operators $\Pi J \Pi$ over rank-two projections Π span all skew operators at (a, b), and rank-one real product effects elsewhere span the symmetric operators there, so if every choice gave zero moment, the sector $\rho_{e_a+e_b}$ would pair to zero with a spanning family and vanish; hence $D_\Omega^{a:b}(\rho) > 0$, and so $D_\Omega^{a:b}(\rho) = \infty$. (iii) Consequently, for $m = 2$ with rebit holders no useful resource has zero rate: charge zero forces $\zeta = 0$, hence real separability by the converse direction of [16] — a two-rebit theorem, the single load-bearing external input of this clause — hence freeness (higher-dimensional two-holder carriers remain open). For $m = 3$ the same holds among resources with $H \neq \{0\}$; whether a charge-zero real-entangled three-holder resource can be useful is open, as is the regime $e_a + e_b \in H(\rho) \setminus \mathrm{supp}_{\mathbb{F}}(\rho)$.

Definition 5.6 (Frame-code states)**.** For an even-weight binary linear code $C \leq \mathbb{F}_2^m$ with chosen generators $g_1, \ldots, g_\ell$ and signs $s_j \in \{\pm 1\}$, let

$$\sigma_C := 2^{-m} \prod_{j=1}^{\ell} (I + s_j G_{g_j}),$$

a valid state ($\prod_j (I + s_j G_{g_j})$ is 2^ℓ times a nonzero projector: independence of the generators keeps $-I$ out of the generated sign group), with $H(\sigma_C) = \mathrm{supp}_{\mathbb{F}}(\sigma_C) = C$. The signs on non-generator codewords are induced through the cocycle and depend on the generating set. For the maximal even-weight code this is, up to conventions, the family of states of Wootters [15] (his Eq. (22): the real part retains exactly the strings with evenly many σ_y's); as a stabilizer construction the family lies in the unlockable class of Wang and Ying [32]. The arbitrary-code frame reading and the graded classification below are, to our knowledge, new.

Theorem 5.7 (Per-pair dichotomy)**.** *For every pair (a, b): if $e_a + e_b \in C$, tracing σ_C to (a, b) leaves $\frac{1}{4}(I \pm J_a J_b)$ — a maximal reference in one shot, hence infinite rate (for the maximal code this recovers Wootters' simultaneous pairwise maximality [15]). If $e_a + e_b \notin C$ then $D_\Omega^{a:b,\mathrm{rSEP}}(\sigma_C) = 0$ for any number of copies, since $H(\sigma_C^{\otimes n}) = C$ contains no (a, b) weight-2 vector. Corollary: C has no weight-2 codeword iff every pair has zero rate; $\tau_4 := \frac{1}{16}(I + J^{\otimes 4})$ (the length-4 repetition code, $d = 4$) is the minimal such example in holder number, since no even-weight code of length ≤ 3 avoids weight-2 words except $\{0\}$; the WGS reference is the maximal code, all weight-2 sectors present.*

Remark 5.8 (Ordinary entanglement points the other way). Partial transpose acts on frame-code states as the frame flip $J_i \mapsto -J_i$, mapping the family to itself, so PPT detects nothing here. The WGS reference is real-entangled with that entanglement bound because its complexification is separable [4]; τ_4 is likewise complex-separable (a mixture of even-parity y-basis product states). The operative obstruction for complexness is charge support, not positivity under the flip.

5.1 Relation to imaginarity monotones

The imaginarity resource theory [20, 21, 22] takes real states as free and quantifies $\rho \neq \rho^{\mathsf{T}}$; its distributed form [23, 19] studies bipartite scenarios under real local operations, with scalar monotones such as $\|\rho - \rho^{T_B}\|_1$ — the norm of the T_B-odd part, which for the globally symmetric states of this paper is exactly the $(1, 1)$ sector — and geometric measures. Our objects differ in three ways. First, every state in this paper is real symmetric, hence *free* for imaginarity theory; the resource here is cross-holder structure inside the skew sectors, invisible to any single-system

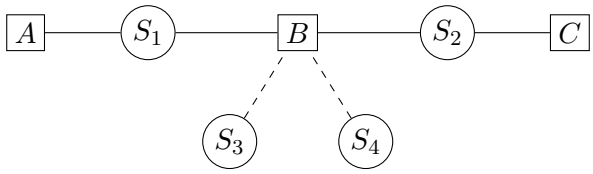


Figure 1: The four-source network of Theorem 7.2. S_1, S_2 run the Renou block through parties A, B, C; the τ_4 shares held by S_3, S_4 are routed to B, coarsening the effective holder partition.

imaginarity measure. Second, our monotone is subgroup-valued, not scalar: $H(\sigma)$ records *which* sectors are occupied, not how much weight they carry, and the weight obstruction (a code with no weight-2 word never acquires one under tensor powers and separable processing) has no scalar counterpart. Third, neither monotone refines the other: the code is magnitude-blind (η_t carries the same code for every $t > 0$, however small), while any scalar sector-weight measure is support-blind (it cannot record *which* pair carries the weight, which is exactly what the per-pair dichotomy above turns on). Magnitude-blindness is no weakness at the asymptotic level: by Lemma 4.5 the rates themselves are blind to t ($D_\Omega(\eta_t) = \infty$ for every $t > 0$), so for two rebits the code is a complete asymptotic invariant ($D_\Omega = \infty$ iff $H \neq \{0\}$ iff the state is real-entangled), and quantitative distinctions live at finite copy number. We are not aware of a prior multipartite sector-support monotone in either literature.

6 Visibility, puncturing, activation

Theorem 6.1 (Charge-flow visibility: necessity)**.** *Attach resource shares to vertices of the* source–party incidence graph *(vertices: sources and parties; edges: each transmitted system), many-to-one allowed. Let $q(c) \in \mathbb{F}_2^V$ be the vertex-charge vector, $q_v(c)$ the mod-2 sum of the charges of v's shares. A charge c can contribute to network statistics only if $q(c)$ is an $\mathbb{F}_2$-boundary — $\partial y = q(c)$ for some edge chain $y \in \mathbb{F}_2^E$ — equivalently, every connected component contains even total charge.*

Proof. Symmetric and skew operators are Hilbert–Schmidt orthogonal, so a nonzero contraction of the sector-c term against source states and effects requires each transmitted system to carry a definite transpose parity y_e, with each source state and each effect forcing even total local parity at its vertex; the resulting parity assignment is an edge chain y with $\partial y = q(c)$, and $\operatorname{im}\partial$ over $\mathbb{F}_2$ is exactly the per-component even-weight vectors. □

Remark 6.2 (Converse, with hypotheses). Conversely, when $\partial x = q(c)$ is solvable *and* the routed J-string has even weight on each source's emitted systems (which is the boundary condition at source vertices), the states $2^{-k}(I + \delta\, G_{\text{string}})$ (k the number of systems carrying the routed J-string), PSD for $|\delta| \leq 1$, realize a network in which c contributes: route the string along the flow and contract. We use only the τ_4 instance below (Proposition 6.3), which is computed exactly; we do not rely on the general converse.

Proposition 6.3 (Visibility witnessed; frame statistics are free)**.** *τ_4 plus two declared Ω ancillas $\Omega_{a_1a_2} \otimes \Omega_{a_3a_4}$, with each holder i measuring $M_i = J_i \otimes J_{a_i}$ (real symmetric, $M_i^2 = I$), yields exactly $P(x) = \frac{1}{16}(1 + x_1x_2x_3x_4)$: weight-4 charge is not statistically inert. However, statistics generated by frame characters alone are always freely simulable: the even-J algebra is commutative,*

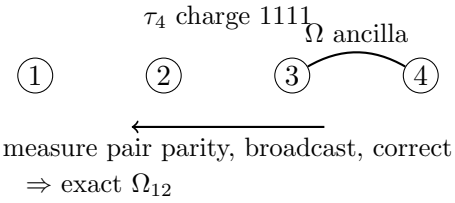


Figure 2: Puncturing: pairing charged holders $3, 4$ with a declared Ω ancilla and measuring the pair parities lowers the charge 1111 to 1100, leaving an exact reference on holders $1, 2$ after the heralded sign correction.

so the outcomes of all *settings admit one joint classical distribution, which shared randomness* λ *can sample, each party outputting its sampled values. (Note the distribution itself can be exotic — on* $\rho_W^{(3)}$ *the three pairwise parities multiply to the operator* $(J_1J_2)(J_1J_3)(J_2J_3) = -I$*, impossible for per-source sign assignments — but a global* λ*-model is not a per-source sign model, and it suffices.)* Frame characters alone cannot constitute a nonfree correlation; genuine advantage requires coupling the character to noncommuting payload.

Lemma 6.4 (Ω-swap)**.** *On* $\Omega_{12} \otimes \Omega_{34}$*, measuring* J_2J_3 *(outcomes* $x = \pm 1$*, probability* $\frac{1}{2}$ *each) leaves holders* $1, 4$ *in exactly* $\frac{1}{4}(I + x\, J_1J_4)$*.*

Proof. Expand $\Omega_{12} \otimes \Omega_{34} = \frac{1}{16}(I - J_1J_2)(I - J_3J_4)$ and apply the projector $\frac{1}{2}(I + xJ_2J_3)$; the terms surviving tr_{23} are I and $x\,(J_1J_2)(J_2J_3)(J_3J_4) \propto J_1J_4$ with coefficient computed via $J^2 = -I$ twice: the conditional state is $\frac{1}{4}(I + xJ_1J_4)$ at outcome probability $\frac{1}{2}$. □

Theorem 6.5 (Puncturing)**.** *(*τ_4 instance, exact.*)* τ_4 *plus one declared* Ω *ancilla on holders* $3, 4$*, each measuring* $\frac{1}{2}(I + x_iJ_iJ_{a_i})$*: the four outcome branches have probability* $\frac{1}{4}$ *each and leave holders* $1, 2$ *in exactly* $\frac{1}{4}(I - x_3x_4\, J_1J_2)$ *— the unit up to a heralded, locally correctable sign. (*General.*) Let* $c \in C$ *with* $a, b \in \mathrm{supp}(c)$*, and pair the (even-sized) set* $\mathrm{supp}(c) \setminus \{a, b\}$ *arbitrarily, placing one declared* Ω *ancilla on each pair; measure the pair parities* $J_iJ_{a_i}$*,* $J_jJ_{a_j}$*. Then some outcome of nonzero probability leaves a resource whose* $e_a + e_b$ *sector is nonzero.*

Proof. The τ_4 instance is a direct computation (the cross terms x_iJ_i alone have odd local charge and vanish; the x_3x_4 term contracts the ancilla's $-J_{a_3}J_{a_4}$ moment). For the general clause (Figure 2; ancillas on non-adjacent holders are obtained along connected paths by Lemma 6.4): after the pair measurements, trace out the holders outside $\{a, b\}$ — this kills every sector charged there. The measured observables commute with each other and with every $G_d \otimes$ (ancilla strings) term, so the unnormalized conditional (a, b)-sector at outcome x is a well-defined function $f(x)$ of the outcome string. The codewords $d \in C$ contributing to f are exactly those with $d_a = d_b = 1$, $\mathrm{supp}(d) \subseteq \{a, b\} \cup \bigcup(\text{pairs})$, and d restricted to each pair equal to 00 or 11; such d is determined by its set of 11-pairs, so distinct contributors carry distinct outcome monomials $\prod_{11\text{-pairs}} x_ix_j$ — linearly independent Walsh characters — and c itself contributes with all pairs 11: its operator collapses to $\pm J_aJ_b$ and its coefficient is $\hat{\sigma}(c)$ times the product of the ancilla moments $\mathrm{tr}[\Omega\,(J \otimes J)] = -1$, hence $\pm\hat{\sigma}(c) \neq 0$. Hence f is a nonzero combination of distinct characters, so $f(x_0) \neq 0$ for some outcome x_0; a zero-probability outcome has identically zero unnormalized conditional state, so $f(x_0) \neq 0$ forces $p(x_0) > 0$, and that outcome carries a nonzero conditional $e_a + e_b$ charge. □

Theorem 6.6 (Activation by declared auxiliaries)**.** *If σ_C is useful, some codeword is used, and for every used codeword c with $|c| \geq 4$ and any $a, b \in \mathrm{supp}(c)$: σ_C plus $\frac{|c|-2}{2}$ declared Ω auxiliaries on a pairing of $\mathrm{supp}(c) \setminus \{a, b\}$ yields, with nonzero probability in a preparation phase, a resource with nonzero $e_a + e_b$ charge — hence positive, and so infinite, (a, b) rate by Theorem 5.5(ii) and Lemma 4.5. When $e_a + e_b \notin C$, neither part alone suffices: the auxiliaries have zero (a, b) rate (their charge code contains no (a, b) weight-2 word, Theorem 5.3) and so does σ_C (Theorem 5.7); when $e_a + e_b \in C$ the pair is one-shot available and activation is moot.*

Proof. Some codeword is used: the realized correlation is linear in the resource operator, so if zeroing each $\hat{\rho}(c)$ individually changed nothing, replacing σ_C by its frame twirl $2^{-m}I$ (zeroing all of them), a free product state, would leave the correlation intact and exhibit a free realization of a nonfree correlation. (If $|c| = 2$ the pair is one-shot available by Theorem 5.7 and nothing needs activating.) The descent for $|c| \geq 4$ is Theorem 6.5; the two alone-cannot clauses are immediate from the cited theorems. □

Remark 6.7. At the code level the mechanism is transparent span arithmetic: c plus the pair vectors equals $e_a + e_b$, so the declared auxiliaries place the target word in the joint code by hand — Theorem 5.3 is respected, not violated. This is the frame-sector transcription of the activation of bound entanglement [25, 27, 29, 30], where auxiliary entangled pairs unlock a bound resource.

7 Bound complexness: partition-relative, activatable, never irreducible (for frame codes)

Definition 7.1 (Boundness notions)**.** Fix a resource σ with holder partition $\mathcal{P}$. σ is *bound relative to* $\mathcal{P}$ if it is useful (§2; usefulness is partition-independent) while $D_\Omega^{a:b,\mathrm{rSEP}}(\sigma) = 0$ for every pair of $\mathcal{P}$-holders, with charge-free ancillas — exactly as Smolin-state boundness is a property of the fine partition while its usefulness lives at a coarse one [26]. Boundness can be lifted by two mechanisms, which we keep separate: *partition coarsening* (a declared network routing that hands two shares to one location, under which the induced code on the coarser partition may acquire weight-2 words) and *auxiliary activation* (Theorem 6.6).

Theorem 7.2 (Bound complexness exists, and is unlocked by coarsening)**.** *Let S_1, S_2 run the two-source Renou network (parties A–S_1–B–S_2–C, Figure 1) targeting the ideal Renou correlation P_R, and let S_3, S_4 send their systems only to B. (i) P_R is nonfree in the four-source network: conditioned on λ, the S_3, S_4 states are products and absorb into B's POVM ($M'_b = \mathrm{tr}_{34}[(I \otimes \rho_3^\lambda \otimes \rho_4^\lambda) M_b]$), so a free four-source realization would average to a free two-source realization of P_R, contradicting [1]; the free class has no preparation phase (Lemma 2.1), so this argument is unaffected by the assisted model's preparation stage. (ii) Give the four sources τ_4, one share each: relative to the four-holder partition, τ_4's only nontrivial character has weight 4 and no separable conditioning with charge-free ancillas ever exposes weight-2 charge (Theorem 5.3). (iii) The declared embedding routes the S_3, S_4 shares to B, which therefore holds both* ab initio *(§2): under the coarsened partition $\{1\}, \{2\}, \{3, 4\}$ the induced code acquires the weight-2 word $e_1 + e_2$, and operationally, in the preparation phase B measures $L_{34} = J_3 \otimes J_4$ on the two shares it holds (legal: resource descendants) and broadcasts h; holders $1, 2$ then share $\frac{1}{4}(I + h\, J_1 J_2)$, and for $h = +1$ the holder of share 1 applies V; every branch yields exactly Ω_{12} before settings are drawn. The round then runs the standard exact WGS/McKague realification of the Renou strategy [4, 5]. Hence τ_4 is bound relative to the four-holder partition and useful*

through coarsening: bound complexness is partition-relative, exactly as for the Smolin state [26], of which this is the frame-sector transcription.

Corollary 7.3 (Used frame-code charges are always accessible)**.** *In any exact frame-code-assisted realization of a nonfree correlation, some codeword is used (Theorem 6.6's twirl argument), and every used codeword of weight ≥ 4 admits the declared-auxiliary descent of Theorem 6.5 on a fresh copy in the preparation phase — the descent is a property of the resource; no simultaneity with the running realization is claimed. We state the scope plainly: this is a corollary of Theorems 6.5 and 6.6, and the descent is available* because *the declared-auxiliary mechanism imports pairwise references; what is ruled out is a frame-code charge resisting both declared mechanisms, not one resisting every conceivable accounting. We describe the outputs structurally (counting units is not a resource accounting here, since Ω replicates exactly, Lemma 4.5): the parity measurements of Theorem 6.5 are nondemolition on the auxiliary pair's character, so the auxiliary survives with its reference intact — for $|c| = 4$ the descent consumes one Ω and yields two (Ω on the target pair* and *on the auxiliary pair, in product). In general the descent of a used weight-$2k$ codeword consumes $k-1$ declared auxiliaries and yields the target-pair reference exactly, on every measurement branch; and when every codeword of the code meets the measured holders in complete auxiliary pairs (*alignment*) and the covered codewords' unmeasured supports are pairwise-disjoint pair blocks (*disjointness*) — both automatic for a single-codeword state, and holding throughout the τ_4 family — the output is the full decoupled product: k pairwise references, each auxiliary returned in kind (the descent catalytic over its auxiliaries), and every overlapping covered codeword yielding an additional reference on its own pair, with the closed correction rule "flip one rebit of a codeword's output pair iff $(-1)^r hs = +1$" (h the outcome product over its measured holders, r the number of auxiliary pairs inside its measured support, s its sign).*[2][3] *The content is* accessibility *of the resource's charge, unconditional at the target pair; by Lemma 4.5, once an exact reference sits on a pair its rate is infinite, so the asymptotic rates are ∞ on every reachable pair and 0 otherwise.*[4] *Noncommutativity can make the frame resource necessary (Proposition 6.3); within the two declared mechanisms it cannot make a weight-≥ 4*

[2]Version 1.0.1 corrects this sentence. Version 1.0.0 stated the accounting as "unit-neutral at best: for $|c| = 4$ the descent consumes one Ω and yields one (τ_4 acts as a relay), and for higher weights it consumes more than it yields," which understates Theorem 6.5's own protocol: its measurements commute with the auxiliary pair's frame character and therefore preserve the auxiliary reference. Found in internal adversarial review (2026-08-12) and verified by explicit computation over all measurement branches at $|c| = 4$ and $|c| = 6$.

[3]Version 1.0.2 scopes the general clause and adds the alignment/disjointness characterization and the correction rule. Version 1.0.1 asserted the decoupled product with in-kind auxiliary return for arbitrary frame codes; continued adversarial verification (two independent engines, exhaustive branch checks up to 12 rebits, independently reproduced) found explicit counterexamples: a code whose covered words' output pairs intersect produces a correlated, rank-obstructed output that no correction — local or global — can factor, and a non-aligned covered word leaves the auxiliaries entangled with the resource. Alignment and disjointness are sufficient unconditionally, and necessary at the level of the code's *surviving* (positive-probability) sector: an exception class exists in which the formal conditions fail while the sign character kills precisely the offending codewords on every surviving branch, so the realized state is the product regardless; evaluated on the effective code, the equivalence holds with no violations (shipped sweep `verify-descent/t6_bigsweep.py`: 120 randomized setups, 1,784 surviving branches, 0 sufficiency failures, 0 effective-code violations, 1 exception-class setup, with the diagnostic in `t7_exception.py`); branch probabilities are uniform iff no codeword lies inside the measured set. The target-pair reference itself is exact unconditionally on every surviving branch, including in every counterexample. No other statement is affected.

[4]Version 1.0.4 replaces "with unit net gain per descent in the aligned regime; asymptotic conversion rates remain open" and the phrase "the accounting is net-positive". The structural statements are unchanged. Unit counts carry no resource meaning because the unit replicates (Lemma 4.5), which also settles the rates; the unit-neutral versus net-positive question of the v1.0.1 note is moot for the same reason.

character inaccessible.

Remark 7.4 (What transfers, and what is new). The skeleton — unit, partial resources, distillation, bound states, unlocking, activation — transfers from multipartite entanglement theory to the frame sector, largely intact; that transfer, and the code-theoretic reason it works (the charge code plays the role that stabilizer separability structure plays in [32, 26, 31, 27]), is the finding. The frame sector adds what the ancestors lack: the graded partial references with their finite-copy/asymptotic separation, the subgroup-valued monotone with its weight obstruction, the per-pair code dichotomy, and the fact that all of it lives inside *complex-separable* states, so it is a resource theory of complexness, not of entanglement. One piece does not transfer: unlike an ebit, the frame unit replicates under the free operations (Lemma 4.5), so the asymptotic layer of entanglement theory (distillable resource, cost, rates) collapses to 0 or ∞ here, and the quantitative content sits at finite copy number and in communication cost.

Open Problem 7.5 (Residual)**.** *Beyond frame codes: whether general real resources admit the same descent; the communication cost of activation and of finite-copy conversions (exact replication of* Ω *uses* $m-1$ *bits, which is optimal); nonfreeness of* η_t*-enabled correlations for* $t \le t_*$*; the fixed-accuracy* $n > 1$ *conversion gap; charge-zero real-entangled three-holder resources; the regime* $e_a + e_b \in H \setminus \mathrm{supp}_{\mathbb{F}}$*; the one-bit lock for noncommuting frame lines; the general converse of Theorem 6.1; and Conjecture 3.7.*

8 Discussion

Operationally: the real/complex network gap is the price of an unshared binary settlement — the relative frame parity of Theorem 3.2 — and that settlement behaves as a genuine resource, graded at finite copy number though asymptotically ungraded (it replicates exactly), with a code-theoretic classification in isolation, reducible to its unit in situ only through declared mechanisms (coarsening; auxiliaries). Structurally: which frame charges can matter at all is a parity law on the network graph, and the classical/quantum boundary inside the theory is Proposition 6.3: frame characters alone are free; complexness advantage is their coupling to noncommuting structure. Most carefully: everything is relative to the declared separable-source-independence axiom; under operational-independence axioms the free class itself changes [8], and what experiments select is complex-equivalent compositional structure relative to the composition axioms they test — we take no position beyond that.

A companion manuscript [34] develops the single-system side: the skew sector is exactly the part of a real operator invisible to single-time statistics, which is the structural reason (Remark 3.4) the frame lock must be a joint observable. A Lean 4/Mathlib artifact accompanies the two manuscripts, verified with no axioms beyond Lean's standard three; for the present paper it formalizes transpose-parity preservation under real conjugation (the single-map germ of Theorem 5.3), the determinant lemma $A^{\mathsf{T}} J A = (\det A) J$ behind Theorem 4.3, and the absence of real product vectors in $\operatorname{ran}\Omega$; the combinatorial arguments of §§6–7 are not formalized, and we make no machine-verification claim for them.

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