Uncomputable critical exponents*

Seth Douglas Nidhal Mghirbi

Version 2.1.0, October 2026

Abstract

We construct translation-invariant nearest-neighbour quantum Hamiltonian families with rational affine coefficient matrices whose existing finite nonnegative critical gap exponents at coupling zero are exactly the nonnegative Δ_3^0 reals, already in two dimensions. Volume limits along factorial tori precede coupling, and gaps are above the entire ground space. Separate finite registers share an energetic ruler, a framed computation and a final-time broadcast to positive-density quantum readers. The inner gap is eventually exact at fixed positive coupling without a general computable settling bound. Three-dimensional broadcast and wetting films give independent routes; tamed rulers and nested limits retain their classifications. A fixed universal noisy rule, with a computable bottom boundary, program origin and infinite cloning, has exactly Δ_3^0 density and counting laws in [0,1] under prescribed time-then-window limits, robust on an open positive-rate kernel region. At computable kernels no effective inquiry converges with positive probability to a fixed such law outside Δ_2^0; the exceptional learnable kernels are null, meager but dense for external finite-cylinder inquiry. Unique classical states of computable specifications have computable local numbers, while growing readouts and state selection permit noncomputable asymptotic content. Restricted response certificates and an exact calibrated-ruler criterion delimit robustness claims. A computable randomized adaptive inquiry reaches a fixed real with positive probability only at Δ_2^0. The beyond-Δ_2^0 gap exponents have no computably bounded relative gap-settling scale, excluding a specified uniform scaling tail. All-size convergence, unique ground states, locally nondegenerate gap identification and robust intrinsic equilibrium exponents remain unestablished.

1 Critical exponents at a computable transition

Local interactions with rational coefficients can have an uncomputable asymptotic exponent at a rational critical coupling. The organizing idea is to make the coupling an energetic ruler. A classical layer selects cells of pitch $p_*(g) = 2r/g + O(1)$; a quantum counter inside each cell then makes the thermodynamic gap equal to a finite box's gap. Finite-size scaling becomes an exact spectral identity in a deliberately constructed model. The size of the local alphabet is fixed for each prescribed exponent and does not grow as g approaches the transition.

Critical gap exponents at this computable transition fill D_3 already in the bulk on $\mathbb{Z}^2$. Separate finite onsite registers let the ruler, framed computation and final-time broadcast share the same two spatial axes. The three-dimensional broadcast and surface-anchored film remain independent routes. A computably tamed planar ruler attains D_2 and its ordered endpoint pairs; the untamed register construction attains the full two-limit ceiling. This is a distinction between selection mechanisms,

*The authors developed this work with AI models from more than one developer and take responsibility for every claim. How the work was done, and what the machine checks cover, is described at the end of the paper.

not between bulk and boundary modes. No dimension-only bound follows from the number of logical registers.

We use the gap above the *entire* ground space, namely the difference between the two lowest distinct eigenvalues. A scalar finite matrix has no such gap and is excluded whenever a gap is required. A family has translation-invariant onsite and oriented nearest-neighbour terms $h_a(g) = h_a^0 + g h_a^1$, with rational Hermitian matrices. Only the onsite term needs to depend on g in our constructions. At rational g every finite-volume matrix is algebraic. On tori, take the commensurate sizes $L = n!$; on the wetting open cubes we also take $L = n!$; the fixed-thickness film uses all $L \in \mathbb{N}$. Define, in this order,

$$\begin{aligned} \Delta_\infty(g) &= \lim_{L\to\infty} \Delta(L, g), \qquad x(g) = \frac{\log(1/\Delta_\infty(g))}{\log(1/g)}, \\ (z\nu)^- &= \liminf_{g\downarrow 0} x(g), \qquad (z\nu)^+ = \limsup_{g\downarrow 0} x(g). \end{aligned} \tag{1}$$

The first limit must exist and be positive at each sufficiently small positive coupling. An existing finite exponent is $z\nu = (z\nu)^- = (z\nu)^+$. General arithmetic ceilings concern rational couplings. In the explicit rulers the gap is constant between rational transition points, so real and rational approaches have the same endpoints. The notation $z\nu$ denotes this gap exponent; it does not assert a dispersion relation or identify a ground-state correlation length.

For an individual real, write S_m and P_m for the left-Σ^0_m and left-Π^0_m classes of its strict lower rational cut. Write $\mathsf{D}_m = \Delta^0_m$ for reals computable in $0^{(m-1)}$. Rational endpoints are treated separately when passing between strict upper and lower cuts. Uniform families throughout the paper are specified by oracle Cauchy names or one-sided approximation names, as explained in Section 10; uniform decidability of exact comparison with a rational is not assumed.

Theorem 1.1 (Bulk ruler)**.** *For every $0 \le a \le b < \infty$ with $a \in \mathsf{S}_2$ and $b \in \mathsf{P}_2$, a fixed rational nearest-neighbour affine family on $\mathbb{Z}^2$ has $((z\nu)^-, (z\nu)^+) = (a, b)$ along commensurate tori. The transition coupling is $g_c = 0$. The gap is at least $1/16$ for $g \le 0$, and tends to zero from the right. For small positive g, the ground configurations fill the plane with cells and*

$$\Delta_\infty(g) = D(p_*(g) - 1), \qquad p_*(g) = \frac{2r}{g} + O(1), \qquad r = \frac{1}{16},$$

off the explicitly specified tie points. Here $D(h)$ is the full quantum counter gap selected by an $h \times h$ box. At ties the gap is the minimum of the two neighbouring values. The ground energy density is 0 for $g \le 0$ and $-g^2/(4r) + O(g^4)$ for $g > 0$. The inner gap limit is computably tamed. Consequently its finite nonnegative existing exponents are exactly D_2.

The proof is in Sections 2 and 11, using the finite-size construction in Sections 3–9. In particular $z\nu$ can be the halting probability Ω_U of any chosen prefix-universal machine. No uncomputable number is an interaction coefficient. There is one counter in every cell, with a positive density of possible excitation locations at fixed g. These gap-closing modes are not supported on the boundary of a finite box. We make no identification with a boundary-independent operator-algebraic gap.

Theorem 1.2 (Three-dimensional bulk broadcast)**.** *For every finite $\alpha \ge 0$ in D_3, a fixed translation-invariant nearest-neighbour rational affine family on $\mathbb{Z}^3$ has critical gap exponent α on cubic tori $L = n!$, with the volume limit taken first and the entire ground space removed. The limiting gap is zero for $g \le 0$, positive for $0 < g < 11/480$, and closes as $g \downarrow 0$. The ground energy density is zero for $g \le 0$ and $-g^2/(4r) + O(g^4)$ from the right, with $r = 1/16$. At each fixed positive coupling in that interval, the gap is carried by at least L^3/p^2 independent bulk reader excitations for all sufficiently large factorial L, where $p = 2r/g + O(1)$ is a gap-attaining pitch. The corresponding local spectral*

weight per volume is at least $1/p^2$. The finite-volume gap is eventually exact at fixed g, without a general computable settling bound. Together with the general two-limit ceiling, the existing finite nonnegative exponents in this convention fill exactly D_3. A selected infinite-volume phase has the same gap above its full zero-energy subspace.

Section 13 gives the complete construction and proof, including the minimum-of-two-gaps rule at pitch ties. The internally anchored computation does not confine the soft quantum modes to a seam: their possible locations have positive density at fixed g. No unique ground state, all-integer-size limit or equality with a locally nondegenerate or KMS-extremized gap is asserted.

Theorem 1.3 (Wetting ruler and film hierarchy)**.** *A surface-anchored rational affine family along factorial open cubes in $\mathbb{Z}^3$ selects a wetting-film height $h_*(g) = (r/g)^{1/2} + O(1)$. Its finite nonnegative existing gap exponents fill D_3, and its ordered endpoint pairs fill $\mathsf{S}_3 \times \mathsf{P}_3$. The gapless side includes $g = 0$. At fixed $g > 0$ the finite-box gap is eventually exactly constant, without a general computable settling bound. Independently, for integer interactions the fixed-thickness film exponents, with the in-plane limit taken first, fill exactly the same classes. More generally k dependent nested box limits, including the outer scaling limit, fill D_{k+1} and its ordered $\mathsf{S}_{k+1} \times \mathsf{P}_{k+1}$ pairs in dimension $k + 1$.*

The wetting statement and its precise selection rules are in Section 14; the separate fixed-thickness theorem and the completed finite local-rule schema are in Sections 12 and 15. Necessity requires every intermediate limit to exist for every remaining finite parameter tuple, and only asserts a class when the relevant final endpoints are finite. Their surface selection remains distinct from the bulk broadcast of Theorem 1.2.

Theorem 1.4 (Planar register broadcast)**.** *For every finite $\alpha \geq 0$ in D_3, a fixed translation-invariant nearest-neighbour rational affine family on $\mathbb{Z}^2$ has critical gap exponent α on square tori $L = n!$, with the volume limit taken first and the entire ground space removed. The limiting gap is zero for $g \leq 0$, positive for $0 < g < 11/480$, and closes as $g \downarrow 0$. Its energy density is zero on the left and $-g^2/(4r) + O(g^4)$ on the right, $r = 1/16$. At fixed positive coupling, eventually at least $(L/p)^2$ independent reader excitations carry the gap, with local spectral weight per area at least $1/p^2$ and $p = 2r/g + O(1)$. The inner gap is eventually exact, without a general computable settling bound. A selected infinite-volume phase has that gap above its full zero-energy subspace. Together with the two-limit ceiling, the existing finite nonnegative exponents in this convention are exactly D_3 already in two dimensions.*

Section 18 proves this theorem with explicit local domains, source predicates and full-sector comparisons. The theorem does not identify a locally nondegenerate or KMS-extremized gap, establish all-size convergence, or supply a unique ground state. Sections 19 and 20 distinguish robust growing-readout information and state selection from intrinsic equilibrium exponents. Section 21 gives the fixed-target reach and hidden-scale obstructions.

Theorem 1.5 (Anchored robust scale laws)**.** *One fixed computable universal noisy rule on $\mathbb{Z}^3$, one open positive-rate kernel region, and fixed computable scale readouts realize exactly $\mathsf{D}_3 \cap [0, 1]$, separately as densities and counting exponents of eventual output sets. Each value is constant across the region for its chosen finite program. The construction has a computable bottom boundary, a distinguished program origin, infinite cloning in two directions, and prescribed time-then-window limits. Expectation and sample iterated limits agree almost surely. At computable kernels no effective inquiry converges with positive probability to a fixed such law outside D_2; for external finite-cylinder inquiry the exceptional learnable kernels form a null, meager but dense subset of the region.*

Section 20.6 proves the anchored range theorem, and Section 20.9 proves the inquiry qualification. This robust counting law is distinct from an intrinsic equilibrium critical gap exponent. Removing the bottom boundary, program origin and infinite cloning is a separate open problem.

The spectral ruler examples are finely tuned. For the earlier two-dimensional and wetting rulers, a specified arbitrarily small onsite perturbation changes the programmed gap exponent to zero (Section 17). The ruler pitch has structural exponent one; a theorem making a conventional correlation length exponent ν uncomputable is not claimed. Finite analytic systems have integer gap exponents. This forces the effective finite system to change infinitely often when its envelope has a noninteger exponent, but does not prove that all noninteger critical exponents in physics require a staircase.

1.1 Relation to earlier work

Spectral-gap undecidability for translation-invariant lattices was established by Cubitt, Pérez-García and Wolf [10, 11]. Bausch, Cubitt and Watson constructed uncomputable phase diagrams [4]. Purcell, Li and Cubitt placed a transition at an uncomputable coupling [39]. Here the coupling is rational and an asymptotic gap exponent contains the uncomputable number. Watson, Onorati and Cubitt proved that a computable renormalisation map can have uncomputably complex flows [46]. None of these statements identifies stability with an effective convergence modulus.

Fisher and Barber developed finite-size scaling for critical phenomena [19]. Our cell decomposition supplies a particular exact transfer, rather than a theorem about the usual scaling ansatz in all models. The commensurate steps belong to the established staircase tradition of Fisher and Selke and of Bak [20, 1]; the arithmetic programming of their gap envelope is the additional feature. Rellich's analytic perturbation theory, as presented by Kato [27], supplies the finite-system obstruction.

The open-edge preference is a form of the boundary-bonus mechanism of Bausch, Cubitt, Lucia, Pérez-García and Wolf [3]; the exact flag identities and all-sector bounds used here are proved below. Watson and Cubitt study complexity of ground energy density [45]. Xu et al. distinguish intrinsic bulk excitations from boundary-dependent gaps [47]. Our bulk broadcast has positive-density local excitations, but its gap removes all zero-energy states. Section 13.7 separates this convention from their locally nondegenerate, KMS-extremized gap. The real hierarchy and density classifications are prior mathematical results of Zheng and Weihrauch and of Downey, Jockusch and Schupp [49, 16]. Meyerovitch classified entropy-dimension values of multidimensional shifts of finite type using arithmetical classes of reals [33]. This is a prior classification of local-rule growth exponents; the observable and realization here concern spectral gaps and their coupling-selected finite-cell envelopes. The companion precision problem [14] is independent of the present proof.

There are closer precedents than undecidable transition locations. Schmidhuber's algorithmic ensemble has the computable critical inverse temperature $\log 2$ and partition value Ω_U there [41]; it is not a local quantum-lattice gap-exponent classification. Balister, Bollobás, Morris and Smith explicitly announce uncomputability of bootstrap critical-probability exponents through a halting reduction [2]. That announcement, referring to work in preparation, concerns failure of a uniform exponent algorithm across update families; it does not by itself exhibit one fixed noncomputable real quantum gap exponent. Watson and Bausch study the complexity of approximating critical boundaries under a finite-size consistency promise [44], a distinct input-size complexity question. Castilla-Castellano and Lucia show that arbitrarily small onsite perturbations remove undecidable behavior from a particular one-dimensional gap family [9]. Our explicit perturbation proves the quantitative change of a programmed exponent to zero; neither result proves computability of every stable exponent. No exhaustive priority claim is made.

2 The coupling as an energetic ruler

The ruler is classical. Its five symbols are W, V, A, S, D: a horizontal wall, a vertical divider, a site above the signal, the signal, and a site below it. Allowed horizontal pairs, in left-to-right order, are

$$WW,\ AA,\ SA,\ DS,\ DD,\ AV,\ SV,\ VS,\ VD.$$

Allowed vertical pairs, in top-to-bottom order, are

$$WS,\ WA,\ WV,\ SW,\ DW,\ VW,\ AA,\ AS,\ SD,\ DD,\ VV.$$

Every other pair costs one. Each W site receives energy $-g$; each vertical VW bond costs $r = 1/16$. These are fixed rational onsite and nearest-neighbour terms, affine in g.

Lemma 2.1 (Forced cell geometry)**.** *On a torus, every violation-free configuration is either all A, all D, or a stack of stripes separated by W rows. A stripe of height $h \geq 1$ consists of repeated columns*

$$V,\ S_0, \ldots, S_{h-1}, \qquad S_o = A^o S D^{h-1-o}.$$

Its cells have pitch $p = h + 1$, interior $h \times h$, and p divides the horizontal circumference.

Proof. The only horizontal pair containing W is WW, so a wall occupies an entire row. Vertical WW is forbidden. Between walls a column containing V is all V. Any other column is A^*SD^*, A^* or D^* by the vertical rules; the boundary rules forbid the last two. If the signal in one column is at offset $o < h - 1$, the pairs SA and DS force the next signal to offset $o + 1$. At $o = h - 1$ the next column must be V: otherwise its last entry would be A above W. After V, the next signal has offset zero. This gives the cycle. With no W, a cyclic vertical column is all A, all D or all V. The horizontal rules admit no cyclic mixture of these column types; only the two uniform vacua remain. □

A perfect pitch-p configuration has energy per site

$$f_g(p) = -\frac{g}{p} + \frac{r}{p^2}, \qquad g_p = \frac{r(2p+1)}{p(p+1)}, \qquad f_g(p) - f_g(p+1) = \frac{g_p - g}{p(p+1)}. \tag{2}$$

Only $p \geq 2$ is geometrically needed. For sufficiently small $g > 0$ the unique minimizer is $p_*(g) = p$ on (g_p, g_{p-1}); at g_p both $p, p+1$ minimize. Completing the square in $1/p$ gives

$$p_*(g) = \frac{2r}{g} + O(1), \qquad \min_p f_g(p) = -\frac{g^2}{4r} + O(g^4). \tag{3}$$

The following estimate controls arbitrary defects, not only the perfect patterns used in this minimization.

Lemma 2.2 (Charging estimate)**.** *Put $f_* = \min(0, \min_{p \geq 1} f_g(p))$. On an $L_1 \times L_2$ torus let $V(\sigma)$ count violated pairs. For every column containing W, let s range over segments from one W to the next, of pitch p_s. Then*

$$E(\sigma) - L_1 L_2 f_* \geq \sum_s p_s (f_g(p_s) - f_*) + (1 - 6r) V(\sigma). \tag{4}$$

In particular a configuration with a violated pair lies at least $5/8$ above $L_1 L_2 f_*$.

Proof. A column with no W has no negative bonus and contributes at least $L_2 f_*$. For the other columns assign each W to the segment it starts. If v_s is the indicator that the segment ends in a VW bond, its energy contribution is $-g + rv_s$. The identity

$$-g - pf_* = p(f_g(p) - f_*) - r/p$$

leaves the additional nonnegative residual $-A_0 f_*$, where A_0 is the area of the W-free columns, and reduces the assertion to $\sum_s (v_s - 1/p_s) \geq -6V(\sigma)$. Declare a segment vertically bad if its closed row range contains a forbidden vertical bond. Group the remaining segments into maximal horizontal runs having the same top and bottom wall rows and no violation on the intervening horizontal bonds. Lemma 2.1 forces the column cycle inside a run. A run of m columns and pitch p contains at least $\lfloor m/p \rfloor$ divider columns; its deficit is greater than -1. A run closing around the torus has zero deficit. Charge only the right end of each nonclosing run. A missing matching wall segment produces a forbidden W–non-W horizontal pair. A matching but vertically bad segment supplies a forbidden vertical bond; otherwise the failed join supplies a forbidden horizontal bond. A violated pair belongs to the closed vertical ranges of at most three segments on that side, and so pays for at most three such run ends. A segment excluded from every good run contains a violated vertical pair; its individual deficit is at least -1. Each such violation is in at most three closed segment ranges. Charging both types uses at most six units of deficit per violation. Substitution proves (4). □

Corollary 2.3 (Selection and separation)**.** *For $g \leq 0$ the vacua have energy zero and every other wall configuration has energy at least r. For small positive nontie g, on a square torus divisible by $p_*(g)$ the minimizers are exactly the perfect pitch-p_* cell lattices. Every other configuration lies at least* $\min(5/8, L\delta(g))$ *above, where*

$$\delta(g) = \min_{p \neq p_*} p(f_g(p) - f_*) > 0.$$

At ties allow both minimizing pitches and omit both from this minimum.

Proof. For $g \leq 0$ a clean nonvacuum stack has energy $L_1 \sum_i (-g + r/p_i) \geq r$, since $p_i \leq L_1$. Defective configurations obey Lemma 2.2. For $g > 0$, clean stripes of a wrong pitch contribute $Lp(f_g(p) - f_*)$; all other configurations have a violated pair. The minimum defining δ is effective and positive because its terms tend to infinity with p. The same argument excludes all pitches except the two minimizers at a tie. Uniform vacua have the positive excess $-L^2 f_*$ and are also above the stated bound for sufficiently large L. □

The threshold for the last statement can be computed at rational $g > 0$: compare the rational energies, find the finite minimizing pitches and a positive separation, then take $L = n!$ large enough that these pitches divide L and $L\delta \geq 5/8$ (and the vacuum excess also exceeds $5/8$). No assertion is made for incommensurate tori.

3 The finite-size engine: statement and setting

Fix a finite local dimension q and three Hermitian matrices h_0, h_h, h_v, of sizes q, q^2, q^2, respectively. On $\Lambda_L = \{0, \ldots, L-1\}^2$, let

$$H_L = \sum_{x \in \Lambda_L} (h_0)_x + \sum_{x, x+e_1 \in \Lambda_L} (h_h)_{x, x+e_1} + \sum_{x, x+e_2 \in \Lambda_L} (h_v)_{x, x+e_2}. \tag{5}$$

The orientation of each bond is fixed; rotation invariance is not required. There are no additional boundary operators. When the spectrum has at least two distinct eigenvalues, set

$$E_0(L) = \min \operatorname{spec} H_L, \qquad \Delta(L) = \min\{\lambda - E_0(L) : \lambda \in \operatorname{spec} H_L,\ \lambda > E_0(L)\}. \tag{6}$$

We assume this gap is defined for all sufficiently large L. Thus scalar finite-volume Hamiltonians are excluded when necessary, and a degenerate ground space does not itself make the gap zero. For $L \geq 2$ put

$$z_L = \frac{-\log \Delta(L)}{\log L}, \qquad z^- = \liminf_{L\to\infty} z_L, \qquad z^+ = \limsup_{L\to\infty} z_L. \tag{7}$$

An existing finite exponent is $z = z^- = z^+$.

For a real a, the conventions $a \in$ left-Σ_2^0 and $a \in$ left-Π_2^0 refer to the arithmetical complexity of the strict lower rational cut $\{r \in \mathbb{Q} : r < a\}$. A real is in Δ_2^0 if it is computable relative to the halting oracle, equivalently if it is a limit of a computable rational sequence. These are representations of real numbers, not conventions about an unspecified digit expansion [49, 16].

Theorem 3.1 (Exact classification)**.** *Consider* (5) *with integer interaction matrices and arbitrary finite local dimension.*

1. *The existing exponents in* $[0, \infty)$ *are exactly* $\Delta_2^0 \cap [0, \infty)$.
2. *The finite pairs* (z^-, z^+) *in* $[0, \infty)^2$ *are exactly the pairs* (a, b) *with* $0 \leq a \leq b < \infty$, $a \in$ *left-*Σ_2^0 *and* $b \in$ *left-*Π_2^0.

Each pair is realized by one fixed Hamiltonian. Its local dimension may depend on the pair. The necessity statements hold more generally for algebraic interaction matrices, in any fixed dimension, whenever the gaps and endpoints in question are defined and finite.

Corollary 3.2 (One fixed uncomputable exponent)**.** *For any chosen prefix-universal machine* U*, there is a fixed model* (5)*, with integer matrices, for which* $\Delta(L) = L^{-\Omega_U + o(1)}$*, where* Ω_U *is the halting probability of* U*. No uncomputable number occurs in a matrix entry.*

The construction and its full spectral estimate are in Sections 5–7. Section 8 provides the required computationally fast words, and Section 9 proves the converse.

4 Two elementary spectral and boundary estimates

We first record estimates that will control all computational and noncomputational sectors. Graph Laplacians use the quadratic form $\sum_{uv\in E} |\psi_u - \psi_v|^2$, with repeated edges counted with their integer weights.

Lemma 4.1 (Penalty localization)**.** *Let* C *be a finite connected graph of maximum degree at most* D*, and let* V *be diagonal with values in* $\{0\} \cup [1, \infty)$*. If all vertices are penalized, then* $L_C + V \geq I$*. If both free and penalized vertices occur, and each connected free cluster has at most* g *vertices, then*

$$L_C + V \ \geq\ \frac{1}{8Dg^2} I. \tag{8}$$

A free path on $n \geq 2$ *vertices has positive gap* $2 - 2\cos(\pi/n) \geq 4/n^2$*; a free cycle on* $n \geq 3$ *vertices has positive gap* $2 - 2\cos(2\pi/n) \geq 4/n^2$*. A free isolated vertex has no positive eigenvalue. The two-vertex cycle with two parallel unit edges has gap* 4*.*

Proof. The all-penalized case is immediate. For each free cluster K_i, choose an edge $u_i p_i$ to a penalized vertex. Each penalized vertex is selected at most D times, so half of its penalty can allocate $1/(2D)$ to each selected cluster. The remaining half pays for its own norm. Therefore the energy of ψ is at least

$$\sum_i \left(\sum_{uv \in E(K_i)} |\psi_u - \psi_v|^2 + |\psi_{u_i} - \psi_{p_i}|^2 + \frac{|\psi_{p_i}|^2}{2D} \right) + \frac{1}{2} \|\psi_{C \setminus F}\|^2,$$

where F is the free set. Let $M_i = \max_{v \in K_i} |\psi_v|$. If $|\psi_{p_i}| \geq M_i/2$, the allocated penalty is at least $M_i^2/(8D)$. Otherwise a path from a maximum to p_i, of at most g edges, has energy at least $M_i^2/(4g)$ by Cauchy–Schwarz. Since $\|\psi_{K_i}\|^2 \leq g M_i^2$, either case bounds the cluster norm by (8). The path and cycle statements follow from their explicit spectra. Keeping parallel-edge weights gives the last case; larger integer weights cannot reduce the lower bounds. □

Lemma 4.2 (Boundary preference)**.** *At each site put four flag bits T, F, B, R. Give a site energy $-(T + F + B + R)$. On a vertical bond, with upper endpoint x and lower endpoint y, add $2T_y + 2B_x$; on a horizontal bond with left endpoint x and right endpoint y, add $2F_y + 2R_x$. Add unit penalties for forbidden onsite and bond patterns. Then*

$$E_{\mathrm{cl}}(\sigma) + 4L = \#\{\textit{incorrect flag bits}\} + \#\{\textit{violated local rules}\}. \tag{9}$$

Here a correct flag is the indicator of its corresponding physical edge. If exactly one configuration has correct flags and no rule violation, it is the unique classical minimizer, of energy $-4L$, and every other configuration has energy at least $-4L + 1$.

Proof. The coefficient of T is -1 on the top row and $+1$ elsewhere, because every other site has one bond from above. Its energy is therefore $-L$ plus the number of errors relative to the top-row indicator. Apply the same argument separately to F, B, R and add the rule penalties. □

All bond contributions in this lemma are genuine two-site matrices, even when one factor is the identity. The edge preference comes solely from missing bonds in (5). We have proved a classical gap *at least* one.

5 A unique classical tableau from uniform interactions

Fix an integer $\beta \geq 1$ and a computable binary word $A(0), A(1), \ldots$ whose bits can be written sufficiently quickly; Section 8 constructs every word we need. Fix a total deterministic one-tape machine M that writes the word on a persistent output track, with bit j at column j, and never crosses the left edge. It can be arranged that the first j bits are written by time $2^{j-1} - 2$ for every sufficiently large j. This is a timing upper bound, not a time supplied as a coupling.

Besides the four flags, the classical alphabet contains a binary digit b, a carry c, a past-one flag f, and the tape, head, persistent output and arrival fields of M. Every listed rule is enforced by a diagonal onsite or nearest-neighbour unit penalty. Conjunctions can be imposed as a single forbidden-pattern projector; their precise grouping does not affect (9).

5.1 Counting the row number

The counter rules are

$$\begin{aligned} T=1 &\implies b=f=0, & F=1 &\implies c=1, \\ F_y=0 &\implies c_y=b_x c_x & &\text{on a horizontal bond } x\to y, \\ T_y=0 &\implies b_y=b_x \,\mathrm{XOR}\, c_x, \quad f_y=f_x \,\mathrm{OR}\, b_x & &\text{on a vertical bond } x\to y. \end{aligned}$$

With correct flags, carries are fixed left to right on each row, and the vertical rule advances the binary row number. Induction gives row r equal to the binary expansion of r, with least significant bit at column zero; f records whether that column was one on an earlier row. Thus the derived label

$$\mathsf{reg} = B\,(f \,\mathrm{OR}\, b) \tag{10}$$

selects exactly columns $0,\dots,\ell(L)-1$ on the bottom row, where

$$\ell(L) = \mathrm{bitlength}(L-1) = \log_2 L + O(1). \tag{11}$$

Indeed a column j has shown a one by row $L-1$ exactly when $2^j \le L-1$. For $L \ge 2$, $\ell(L) < L$.

5.2 A nearest-neighbour computation tableau

Let the tape field be γ, the head state be $h \in Q \cup \{\perp\}$, the output be $o \in \{\perp, 0, 1\}$, and the two arrival signals be $a_L, a_R \in Q \cup \{\perp\}$. The top row has blank tape and output, with the initial head only at its leftmost site. On each horizontal bond, the left site's transition fixes the right site's a_L; the right site's transition fixes the left site's a_R. A missing source gives a blank arrival. Require $a_L = \perp$ at $F = 1$ and $a_R = \perp$ at $R = 1$. These equalities fix every unused arrival signal as well.

On a vertical bond, the upper site's own transition fixes the tape and output below it. Its staying-head transition, or its a_L or a_R, fixes the next head; two incoming heads are forbidden. This is a finite rule table determined by the transition function of M. In particular, the arrival fields implement the update by horizontal and vertical *two-site* tests; a three-site update is not an interaction.

Given one row's tape, head and output, first fix all arrival fields by the horizontal tests and edge initialization, then fix the next row by the vertical tests. This proves uniqueness by induction, including the last row's arrival fields. At time t the head column is at most t. A transition to another included row cannot leave the right side of an $L \times L$ tableau. An outgoing move after the final row places no condition on a nonexistent next row. Consequently the tableau exists and is unique for every $L \ge 2$, regardless of whether all desired output bits have yet appeared.

Together with the counter, this gives a unique correct classical configuration σ_L^* and the unit separation of Lemma 4.2. On the bottom register put

$$m_j = \beta \mathbf{1}_{\{o_j=1\}}, \qquad r_j = 2^{m_j}, \qquad 0 \le j < \ell(L). \tag{12}$$

For all sufficiently large L, the timing assumption gives $o_j = A(j)$ for $j < \ell(L)$: the first ℓ bits appear by row $2^{\ell-1} - 2$, and $L - 1 \ge 2^{\ell-1}$. Therefore

$$k(L) = \sum_{j<\ell(L)} m_j = \beta\,|A \cap [0, \ell(L))|. \tag{13}$$

6 The quantum counter and all its sectors

At every site use the same additional space with digit $d \in \{0, \dots, 2^\beta - 1\}$ and head-side symbol $s \in \{\mathsf{NL}, \mathsf{NR}, \mathsf{INC}, \mathsf{RET}\}$. The symbols NL and NR mean left and right of a head; the other two are head states. Outside reg, penalize every dynamic field other than $(0, \mathsf{NL})$ by one and apply no transition. Inside reg, penalize digits $d \geq r_j$ and, on two consecutive register sites, the side-symbol patterns

$$(\mathsf{NR}, \mathsf{NL}), \qquad (\text{head}, \mathsf{NL}), \qquad (\mathsf{NR}, \text{head}), \qquad (\text{head}, \text{head}).$$

Either head state is meant here. All these penalties are nonnegative.

For each transition $u \leftrightarrow v$ below, add its edge Laplacian $(|u\rangle - |v\rangle)(\langle u| - \langle v|)$. Digits not explicitly changed are retained. The rules on the intended register are

$$(d, \mathsf{INC}) \leftrightarrow (d+1, \mathsf{RET}) \qquad (d+1 < r_j), \tag{14}$$

$$(r_j - 1, \mathsf{INC})(e, \mathsf{NR}) \leftrightarrow (0, \mathsf{NL})(e, \mathsf{INC}) \qquad (j+1 < \ell), \tag{15}$$

$$(r_j - 1, \mathsf{INC}) \leftrightarrow (0, \mathsf{RET}) \qquad (j = \ell - 1), \tag{16}$$

$$(0, \mathsf{NL})(e, \mathsf{RET}) \leftrightarrow (0, \mathsf{RET})(e, \mathsf{NR}) \qquad (j+1 < \ell), \tag{17}$$

$$(d, \mathsf{RET}) \leftrightarrow (d, \mathsf{INC}) \qquad (j = 0). \tag{18}$$

The conditions involving j in this notation are implemented as follows. Rules (14) and (18) are onsite, guarded by $\mathsf{reg} = 1$, with the latter also requiring $F = 1$. Rules (15) and (17) require both endpoints to be register sites. Rule (16) is a horizontal bond term: its left endpoint has $\mathsf{reg} = 1$ and its right endpoint has $\mathsf{reg} = 0$; it changes only the left dynamic field and is the identity on the right dynamic field. Such a right neighbour exists on σ_L^* since $\ell(L) < L$. Every digit test uses finite local classical labels. Thus every rule is onsite or nearest-neighbour, including the last-cell test. No rule queries coordinates or the absence of an unrepresented neighbour.

Write K_L for this nonnegative dynamic operator on the correct classical block. In the next lemma set $B_* = \prod_{j<\ell} r_j = 2^k$; this symbol is unrelated to the bottom flag B.

Lemma 6.1 (Full counter gap)**.** *For $\ell \geq 1$ the dynamic kernel is nonempty, and the smallest positive eigenvalue obeys*

$$\frac{1}{64\ell^2 B_*^2} \;\leq\; \Delta^+(K_L) \;\leq\; \frac{4\pi^2}{B_*^2}. \tag{19}$$

There is a free counter orbit of length

$$N = 2\sum_{j=0}^{\ell-1} \frac{B_*}{r_0 \cdots r_{j-1}}, \qquad 2B_* \leq N \leq 2\ell B_*. \tag{20}$$

The empty product is one. If $N = 2$, the orbit retains two parallel edges.

Proof. First fix the exterior fields. A wrong exterior field cannot change under any transition and carries a penalty at least one, so the entire corresponding block is at least one. With the exterior correct, head number on the register is conserved.

With zero heads there are no transitions. Free isolated configurations, including the all-zero, all-NL configuration, give ground states; penalized isolated configurations cost at least one. With two or more heads every configuration is penalized: between consecutive heads the side-symbol rules would require NR immediately after the first and NL immediately before the second, forcing a forbidden switch. Adjacent heads are forbidden directly. The all-penalized part of Lemma 4.1 applies even if this sector has large degree.

With one head, there is at most one forward match and one backward match. At an incrementing head the forward choice is increment, carry or wrap; its predecessor is a carry or restart. At a returning head, its digit distinguishes an increment predecessor (nonzero) from a wrap or return predecessor (zero, at or away from the last site). The return or restart successor is also unique. Illegal digits or side strings remove possible matches. Hence transition degree is at most two, counting weights.

There are exactly $2\ell B_*$ free one-head configurations: choose a head position, one of its two states, and legal digits; the side symbols are forced. On a component meeting penalties, a free cluster has size at most $2\ell B_*$, so Lemma 4.1, with $D = 2$, gives $1/[16(2\ell B_*)^2]$. On a completely free component, a nontrivial path or cycle has at most $2\ell B_*$ vertices and the positive spectral gap is larger than the same bound. Free singletons only enlarge the ground space. These arguments prove the lower bound in every sector.

For the upper bound start with an incrementing head at site zero and any legal digit string. It increments the mixed-radix value, carries over maximal digits, and returns over the zeros it has created. Overflow wraps to zero. It returns to the initial digit value after B_* increments. Each cell j is visited in the forward sweep $B_*/(r_0 \cdots r_{j-1})$ times and contributes twice that many transitions in a complete orbit. This gives (20). The orbit is free and its positive gap is $2 - 2\cos(2\pi/N) \le 4\pi^2/B_*^2$. When $\ell = 1, r_0 = 1$, wrap and restart contribute to the same pair of states, giving the double-edge cycle with gap four. This case satisfies the same inequalities. □

We use the orbit only for an upper bound. Equality of the full gap with its cycle eigenvalue is unnecessary and is not asserted. In particular, other free one-head components and their ground states are allowed.

7 Assembly and the realization formula

The complete local space is the tensor product of the finite classical alphabet and the dynamic alphabet. Add the classical terms to the guarded dynamic terms. All matrix entries are integers: diagonal rules are integer projectors and each transition contributes diagonal $+1$ and off-diagonal -1. The matrices are fixed once M and β are chosen. They have exactly the form (5).

Theorem 7.1 (Boundary realization)**.** *For every word satisfying the timing condition of Section 5 and every integer $\beta \ge 1$, the constructed Hamiltonian has, for all sufficiently large L,*

$$\frac{4^{-k(L)}}{64\ell(L)^2} \;\le\; \Delta(L) \;\le\; \min\{1, 4\pi^2\, 4^{-k(L)}\}. \tag{21}$$

Consequently, writing $\rho_J(A) = |A \cap [0, J)|/J$,

$$z_L = 2\beta\rho_{\ell(L)}(A) + O\Big(\frac{\log\log L}{\log L}\Big). \tag{22}$$

In particular $z^- = 2\beta \liminf_J \rho_J(A)$ and $z^+ = 2\beta \limsup_J \rho_J(A)$.

Proof. All classical labels are conserved. The exact block decomposition is

$$H_L = \bigoplus_\sigma (E_{\mathrm{cl}}(\sigma) I + K_\sigma), \qquad K_\sigma \ge 0. \tag{23}$$

The correct block has energy $-4L$ and a nonempty kernel. Every other classical block is at least $-4L + 1$, independently of its apparent register, head pattern or output. A superposition across

blocks cannot evade this operator inequality. Within the correct block exterior defects, zero-head excitations, multi-head sectors and all one-head components were covered in Lemma 6.1. Thus

$$\min\{1, \Delta^+(K_L)\} \le \Delta(L) \le \Delta^+(K_L).$$

The lower estimate in (19) is at most one and proves the lower bound in (21). Changing a single frozen exterior field from $(0, \mathsf{NL})$ to $(0, \mathsf{NR})$ in a dynamic ground state produces a unit-energy eigenvector. There are no register/exterior adjacency penalties and the wrap term is the identity on the exterior dynamic field, so this gives $\Delta(L) \le 1$. The orbit gives the other upper bound.

Taking base-two logarithms of the bracket gives $\log_2(1/\Delta(L)) = 2k(L) + O(\log \ell(L))$. Equations (11) and (13) yield (22). Every sufficiently large integer occurs as $\ell(L)$, and $\ell(L)$ increases only by unit steps, which proves the endpoint equalities. □

Wrong tableaux, fake interior boundaries and arbitrarily long apparent computations are covered by the classical unit separation, without any runtime bound on those wrong blocks. The only true computation is static. The closing eigenvectors keep that tableau fixed, freeze the exterior, and evolve the quantum counter on the logarithmic boundary segment.

8 Fast words with prescribed density endpoints

The usual computable-density classification supplies the endpoint arithmetic [16]. The following timing refinement is needed because only L rows are available to write approximately $\log_2 L$ output bits.

Lemma 8.1 (Fast density word)**.** *If $0 \le a \le b \le 1$, $a \in$ left-Σ^0_2 and $b \in$ left-Π^0_2, there is a binary word A with lower density a and upper density b whose jth bit is computable in time polynomial in $\log(j+2)$. The polynomial and its constants may depend on the endpoints' programs.*

Proof. The sequence characterizations in [16, Theorem 5.1 and Corollaries 5.4–5.5] give computable rational sequences with liminf a and limsup b. If $a < b$, choose a fixed rational c with $a < c < b$. Clip the first sequence to $[0, c]$ and the second to $[c, 1]$, and interleave them. Continuity and monotonicity of clipping preserve the desired endpoint of each sequence. The result is a computable rational sequence $a_n \in [0, 1]$ with endpoints a, b. If $a = b$, membership of both cuts gives a Δ^0_2 real, so use a convergent computable rational sequence clipped to $[0, 1]$. Choosing the separator uniformly from arbitrary endpoint names is not claimed.

Fix $a_0 = 0$. Put $\sigma(0) = 0$. At stage s, simulate the canonical algorithm for $a_{\sigma(s)+1}$ from its start for s steps, and advance σ by one exactly when the output has appeared. Otherwise retain its value. Totality makes σ unbounded, and unit advances preserve liminf and limsup for $b_s = a_{\sigma(s)}$. Every selected output other than a_0 has $O(s)$ bits and comes from a bounded simulation. Reconstructing all advances to stage s takes $O(s^2)$ simulated steps, with polynomial universal-simulation overhead. This explicit output bound prevents a large rational name from defeating the slowdown.

Let $N_0 = 1$ and $N_{s+1} = (s+2)N_s$. Set $A(0) = 0$ and, for $0 \le t < N_{s+1} - N_s$, put

$$A(N_s + t) = \lfloor (t+1)b_s \rfloor - \lfloor t b_s \rfloor. \tag{24}$$

The number of ones in the first t letters of that stage is $\lfloor t b_s \rfloor$. If $J = N_s + t$, its density differs by less than $1/J$ from

$$\frac{N_s}{J}\rho_{N_s}(A) + \frac{t}{J}b_s. \tag{25}$$

At the end of the stage, the density differs from b_s by at most $1/(s+2) + 1/N_{s+1}$. Endpoints therefore attain the liminf and limsup of b_s. The convex formula shows that intermediate densities cannot create additional limit extrema. This proves both density assertions, including $a = 0$ or $b = 1$.

To compute $A(j)$, find its factorial-sized stage, reconstruct $\sigma(s)$ and its bounded rational output, then evaluate the two floor values in (24). The stage has $s = O(\log(j+2))$, all index arithmetic has $O(\log(j+2))$ bits, and the selected rational has polynomially bounded length. All steps take polynomial time in $\log(j+2)$. □

Realization in Theorem 3.1. Given $(z^-, z^+) = (a, b)$ as in the theorem, choose an integer $\beta \geq \max\{1, b/2\}$. Positive rational scaling preserves the cut classes. Apply Lemma 8.1 to $(a/(2\beta), b/(2\beta))$. A one-tape machine can compute the bits and write them successively on a persistent track, using separate work symbols. Simulation and travel to column j cost at most polynomial time in j per bit. Its prefix runtime is therefore eventually less than $2^{j-1} - 2$. Theorem 7.1 gives the prescribed pair. Taking $a = b$ gives every existing nonnegative Δ_2^0 exponent. □

Proof of Corollary 3.2. Dovetail the prefix-universal machine's programs and enumerate its halting weights, giving computable increasing rational sums $\Omega_{U,s}$ converging to Ω_U. Use the convergent sequence $\Omega_{U,s}/2$ in the word construction and set $\beta = 1$. The resulting exponent is $2(\Omega_U/2) = \Omega_U$. The finite simulator and word algorithm determine the finite classical alphabet and integer matrices. The real is encoded in an asymptotic density, not as a local coefficient. □

An exponent zero does not assert a uniform positive gap. For example, the all-radix-one counter has orbit length $2\ell(L)$ and a closing gap of order $1/\ell(L)^2$, still with exponent zero.

9 Necessity from finite algebraic spectra

Proposition 9.1. *For fixed algebraic local matrices, every defined finite existing exponent is Δ_2^0. If both endpoints are finite, the lower endpoint is in left-Σ_2^0 and the upper endpoint is in left-Π_2^0. This holds in any fixed lattice dimension and for any uniformly computable choice of finite boxes.*

Proof. For each L, construct H_L exactly over its algebraic coefficient field. Exact characteristic polynomials and real-root isolation compute its distinct eigenvalues and decide multiplicities. Thus the gap above the entire ground space is a uniformly computable positive algebraic number whenever defined. The hypothesis that it is eventually defined is essential; the zero Hamiltonian has no second distinct eigenvalue. We do not replace algebraic entries by arbitrary computable real entries while retaining a multiplicity algorithm.

Compute rational q_L within $1/L$ of z_L. This preserves all finite endpoints. For rational r,

$$r < \liminf_L q_L \quad \Longleftrightarrow \quad \exists t \in \mathbb{Q}\ \exists N \quad (r < t \text{ and } \forall L \geq N :\ q_L > t).$$

This is a Σ_2^0 lower cut. Dually the strict upper cut of $\limsup_L q_L$ is Σ_2^0. If that endpoint is irrational, its strict lower rational cut is the complement of its strict upper cut, hence Π_2^0; if it is rational, its lower cut is computable and is also Π_2^0. This avoids a false complement identity at a rational endpoint. An existing limit is computable in the halting oracle: that oracle can find arbitrarily accurate tail-Cauchy intervals of the computable sequence. Equivalently it is a Δ_2^0 real.

For a d-dimensional box the norm estimate $\Delta(L) \leq 2\|H_L\| \leq CL^d$ gives an eventual lower bound for z_L. A finite limsup gives bounded tails when needed. No efficient diagonalization or computable rate of asymptotic convergence is claimed. □

This proves the remaining direction of Theorem 3.1. More generally, the same argument applies to total computable postprocessing of uniformly computable finite-scale spectral data. Any logarithm argument must be positive and every scale denominator must be nonzero. Merely naming a scale does not supply these domain conditions.

10 Arithmetic of limits and density readers

The arithmetic claims depend on the representation of a *uniform* family. For $X = 0^{(m-1)}$, uniformly D_m data mean that an X-oracle algorithm approximates x_n within 2^{-p} uniformly in (n, p). Uniform S_m data have increasing rational X-computable names, and uniform P_m data have decreasing ones. For an individual real these agree with the one-sided cut classes, with a separate rational-endpoint case. They do not imply that exact strict lower cuts are uniformly decidable at the same level.

For example let $x_e = 2^{-t}$ if machine e first halts at step $t \geq 1$, and let $x_e = 0$ otherwise. Running for s steps and returning the value found, or zero, gives error at most 2^{-s}. This is a uniformly computable family, but $\{e : x_e > 0\}$ is the halting set. A theorem about names cannot silently be replaced by a uniform exact-comparison theorem.

Proposition 10.1 (Limit calculus)**.** *For uniform names, finite outputs, and the stated existing limits, the following are arithmetic ceilings. Every entry is sharp for abstract data; physical attainment is asserted separately.*

Input	*limit*	*lower limit*	*upper limit*
D_m	D_{m+1}	S_{m+1}	P_{m+1}
S_m	P_{m+1}	S_{m+2}	P_{m+1}
P_m	S_{m+1}	S_{m+1}	P_{m+2}

Infimum and supremum send D_m data respectively to $\mathsf{P}_m, \mathsf{S}_m$. For S_m data they give $\mathsf{P}_{m+1}, \mathsf{S}_m$; for P_m data they give $\mathsf{P}_m, \mathsf{S}_{m+1}$. A limit with an effective error bound preserves the input class. Computable continuous increasing maps preserve the side; decreasing maps exchange it, on their domains.

Proof. Work relative to X. For a convergent computable rational sequence an X' search finds arbitrarily accurate tail-Cauchy intervals. Oracle Cauchy names reduce asymptotic limits to rational sequences by errors 2^{-n}. Moreover

$$q < \liminf_n a_n \iff \exists r > q \; \exists N \; \forall n \geq N : \; a_n \geq r.$$

This is a Σ^X_2 lower cut; the upper endpoint is dual. The rational-endpoint distinction avoids incorrectly complementing strict cuts at equality. For increasing names $a_{n,s}$, the analogous lower cut of the liminf has an additional existential approximation witness after the universal n quantifier, hence Σ^X_3. The strict upper cut of their limsup is Σ^X_2, giving a P_{m+1} real. An existing limit is its own limsup. Negation proves the other row.

Dovetailing preserves suprema of left-c.e. and infima of right-c.e. names. An infimum of left-c.e. values has a Σ^X_2 upper cut: there is an index and a rational buffer below the tested threshold bounding every increasing approximant. The dual gives the supremum rule. If $|x_i - x| \leq 2^{-i}$, use $x = \sup_i(x_i - 2^{-i}) = \inf_i(x_i + 2^{-i})$ to preserve one-sided names. Computably continuous monotone maps preserve or reverse these names on their domains. For a positive two-sided name, search for an interval bounded away from zero; for a positive increasing name, search for a positive approximant. For a decreasing name q_s converging to a positive value, each q_s is positive, so apply logarithms

or reciprocals directly to these upper approximants and take monotone rational roundings. This requires no computable positive lower bound on the limit.

Sharpness of the two-sided row is the relativized real hierarchy [49, 16]. For the left-c.e. limit and limsup entries, write a right-c.e.-in-X' value as the limsup of an X-computable sequence, and take its tail suprema, which are left-c.e. in X. Sharpness of the left-c.e. liminf entry follows from the characterization of lower densities of c.e. sets [16, Theorem 5.7], relativized to X; their finite prefix densities have increasing X-names. Negation handles the dual entries. For an infimum target with Σ^X_2 upper cut, write its membership as $\exists a \forall b\, R(q, a, b)$. An increasing name starts at q and jumps to a common upper bound on the first failed witness. The infimum over (q, a) is the target. Supremum is dual. Bounded targets can be handled in one fixed compact rational interval. □

Lemma 10.2 (Relative density realization)**.** *For $X = 0^{(k-1)}$ and $0 \le a \le b \le 1$ in $\mathsf{S}_{k+1} \times \mathsf{P}_{k+1}$, an X-computable set A has lower density a and upper density b. When $a = b$, its density is that D_{k+1} real. A finite prescribed prefix can be added.*

Proof. The relativized sequence characterizations [16, Theorem 5.1 and Corollaries 5.4–5.5] provide a rational X-computable sequence with those endpoints: for unequal endpoints, interleave lower and upper sequences clipped on opposite sides of a fixed rational separator. For equal endpoints use a convergent sequence. Clip all values to $[0, 1]$. On $[N_s, N_{s+1})$, where $N_0 = 1$ and $N_{s+1} = (s + 2)N_s$, write the mechanical word of slope b_s. Its first t letters contain $\lfloor tb_s \rfloor$ ones. At a block endpoint its density differs from b_s by at most $1/(s + 2) + 1/N_{s+1}$. Between endpoints it differs by at most $1/J$ from a convex combination of the preceding density and b_s. Thus no new extreme limit values are introduced, and both endpoints are attained. Each bit is X-computable. Finite changes preserve density. There is no claim of uniformly choosing a separator from arbitrary endpoint descriptions. □

Lemma 10.3 (A value beyond either preceding side)**.** *For every $k \ge 1$ some $x \in \mathsf{D}_{k+1} \cap (0, 1)$ lies in neither S_k nor P_k.*

Proof. Enumerate all Σ^0_k and Π^0_k subsets of $\mathbb{Q}$, including those that are not cuts. Starting with $[1/4, 3/4]$, test its midpoint against the next set with $0^{(k)}$. If it belongs, keep the left third; otherwise keep the right third. If the tested set is the strict lower cut of a value v, that third excludes v. The nested intervals have length $3^{-n}/2$ and $0^{(k)}$-computable endpoints. Their intersection is therefore in D_{k+1} and avoids every listed cut. □

Two qualifications will recur. Independent limits do not add their oracle levels: $0'$ computes both limits of two computable sequences and their defined computable combination. Additional levels arise from dependent limits with the required uniform names. Also, a syntactic ceiling does not prove a value has exactly that least level or that a physical model attains it.

11 Exact transfer to a bulk critical exponent

Superimpose the finite-size engine on the ruler. Interior sites carry the four tableau flags T, F, B, R, tied by two-site unit penalties to the conditions that their upper, left, lower and right neighbours are the corresponding W, V, W, V walls. Wall sites carry one dummy tableau label. Tableau rules act only between interior sites of one cell. Every quantum site outside the cell's marked bottom-row segment is frozen. The classical wall bonus replaces the physical-box flag reward inside the cell; all additional tableau tests are nonnegative. In a perfect lattice each cell consequently has the unique tableau of its $h \times h$ box, where $h = p - 1$.

Let A be a fast density word from Lemma 8.1, modified so $0, 1, 2 \in A$. Let $D(h)$ denote the *full* counter gap selected by that tableau. With $\ell(h) = \text{bitlength}(h-1)$ and $k(h) = \beta|A \cap [0, \ell(h))|$, Lemma 6.1 gives

$$\frac{4^{-k(h)}}{64\ell(h)^2} \le D(h) \le 4\pi^2 4^{-k(h)}, \qquad D(h) \le \frac{\pi^2}{\ell(h)^2}. \tag{26}$$

The last inequality follows from the free orbit of length at least 2ℓ: each summand in (20) is at least one. It guarantees gap closure even when the density, and hence the exponent, is zero.

Proof of Theorem 1.1. All classical labels are conserved and each dynamic block is nonnegative. For $g \le 0$ the vacua have no quantum region; their dynamic Hamiltonian is a sum of frozen-site penalties with positive eigenvalues at least one. Every other wall block has offset at least r, and a wrong tableau adds a unit penalty. Thus the gap is at least r on every torus.

Fix sufficiently small positive rational g, first off ties. Choose $L = n!$ large enough for Corollary 2.3. All nonminimizing classical blocks then lie at least $5/8$ above the minimum. The finitely specified word machine has written the required prefix in every cell once h exceeds its fixed timing cutoff. Each minimizing block has a tensor sum of identical disjoint cell counters and frozen exterior sites. Its first positive excitation is exactly $D(h)$. Since $k(h) \ge 3$, $D(h) \le 4\pi^2/64 < 5/8$, so no other classical sector undercuts it. At a tie, minimizing stripe stacks use the two pitches; the least positive dynamic excitation over all minimizing blocks is $\min(D(p-1), D(p))$. Pure stacks show that both candidate values occur. Rational wall energies and the finite timing cutoff give a computable factorial index beyond which this identity holds, also at rational ties.

Every classical block has a zero-energy dynamic state, so the density is the wall minimum. Equations (2) and (3) prove its asserted expansion and nonanalyticity at zero. The positive side has commensurate first-order kinks accumulating at zero; the gap at zero itself remains positive.

For $g \in (g_p, g_{p-1})$, $\log(1/g) = \log p - \log(2r) + o(1)$ and $\ell(p-1) = \log_2 p + O(1)$. Taking logarithms in (26) gives

$$x(g) = 2\beta\rho_{\ell(p_*(g)-1)}(A) + O\left(\frac{\log\log(1/g)}{\log(1/g)}\right).$$

The pitch takes every sufficiently large integer. Its sampled bitlength is nondecreasing, unbounded and advances by unit steps. Ties use the minimum of two gaps; the two neighbouring normalized logarithms differ by $o(1)$, so ties do not alter either endpoint. The endpoints are therefore 2β times the two densities. Choose $\beta \ge \max(1, b/2)$ and use Lemma 8.1 with densities $a/(2\beta), b/(2\beta)$. This proves attainment. The computably tamed inner limit and Proposition 16.1 give necessity. □

The canonical *structural* length of this construction is its pitch: $\xi(g) = p_*(g) = g^{-1+o(1)}$. If one defines a structural ν from this length, it equals one, leaving the uncomputability in $z\nu$. Ground space degeneracy means that correlation lengths of chosen ground states need not equal this pitch. An equality with a GNS gap, or an assertion about all torus sizes, requires a separate argument and is not used. Different stripes may choose independent horizontal phases, so not every minimizing configuration has one common vertical period. The structural length denotes the cell pitch, or the period of a chosen periodic representative.

12 A film whose inner gap limit becomes exact

The finite-size engine writes a fast *computable* word because its available time and read length grow together. A film separates them. Use boxes $[0, L)^2 \times [0, M)$ with coordinates (x, y, z), take $L \to \infty$

first, and only then scale M. The film gap and its normalized exponent sequence are

$$Q(M) = \lim_{L\to\infty} \Delta(L, M) > 0, \qquad z_M = \frac{\log(1/Q(M))}{\log M}. \tag{27}$$

All intermediate gaps and limits used in a necessity assertion must be defined. The construction below makes the inner limit eventually exact.

12.1 The local fields and their determination

Put two flag bits on each coordinate axis, indicating its near and far faces. For each flag use onsite energy -1 when set and a bond contribution $+2$ at the endpoint that is not on the designated face. The six independent copies of Lemma 4.2 give

$$E_{\mathrm{cl}}(\sigma) = E_* + \#\{\text{flag errors}\} + \#\{\text{rule violations}\}, \qquad E_* = -2(L^2 + 2LM). \tag{28}$$

This is an exact identity, including coincident opposite faces when a side has length one. Every additional forbidden onsite or bond pattern has unit nonnegative cost.

Let A be a Δ_2^0 set, with a total computable approximation $a(j, s) \to A(j)$. Modify the first three bits to be one at every stage. On $z = 0$, a fixed ordinary machine uses y as time and x as tape. It first initializes output cells $0, 1, 2$ permanently to one in a known finite time. It then computes $a(j, s)$ for $j = 0, \ldots, s$ at stage s and writes each answer to its own cell on a persistent output track. Work and output are separate tracks. Each stage finishes. For each fixed j there is therefore a physical time t_j after which every subsequent output at cell j equals $A(j)$. Blank output is read as zero.

The two-site arrival compiler of Section 5 specifies this tableau. Initialize at $y = 0$ with the head at $x = 0$; fix both missing-neighbour arrival fields at $x = 0, L - 1$. Freeze *every* machine field to one dummy value off $z = 0$, and guard both endpoints of machine bond rules by $z = 0$. The head cannot travel farther than elapsed time, so all included rows coincide with the infinite one-sided computation. This proves existence and uniqueness without a bound on the runtime of a stage.

In every (x, z) slice put a separate binary thickness counter (b, c, f). At $z = 0$ set $b = f = 0$; at $x = 0$ set $c = 1$. Across an x bond set $c_{x+1} = b_x c_x$, and across a z bond set

$$b' = b \,\mathrm{XOR}\, c, \qquad f' = f \,\mathrm{OR}\, b.$$

Functional induction shows that row z encodes z in binary, and at $z = M - 1$ the bit $b_j \,\mathrm{OR}\, f_j$ equals $\mathbf{1}_{\{2^j \le M-1\}}$. Copy the machine's output bit along z on another field, initialized at $z = 0$. These recurrences determine all counter and transport fields, including those not read by the quantum layer.

The quantum region is

$$\mathsf{reg} = Y_1 Z_1 (b \,\mathrm{OR}\, f), \qquad \ell(M) = \mathrm{bitlength}(M - 1).$$

For $L > \ell(M)$ it is one segment at $(y, z) = (L - 1, M - 1)$. Its radices are $2^{\beta o'_j}$, where o' is the transported bit. All other quantum fields are frozen. Use exactly the guarded rules (14)–(18). In particular wrap tests an x bond with region on the left and nonregion on the right; it changes only the left dynamic field. There is a physical such neighbour in every asymptotic box. No three-site interaction or absent-neighbour oracle is introduced. All coefficients are integers.

Theorem 12.1 (Exact fixed-thickness gap)**.** *For every Δ_2^0 set A containing $0, 1, 2$ and integer $\beta \ge 1$, the construction has, for all sufficiently large fixed M and all $L \ge L_0(M)$,*

$$\Delta(L, M) = Q(M) = \Delta^+\big(K[2^{\beta A(j)} : j < \ell(M)]\big). \tag{29}$$

Here K is the full counter operator, not only its main orbit. The threshold need not have a computable bound. Its exponent sequence satisfies

$$z_M = 2\beta\rho_{\ell(M)}(A) + O\Big(\frac{\log\log M}{\log M}\Big). \tag{30}$$

Proof. Flags, counters, the machine and transport are uniquely determined in that order. Equation (28) puts all other classical blocks at least one above this block. Their quantum operators are nonnegative. Incorrect exterior quantum fields also cost at least one and cannot move. The counter analysis in Lemma 6.1 covers zero heads, multiple heads, penalized components, and all free paths and cycles, retaining parallel-edge multiplicities.

At fixed M, choose $L_0(M) > \max(\ell(M)+1, \max_{j<\ell(M)} t_j)$. This is finite because only finitely many output cells are read. Above it the radix vector is exactly the same for every L. For large M its first three radices are 2^β, so its gap is at most $4\pi^2/64 < 1$. It is therefore the gap of the entire box. Taking logarithms of Lemma 6.1, with $k = \beta|A \cap [0, \ell(M))|$, proves (30). Since $\ell(M)$ takes every large integer through unit steps, the two endpoints are exactly 2β times the density endpoints of A. □

Corollary 12.2 (Film classification)**.** *For translation-invariant integer nearest-neighbour interactions on $\mathbb{Z}^3$, existing nonnegative finite film exponents are exactly $\mathsf{D}_3 \cap [0,\infty)$. Ordered finite nonnegative pairs are exactly $\mathsf{S}_3 \times \mathsf{P}_3$. Some fixed model has an existing exponent in neither S_2 nor P_2. Necessity holds for algebraic interactions in any fixed dimension, under the stated existence and positivity promises.*

Proof. For necessity, isolate the distinct finite algebraic eigenvalues, as in Proposition 9.1. A fixed-thickness inner limit has a uniform $0'$-Cauchy name. Its promised positivity permits computation of its logarithm in that oracle. Proposition 10.1 then gives the three outer ceilings. For attainment choose $\beta \geq \max(1, b/2)$ and apply Lemma 10.2 at level $k = 2$ to $a/(2\beta), b/(2\beta)$. Theorem 12.1 applies. For an existing value outside both preceding sides, use the diagonal real of Lemma 10.3 and a positive rational scaling. □

No uniform gap equality with the main counter cycle is asserted. For example the legal radix vector $(2,2,2,1^{15})$ has a main cycle of length 58, but also a closed free path of length 30 with gap $2 - 2\cos(\pi/30) < 2 - 2\cos(2\pi/58)$. Fix digits $d_0 = d_1 = 0, d_2 = 1$ and all later digits zero. An incrementing head travels from position 3 to 17, wraps, and returns to 3; it can neither enter nor leave through digit $d_2 = 1$. Its forced side symbols are legal throughout. The independent component check also finds a longer free path. The full-counter bracket, exact formula (29), and density exponent do not require identifying the minimizing component.

12.2 A classical moving-corner observable

The same arithmetic level occurs without a quantum layer. In a square tableau use the limit machine above, and index a site by (j,i) back from the *far-time* corner. Its physical time is $L-1-i$. At fixed (j,i) the output tends to $A(j)$ as $L \to \infty$. The $J \times J$ average of that limiting configuration is

$$J^{-2} \sum_{i,j<J} A(j) = J^{-1} \sum_{j<J} A(j).$$

Thus its existing density and ordered endpoints fill the same level-three classes on the allowed interval. Necessity is one pointwise limit of computable finite tableaux followed by the window-size limit. A fixed site measured from the initialized corner would instead remain at fixed computation time. This is a moving-corner boundary-selected observable, not a claim about a canonical translation-invariant bulk state density.

13 A final-time broadcast to three-dimensional bulk counters

The film separates the read length from the available computation time, but its readers lie on an edge. We now retain the unbounded classical computation and distribute its final word to an extensive set of finite quantum readers. All coordinates below are periodic, with $T_L = (\mathbb{Z}/L\mathbb{Z})^3$ and $L = n!$. There is no external boundary term, designated site or size-dependent interaction. The positive-coupling statements in this section concern only

$$0 < g < \bar{g} = g_5 = \frac{11}{480}.$$

Thus all selected ruler pitches satisfy $p \geq 6$. The Hamiltonian itself is defined for every real g.

Fix a set $A \leq_T 0'$ containing $0, 1, 2$, an integer $\beta \geq 1$, and a total computable approximation $a(j, s) \to A(j)$. Use an ordinary one-sided-tape machine with separate work and persistent output tracks. At stage s it computes and writes $a(j, s)$ for all $j \leq s$. Every stage finishes. For each fixed j, its physical output $o_t(j)$ is consequently constant and equal to $A(j)$ after some finite time t_j. Blank output means zero. A later write to an already settled cell has the same value. This does not require any bound on t_j.

13.1 An internally created frame and a protected ruler

We use one precise classical tiling import. The periodic-to-open construction of Cubitt, Pérez-García and Wolf [11, Section 6.3, equation (133), Lemma 55], in the version 5 full text, is an integer-weight nearest-neighbour layer on a square torus with minimum energy four. Every minimizer has a single frame row and frame column, whose complement is a valid modified Robinson tiling of an open $(L-1) \times (L-1)$ square. Every other configuration costs at least five. We import this finite tile set and lemma, not a new assertion about all periodic tilings.

Put this layer in each xy plane and copy its entire tile label on every z bond. If F_z is the frame energy in a plane, then

$$E_{\text{frame}} = \sum_z F_z + \#\{\text{failed tile copies}\} \geq 4L.$$

Equality forces common frame seams at all heights; failure of equality costs at least one. Translation of the frame remains free. Its row and column types define four adjacent-interior flags X_0, X_1, Y_0, Y_1. For example, on an x bond require

$$X_0(\text{right}) = \mathbf{1}_{\{\text{left in frame column, right outside frame column}\}}.$$

The opposite-side test defines X_1, and y bonds define Y_0, Y_1. Each false flag has a unit diagonal penalty. These are tests on actual torus bonds. The frame sites are tensor factors of the physical system, not deleted sites; only the computation rules are disabled across the marked internal edges.

In every xz plane put the five-symbol ruler of Section 2, with its displayed horizontal pairs along x and vertical pairs along increasing z. Add unit y-bond penalties for unequal ruler symbols. Its perfect pitch-p density is $f_g(p) = -g/p + r/p^2$, $r = 1/16$. Write

$$f_* = \min(0, \min_{q \geq 1} f_g(q)), \qquad \mathcal{P}(g) = \{q \geq 2 : f_g(q) = f_*\}.$$

In the stated interval $\mathcal{P}(g)$ is either $\{p\}$ or, at $g = g_p$, $\{p, p+1\}$, with g_p as in (2). The auxiliary $q = 1$ term is useful for defective segments and never minimizes here.

For completeness, the exact charging identity on one square layer is

$$E - L^2 f_* = \sum_s p_s(f_g(p_s) - f_*) + r \sum_s (v_s - 1/p_s) + V_{\mathrm{err}} - A_0 f_*. \tag{31}$$

Here segments start at a W, v_s indicates an ending VW bond, V_{err} counts forbidden pairs, and A_0 is the area of W-free columns. In particular $-A_0 f_*$ is retained in the equality. It is nonnegative and may then be dropped for a lower bound. The segment/run proof of Lemma 2.2 applies literally: a vertically bad segment has deficit at least -1, a good open run has deficit greater than -1, and a closing run has zero deficit. A vertical violation receives at most three bad-segment and three run-end charges, while a horizontal violation receives at most three run-end charges. Therefore

$$E - L^2 f_* \geq \sum_s p_s(f_g(p_s) - f_*) + \frac{5}{8} V_{\mathrm{err}}. \tag{32}$$

This controls a putative extensive gain from defects, without a small-perturbation estimate in the volume.

Define the nonoptimal stripe separation by excluding *every* minimizing pitch:

$$\delta(g) = \min_{q \geq 2,\ q \notin \mathcal{P}(g)} q(f_g(q) - f_*) > 0.$$

Indeed $q(f_g(q) - f_*) = q|f_*| - g + r/q \to \infty$. At rational g, a tested value M excludes all q with $q|f_*| - g > M$, so the search is finite and effective. A sufficient fixed-g size cutoff includes

$$L\delta(g) \geq 5/8, \qquad L^2|f_*| \geq 5/8, \qquad q \mid L \quad (q \in \mathcal{P}(g)). \tag{33}$$

The second condition excludes the uniform W-free vacua. Summing the layer estimates over y, then adding copy penalties, leaves every nonminimal three-dimensional ruler at least $5/8$ above its minimum. The cutoff depends on g and is not asserted uniform near zero.

Add a Boolean marker v constant on z and y bonds, equal to the indicator of ruler symbol V on non-W sites. On a W site in the frame column require $v = 1$. In a clean stripe of pitch q there are L/q marked columns. Constancy of v makes these columns common to all stripes and all y planes, so different pitches cannot coexist in a zero-penalty configuration. Conversely, any uniform pitch dividing L can be translated so that one of its V columns is the frame column. At a tie both uniform pitches remain attainable; their mixtures incur at least one auxiliary penalty. The frame-column condition is gated by W and imposes no divider in a W-free vacuum.

13.2 Three binary fields broadcast a periodic word

Lemma 13.1 (Nearest-neighbour broadcast)**.** *Suppose the ruler walls are $z = z_0$ modulo p, with $p \mid L$. Introduce bits B, C, D on every site, including frame and wall sites, and impose*

$$\begin{aligned} B(x, y+1, z) &= B(x, y, z), & B(x, y, z+1) &= B(x, y, z), \\ D(x, y, z) &= C(x-1, y, z), & C(x, y, z+1) &= D(x, y, z), \\ C(x, y, z) &= B(x, y, z) & &\text{on a } W \text{ plane.} \end{aligned}$$

The solutions are in bijection with arbitrary p-periodic words $b(x)$, with $B(x, y, z) = b(x)$. Each word has a unique C, D filling.

Proof. The first two equalities make B independent of y, z. Starting at a wall, the next two give

$$C(x, y, z_0 + t) = b(x - t), \qquad D(x, y, z_0 + t) = b(x - t - 1).$$

At the next wall $C = B$ says exactly $b(x - p) = b(x)$. Conversely these formulas fill each stripe consistently and close around the torus because $p \mid L$. There is one free bit for each of the p residues. □

Every equality is onsite or on one ordinary oriented bond. The apparent diagonal shift is split into an x bond and a z bond by D. There is no entire word stored onsite. On noncommensurate periodic test graphs the free dimension can instead be $\gcd(L, p)$, identifying different source residues. Thus this lemma is not an all-size copying assertion. Factorial tori eventually accommodate both pitches at each fixed tie.

13.3 Two prefix masks and the final-time source

Introduce an inert/active bit I, with unit equality penalties along all three axes. Inert sites allow only ruler symbols A, D and fix all computation, mask and broadcast fields to prescribed dummy values. The frame and its geometric flags remain. Active sites enable the following predicates. Mixed phase bits are ordinary classical configurations with positive penalties, not excluded sectors.

Let F_z flag the non-W row immediately after a W row, and T_z the non-W row immediately before the next W. Let F_x flag the non-V site immediately after a V column. Each flag has a defining two-site truth table. At pitch p, each stripe has height $h = p - 1$. Two copies of the Boolean thickness counter from Section 12 will mark

$$\ell(p) = \text{bitlength}(p - 2).$$

Within its own horizontal interval initialize $b_j = f_j = 0$ on F_z and carry $c_0 = 1$ at the first site. The rules are

$$c_{j+1} = b_j c_j, \qquad b'_j = b_j \text{ XOR } c_j, \qquad f'_j = f_j \text{ OR } b_j. \tag{34}$$

The first is an x-bond predicate; the latter two are z-bond predicates. At stripe row t the bits encode t in binary and f_j records whether bit j was previously one. At T_z, $t = p - 2$, so

$$b_j \text{ OR } f_j = \mathbf{1}_{\{2^j \le p-2\}} = \mathbf{1}_{\{j < \ell(p)\}}. \tag{35}$$

The *local* copy restarts after each V column, has width h, and has dummy b, c, f fields on V and W. Carry is enforced only inside that interval; increment is enforced only between non-W rows. Copy the top indicator into P_{cell} along *all* z bonds, including through W, and set it to zero in V columns. Do not make P_{cell} dummy on W: its vertical copy is needed there. Alignment makes the repeated top indicators agree, so it selects the first $\ell(p)$ sites of every local cell.

The *global* copy restarts only at X_0 and spans the entire frame interior of width $L - 1$, without restarting at V. Initialize it on each F_z; impose its carry only inside the frame interior and its increment only inside the stripe. Copy its top indicator in P_{global} within that non-W stripe only, using dummy zero on W and the frame column. All other unused counting fields are dummy. The resulting floor mask selects precisely the first $\ell(p)$ tape indices after the frame column. Since that column is a V column, these are also local indices $j = 0, \ldots, \ell(p) - 1$ in the first cell. The identity holds after every translation and coordinate wraparound. Moreover $\ell(p) < p - 1$ for $p \ge 6$, leaving an actual nonreader neighbour after the last reader site.

At every stripe floor F_z run the fixed machine in the internally open xy square, using x as tape and y as time. Its rows are $0, \ldots, L - 2$ and its width is $L - 1$. The two-site arrival compiler

of Section 5 applies: horizontal predicates fix the incoming head signals from neighbours, and vertical predicates fix the next tape, head and persistent output from that finite onsite tuple. Both missing-edge arrival fields are blank, including fields in the final row; all inactive machine fields are dummy. Work and output tracks are distinct. Initialize the head at X_0, Y_0. At physical time t its position is at most t, so an included transition cannot require tape beyond the right edge. Functional induction gives the ordinary infinite-tape computation on all included rows. The divider columns do not cut this giant tableau.

Impose the onsite source equality only at $F_z \wedge Y_1 \wedge P_{\mathrm{global}}$:

$$B = o_{L-2}(j), \qquad 0 \le j < \ell(p). \tag{36}$$

Also impose $B = 0$ wherever $P_{\mathrm{cell}} = 0$. There is no equality between B and the machine output at earlier time rows. Computations on different stripe floors have the same initialization and final time, so their demands agree. Lemma 13.1 now forces this final prefix, repeated in every cell throughout the volume, with zeros on the other residues.

Lemma 13.2 (Simultaneous completion)**.** *For each minimum frame and each aligned uniform minimum pitch dividing L, the active auxiliary fields have a zero-penalty filling. Every such filling has the same repeated radix word $[2^{\beta o_{L-2}(j)} : j < \ell(p)]$.*

Proof. Fill the two thickness counters functionally, copy their top masks with their respective boundary conventions, and fill each identical machine tableau. Set the p-periodic word to the final prefix and zeros elsewhere. Its C, D extension exists by the broadcast lemma. All source, mask and alignment predicates are satisfied. Conversely, functional counters and tableaux fix both masks and the source prefix, and the zero mask fixes every remaining residue. The broadcast lemma fixes C, D. The Robinson interior need not be unique; it does not enter the radix predicates. □

All added constraints have unit nonnegative weight. Any Boolean fact derived from a neighbour is stored onsite and checked on its defining bond. Later predicates use only that stored fact and other onsite fields. Hence the compiler uses a fixed finite alphabet and only onsite or nearest-neighbour rational projectors. Neither an oracle nor a volume-dependent register enters an interaction coefficient.

13.4 The full Hamiltonian and its protected positive gap

Activate the guarded counter of Section 6 on every x segment with $I = 1$, $T_z = 1$ and $P_{\mathrm{cell}} = 1$, for every y, including the internal frame row. Its first-site rule uses F_x and its radix is $2^{\beta B}$. Wrap uses the actual bond from the final reader site to its nonreader neighbour. Freeze every exterior dynamic field to $(0, \mathsf{NL})$ with a unit penalty. Retain all parallel local edges. On inconsistent classical labels, each controlled transition is still a nonnegative edge Laplacian times a classical projector. Thus every quantum block remains nonnegative.

Put a qubit at each site as well. When $I = 1$, penalize its state $|1\rangle$ by one. On a y bond with both endpoints inert put

$$(|01\rangle - |10\rangle)(\langle 01| - \langle 10|),$$

and otherwise no qubit bond term. Inert counter fields are frozen. This completes a fixed nearest-neighbour family with rational coefficient matrices, affine in g only through the ruler bonus.

Lemma 13.3 (Eventual distinct-level identity)**.** *For each fixed $0 < g < \bar{g}$, for all sufficiently large $L = n!$,*

$$\Delta(L, g) = \min_{q \in \mathcal{P}(g)} d_q, \qquad d_q = \Delta^+\big(K[2^{\beta A(j)} : j < \ell(q)]\big). \tag{37}$$

The threshold is finite but has no general computable bound. For $g \le 0$, $\lim_n \Delta(n!, g) = 0$.

Proof. All classical fields are diagonal and unchanged by quantum moves, so

$$H_L(g) = \bigoplus_{\sigma} (E_{\mathrm{cl}}(\sigma; g)\mathbf{1} + K_\sigma), \qquad K_\sigma \ge 0. \tag{38}$$

The frame and ruler lower bounds, together with zero auxiliary penalties, are simultaneously attained by Lemma 13.2. Quantum kernels are nonempty. Consequently the ground energy is exactly

$$E_*(L, g) = 4L + L^3 f_*.$$

After (33), any nonminimum frame costs at least one and any nonminimum ruler costs at least $5/8$. A failed flag, tableau, copy, alignment or phase equality costs an integer at least one. The bare ruler never lies below its own minimum, so a changed phase cannot cancel such a penalty. An all-inert configuration cannot attain $f_* < 0$. Every nonminimizing classical block therefore starts at least $5/8$ above E_*. Nonnegativity of K_σ excludes even an arbitrarily long fake reader in those blocks; superposition across diagonal blocks cannot lower an offset.

In a minimum block of uniform pitch q, the number of independent identical readers is

$$N_L = (L/q)(L/q)L = L^3/q^2. \tag{39}$$

The factors count stripes in z, intervals in x, and all y rows. Other quantum factors have unit gaps. The first positive eigenvalue of a tensor sum with nonempty factor kernels is the least positive factor gap, even when the kernels are degenerate.

Set $T(q) = \max_{j<\ell(q)} t_j$. For $L - 2 \ge T(q)$ the finite radix word is exactly the one defining d_q. There are only one or two pitches to settle. With $k_q = \beta|A \cap [0, \ell(q))|$, Lemma 6.1 gives

$$\frac{4^{-k_q}}{64\ell(q)^2} \le d_q \le 4\pi^2\, 4^{-k_q} \le \frac{\pi^2}{16} < \frac{5}{8}. \tag{40}$$

The last step uses $q \ge 6$, $\beta \ge 1$ and $0, 1, 2 \in A$. Thus the full ground-space gap is supplied by a reader. At a tie alignment retains the two uniform classical minima and removes their mixtures. Taking the least positive energy over both ground blocks gives $\min(d_p, d_{p+1})$, as claimed. No equality between d_q and the main counter cycle is used: closed free paths, zero-head states, penalized components and multiple-head sectors are all covered by Lemma 6.1. The legal path counterexample in Section 12 still applies.

For $g \le 0$ the bare ruler minimum is zero, attained by a uniform A or D vacuum. With $I = 0$ and dummy auxiliaries the frame minimum $4L$ is attained, and all quantum terms are nonnegative. The inert qubit layer has the all-zero ground vector. On one of its y rings the single-excitation Fourier vector has positive eigenvalue

$$\epsilon_L = 2 - 2\cos(2\pi/L) \quad (L \ge 3).$$

Therefore $0 \le \Delta(L, g) \le \epsilon_L \to 0$. Only this positive upper witness is needed, without an assertion about the ring's full gap or a unique ground vector. □

The machine settling time in this proof need not be computably bounded as q varies. The geometric cutoff is effective at rational g; the further settling cutoff is not. The volume limit is taken at fixed g before the pitch grows. In particular it is not an infimum over the machine's intermediate guesses.

13.5 The critical law and every prescribed third-level value

The logarithm of (40) yields

$$2k_p - \log_2(4\pi^2) \le \log_2(1/d_p) \le 2k_p + 6 + 2\log_2 \ell(p).$$

Since $\ell(p) = \log_2 p + O(1)$ and $p = 2r/g + O(1)$, off ties,

$$\frac{\log(1/\Delta_\infty(g))}{\log(1/g)} = 2\beta\rho_{\ell(p)}(A) + O\Big(\frac{\log \ell(p)}{\ell(p)}\Big). \tag{41}$$

Every sufficiently large pitch, and every sufficiently large bitlength, is visited. At a tie the inverse logarithm is the larger of the neighbouring values. Both have the same density limit and vanishing error, including when the bitlength increases. Thus if A has density ρ, the exponent is $2\beta\rho$.

Here is the needed density realization explicitly. For a finite $\alpha \ge 0$ computable in $0''$, choose a fixed integer $\beta \ge \max(1, \alpha/2)$ and put $\rho = \alpha/(2\beta)$. The relativized limit lemma supplies a $0'$-computable rational sequence $u_s \in [0,1]$ tending to ρ. Starting with $N_0 = 1$, put $N_{s+1} = (s+2)N_s$ and on $[N_s, N_{s+1})$ use the mechanical word of slope u_s: its first t bits have sum $\lfloor tu_s \rfloor$. For an intermediate prefix the density differs by at most $1/(N_s + t)$ from

$$\frac{N_s\rho_{N_s}(A) + tu_s}{N_s + t}.$$

At the endpoint its difference from u_s is at most $1/(s+2) + 1/N_{s+1}$. Endpoints converge to ρ, and the displayed convex combination then proves convergence of all intermediate densities. The resulting set is computable in $0'$; changing its first three bits to one preserves the density. The ordinary limit lemma gives the total computable approximation used by the machine. The ordinary algorithm for that approximation, rather than an oracle, is encoded in the finite local alphabet. Equation (41) proves the required exponent α.

The gap closes even for $\alpha = 0$. The orbit in (20) has length at least 2ℓ, so its upper witness gives $d_p \le \pi^2/\ell(p)^2 \to 0$. The ground energy per volume is zero on $g \le 0$, and for small $g > 0$ it is f_*. Indeed the frame contribution is $4/L^2 \to 0$, and completing the square gives

$$-\frac{g}{p} + \frac{r}{p^2} = -\frac{g^2}{4r} + r\left(\frac{1}{p} - \frac{g}{2r}\right)^2.$$

Rounding p near $2r/g$ makes the last term $O(g^4)$. The two different one-sided quadratic behaviors preclude real analyticity at the computable point zero. Together with the gapless inert side and the positive gaps closing from the right, this proves the critical-point assertion.

For necessity, at rational g a finite torus matrix is rational. Multiplicity-aware algebraic root isolation computes its two lowest distinct eigenvalues whenever defined. Replace their gap by rational approximants with effective errors tending to zero. For an existing inner limit a halting oracle finds tail-Cauchy indices, giving a uniform $0'$-Cauchy name. Promised positivity allows a search for a certified interval bounded away from zero in that same oracle, and then computation of the logarithm. One further existing coupling limit is computable in $0''$, as in Proposition 16.1. A computable critical point can be approached by certified rational points on its required side without adding an oracle. No decidability of strict rational cuts of arbitrary computable real arrays is assumed.

Corollary 13.4. *Some fixed three-dimensional family in Theorem 1.2 has an existing positive gap exponent outside* $\mathsf{S}_2 \cup \mathsf{P}_2$.

Proof. Enumerate all Σ_2^0 and Π_2^0 subsets of $\mathbb{Q}$, including noncuts. Oracle $0''$ decides membership of a chosen midpoint in each candidate. Starting inside $[1/4, 3/4]$, choose the left third of the current interval when its midpoint belongs to the candidate, and the right third otherwise. If the candidate is a strict lower cut, its real lies above the midpoint in the first case and at or below it in the second, so the chosen third excludes that real. The nested intervals have a unique $0''$-computable point, different from every listed cut-real. Apply the attainment just proved. □

13.6 Extensive local excitations and a selected infinite-volume phase

Fix $g \in (0, \bar{g})$ and a pitch p attaining (37); at a tie choose a gap-attaining one. In its minimum classical block choose normalized counter vectors $\phi \in \ker K$ and $K\psi = d_p\psi$. Use ϕ on every reader and the prescribed states on all exterior factors, giving a product ground vector Ω_L. Replacing ϕ by ψ at any of the $N_L = L^3/p^2$ reader locations gives mutually orthogonal eigenvectors at $E_* + d_p$, since $\langle \phi, \psi \rangle = 0$. Their support length $\ell(p)$ is independent of L at fixed g. They occupy every y row and a periodic array in xz, including locations arbitrarily far from the frame seams. Linear combinations give extended modes in the same extensive flat band.

More quantitatively, let $a = |\psi\rangle\langle\phi|$ on one reader's dynamic factors and identity on other factors. It has norm one and finite support. If P_d projects onto the first positive energy in this classical block and τ_u denotes spatial translation, then

$$\frac{1}{L^3} \sum_{u \in T_L} \|P_d \tau_u(a) \Omega_L\|^2 \geq \frac{1}{p^2}. \tag{42}$$

Every aligned reader translate contributes one. The bound persists as volume tends to infinity at fixed g; neither its value nor the support size is asserted uniform as $g \downarrow 0$.

To obtain an infinite-volume phase, translate the common frame so its two seams recede from every fixed window, retain a fixed ruler phase and the settled word, and pass to a locally convergent subsequence of the remaining finite-alphabet patterns. Choose the same reader product vectors. Finite-range dynamics converge on local observables uniformly on compact time intervals, by the standard Lieb–Robinson thermodynamic-limit construction [36]. Every finite-volume ground-state excitation spectral measure is supported in $\{0\} \cup [d_p, \infty)$. Its weak limit excludes $(0, d_p)$. Local vectors are dense in the ground-state GNS representation, so its Hamiltonian has no spectrum there. A fixed aligned reader operator retains unit spectral weight at d_p (its time correlation is the corresponding exponential). Thus the gap in this selected representation is exactly d_p above the *entire* zero-energy subspace. At a tie the same argument uses the selected gap-attaining pitch. This completes the proof of Theorem 1.2.

13.7 Gap conventions and effective selection

The positive-density statement is not an identification with every operator-algebraic gap definition. The locally nondegenerate criterion of Xu et al. [47, equation (4), Definition 1.11] requires

$$\omega(a^*[H, a]) \geq \gamma(\omega(a^*a) - |\omega(a)|^2)$$

and their bulk gap takes a supremum over KMS ground states satisfying that criterion. Our readers have local zero-energy degeneracy. Choose orthogonal $\phi_0, \phi_1 \in \ker K$ and the local map $|\phi_1\rangle\langle\phi_0|$ in the product state using ϕ_0. Its energy is zero and its variance one, excluding every positive γ in this state. Already $\mathrm{diag}(0, 0, 1)$ has distinct-level gap one but zero locally nondegenerate threshold. Neither a unique zero-energy vector nor the KMS-state supremum is computed here.

The certification theorem for that other convention therefore supplies no contradictory ceiling for the present result.

There is also a useful sufficient effective-collapse condition. If, at every rational g, a computable procedure supplies the complete finite list of limiting reader operators with their correct radix labels, and a competing-sector threshold above their gaps, then it computes the limiting gap. Its existing exponent is D_2. An effective volume settling bound suffices, but is not necessary if these operators are already known. Here the pitch is computable; its settled word $A \upharpoonright \ell(p)$ is not uniformly computable in general. The finite periodic quantum readers do not provide an effective list of their limiting classical labels.

Two obstructions explain the construction choices. Retaining independent readers for every previous computation stage would give $\min_{s \le s(L)} d_s$, tending to $\inf_s d_s$, a right-c.e. quantity for computable stage gaps. The history $1/8, 1/2, 1/2, \dots$ would retain $1/8$, not the final value $1/2$. The final-time source withdraws old guesses instead. If the union $\mathcal{A}_L$ of all available reader supports were confined to one seam, a norm-bounded local operator of radius R would have spectral weight into that factorized band at most $\|a\|^2 |\mathcal{A}_L^{(+R)}|/L^3$, tending to zero. The relevant union is over spatially different reader locations, not the degeneracy of a single reader. Equation (42) establishes the extensive union here.

Finally, coupling equicontinuity alone does not compute a volume limit even when it is pointwise eventually exact. For the same computable approximation let

$$k_{n,s} = \sum_{j<n} a(j,s), \qquad q_n = 2^{-n}, \qquad y_{n,s} = 2^{-n-k_{n,s}}.$$

On $[0,1]$, let $F_s(0) = 0$, interpolate the knots $(q_n, y_{n,s})$ for $0 \le n \le s$, and join the last knot to zero linearly. A knot-interval slope is

$$2^{-k_{n,s}}\bigl(2 - 2^{-a(n,s)}\bigr) \le 3/2;$$

the terminal slope is at most one. These uniformly computable piecewise-linear functions have a common Lipschitz constant. At each fixed $g > 0$ their values eventually settle. But the limiting value at q_n identifies k_n among the finitely many separated powers of two, and successive k_n recover A. A computable limit would therefore compute A. At the knots its normalized inverse logarithm is $1 + \rho_n(A)$. Neighbouring knot ratios lie between two and four, so interpolation changes this by $O(1/n)$. When the density exists the exponent is $1 + \rho(A)$, potentially outside both one-sided level-two classes. This is a scalar-data example, not another Hamiltonian realization. Effective boundary-to-volume brackets for energy density remain an additional argument.

13.8 Finite checks and the exact scope of the theorem

The ancillary programs `check_bulk_broadcast.py` and `check_bulk_independent.py` implement separate finite checks. The first checks literal copying components, both prefix counters, exact ruler minimization and small-torus charging, tie alignment, the inert-ring witness, tensor gaps and the scalar modulus example. The second uses independent binary row elimination on translated geometries and all tested source fillings, forces contradictions from changed source bits, checks defective rulers and complete native counter products, and reproduces the long legal free paths and degenerate-kernel tensor sums. It also checks tie arithmetic, physical endpoint guards and the reader count. The expected noncommensurate collapse and failure of universal main-cycle equality are included as counterexamples to those stronger assertions.

These programs do not generate the full onsite matrix for an arbitrary target machine, diagonalize a three-dimensional many-body Hamiltonian, reprove the imported frame lemma, or numerically

certify a noncomputable limit. Floating-point trigonometric checks are corroboration, not interval-certified proofs. The finite-rule compiler and block argument supply the general statements. There is no conclusion here about all integer sizes, robustness, unique ground states, a separate correlation length, minimum spatial dimension, or an extremum over all infinite-volume phases. The classical computation still uses an internal anchor. Positive-coupling inner limits outside $(0, \bar{g})$ are not claimed.

14 A surface-anchored wetting ruler

The film engine can also read a coupling-selected thickness. Here the volume convention requires particular care at height ties. We use open cubes with $L = n!$ for the exact all-coupling statement. Away from height ties the same inner gap identity holds along all sufficiently large open cubes. The anchors and closing modes are on faces and an edge; this is not an intrinsic bulk-gap realization.

14.1 Local geometry and energy

Use the six face flags of Section 12 and a wetting track with symbols F, A, S, D, V, W, O, denoting floor, signal surroundings, signal, divider, ceiling and vacuum. Require F exactly at Z_0. Allowed x pairs are

$$FF,\ WW,\ OO,\ AA,\ SA,\ DS,\ DD,\ AV,\ SV,\ VS,\ VD.$$

Allowed z pairs, written in order of decreasing z, are

$$WS,\ WA,\ WV,\ SF,\ DF,\ VF,\ AA,\ AS,\ SD,\ DD,\ VV,\ OW,\ OO.$$

Allowed y pairs are FF, WW, OO and every pair within $\{A, S, D, V\}$. Require the far Z_1 face to carry O or W. Every other pair or onsite pattern costs one. The coupling contributes g per interior symbol A, S, D, V, and a VF bond costs $r = 1/16$. All interactions are rational and affine in g.

In a clean configuration the floor is $z = 0$, a flat ceiling is at $z = h + 1$, $1 \le h \le L - 2$, and vacuum fills the space above it. The signal cycle cuts each (x, z) interior slice into periods $h + 1$, with a free horizontal phase in each y plane. The floor itself is not cut. The energy of one such plane, after subtracting the flag energy, is

$$E_L(h) = Lgh + r \left\lfloor \frac{L}{h+1} \right\rfloor \tag{43}$$

for an optimal phase; an extra divider costs another r. The limiting plane energy per width is

$$\phi_g(h) = gh + \frac{r}{h+1}, \qquad t_h = \frac{r}{(h+1)(h+2)}, \qquad \phi_g(h) - \phi_g(h+1) = t_h - g. \tag{44}$$

Thus off ties $h_*(g) = h$ on (t_h, t_{h-1}) and $h_*(g) = (r/g)^{1/2} + O(1)$.

Lemma 14.1 (Wetting selection with commensurate open cubes)**.** *Fix sufficiently small positive rational g. Along sufficiently large $L = n!$, the classical minimizing heights are exactly the minimizers of ϕ_g. There is one such height off ties, and two adjacent heights at a tie. All other classical configurations have a positive separation independent of L. The same assertion off ties holds along all sufficiently large L, with a possibly smaller positive separation than $1/2$.*

Proof. The flag identity puts every wrong flag at unit cost. Decompose a fixed y plane into maximal clean runs of columns with equal floor and ceiling heights and no violated intervening x bond. A run

of m columns has interior cost mgh and at least $\lfloor m/(h+1) \rfloor$ divider corners. Its energy is therefore at least $m\phi_g(h) - r$. A bad column or broken horizontal bond terminates a run. For $0 < g \le 1/8$, a bad column has unit error cost and nonnegative interior and corner energies, whereas the minimum plane energy per width is at most $\phi_g(1) \le 1/4$. Charging each extra run to its terminating error gives

$$E(R_y) \ge L\phi_* - r + (3/4 - r)(\#\text{bad columns} + \#\text{broken } x \text{ bonds}).$$

Additional y violations only raise the energy. Correct flags and clean rules are attainable at an optimal height and phase, with energy at most $L\phi_*$ per plane. A defective plane consequently costs at least $5/8$ relative to that comparison. Flag errors outside the plane decomposition are also nonnegative and charged once.

At fixed nontie g, put $\delta(g) = \min_{h \ne h_*}(\phi_g(h) - \phi_*) > 0$. The minimum is effective because $\phi_g(h) \to \infty$. A clean wrong-height plane lies at least $L\delta(g) - r$ above an optimal one, so it is excluded for large L. A nonoptimal phase at the selected height may cost only r and must not be assigned a universal $1/2$ separation. Changing ceiling heights between y planes creates violated y bonds; the same positive comparison excludes that case.

Along factorial sizes all minimizing periods $h+1$ divide L. Their clean row energies are exactly $L\phi_*$, also at ties. For any other fixed height, the same $L\delta - r$ estimate applies; the finitely many low candidate heights can be included in the factorial divisibility threshold. Heights beyond them have an increasing positive bulk excess. This proves a positive uniform separation from the set of minimizing clean configurations. □

For the quantum readout introduce an additional classical bit C. Set $C = 0$ at Z_1 and at every noninterior symbol. On a z bond whose lower endpoint is interior, require its C to equal the indicator that the upper endpoint is W. This is a two-site diagonal test and uniquely determines C everywhere in a clean configuration.

The machine runs on the uncut floor with tape x and time y. Use the limit program for a Δ_2^0 set A. The thickness counter starts at the floor, the output is transported through the film, and the quantum segment is selected onsite by $Y_1 C(b \,\mathrm{OR}\, f)$. Its length is $\ell(h) = \mathrm{bitlength}(h)$ and its radices, after settling, are $2^{\beta A(j)}$. Freeze all unused machine, transport and dynamic fields, with blank output interpreted as zero. This is exactly the functional film compiler with its far thickness face replaced by the selected ceiling; it does not truncate the floor computation.

Theorem 14.2 (Wetting transfer)**.** *For any Δ_2^0 set A containing $0, 1, 2$ and integer $\beta \ge 1$, this rational affine family has, for every sufficiently small fixed $g > 0$,*

$$\Delta_\infty(g) = \min_{h \in \operatorname{argmin} \phi_g} \Delta^+(K[2^{\beta A(j)} : j < \mathrm{bitlength}(h)]), \tag{45}$$

along factorial open cubes. The lower and upper gap exponents are β times the lower and upper densities of A. Every nonnegative finite D_3 value and ordered $\mathsf{S}_3 \times \mathsf{P}_3$ pair occurs. For $g \le 0$ close to zero the gap tends to zero. The bulk energy density is g for $g < 0$ and zero for $g > 0$, with a kink at zero.

Proof. Initialize the first three output cells permanently to one before running the approximation program, as in Section 12. For every read length $\ell \ge 3$ the first three radices are then at least two. The free orbit has $N \ge 2(8 + 4 + 2) = 28$, so its gap, and hence the full counter gap, is at most $4\pi^2/N^2 \le \pi^2/196 < r$. This is below the conservative classical separation r of Lemma 14.1. Equality with the orbit gap is not required. At each fixed selected height only finitely many output bits are read. They settle as $L \to \infty$, so the full-counter identity and the all-sector proof of Theorem 12.1

give (45). The settling time is not generally computable from g; it is not the computable classical selection threshold.

Since $\log(1/g) = 2\log h + \log(1/r) + o(1)$ and $\log_2(1/D(h)) = 2\beta\ell(h)\rho_{\ell(h)}(A) + O(\log \ell(h))$, the normalized gap logarithm is $\beta\rho_{\ell(h_*(g))}(A) + o(1)$. Every sufficiently large integer height occurs. Adjacent heights at a tie have either the same read length or lengths differing by one, which gives the same endpoint limits. Apply Lemma 10.2 with densities $a/\beta, b/\beta$ and $\beta \geq \max(1, b)$. Proposition 16.1 gives the matching ceiling.

For $g = 0$, the lowest clean corner count is one and minimizing heights satisfy $h + 1 > L/2$, equivalently $h \geq \lfloor L/2 \rfloor$. For fixed negative g sufficiently close to zero, the maximal height $L - 2$ minimizes once $L|g| > r$. The same run charging uses $1 + gL - \phi_g(L - 2) \geq 3/4$ for sufficiently small $|g|$ to exclude defective competitors. In either case the minimizing quantum segments have diverging bitlength, and their main-orbit upper bound is $\pi^2/\ell^2 \to 0$, independent of the guessed bits. Filling the cube at negative g gives energy density g; a bounded-thickness film at fixed positive g has only surface energy and density zero. The nonnegative dynamic terms have kernels, so they do not change these densities. □

The volume convention is substantive. At the tie $t_7 = 1/1152$, formula (43) selects height 8 at $L = 72n + 8$ and height 7 at $L = 72n + 9$. Their read lengths differ, and a legal word can give different full counter gaps. Thus the all-size open-cube gap limit need not exist at a tie. Neither an all-size identity there nor a universal classical separation of $1/2$ is used. The factorial convention removes this residue obstruction without changing rational coefficients or the coupling exponent.

At a single fixed coupling the eventual counter is one finite algebraic matrix and its gap is individually computable. The failure is uniform in the coupling. To prove that no general computable scalar-gap settling modulus exists, choose a realized exponent outside D_2. A uniform computable inner-gap modulus would make the scalar gaps uniformly computable and Proposition 16.1 would put this exponent in D_2, a contradiction. No injectivity of the map from radix vectors to their scalar gaps is assumed.

15 Dependent nested limits at every finite level

For $k \geq 1$ define

$$Q(M) = \lim_{L_1} \cdots \lim_{L_{k-1}} \Delta(M, L_1, \ldots, L_{k-1}), \qquad z_M = \frac{\log(1/Q(M))}{\log M}. \tag{46}$$

The rightmost limit is taken first. Require every intermediate limit to exist at each fixed tuple of remaining parameters. Require the final $Q(M)$ to be finite and positive for every sufficiently large M; earlier intermediate limits need not be positive because their logarithms are not taken.

Theorem 15.1 (Nested-box classification)**.** *For algebraic finite-range interactions, finite lower, upper and existing exponents in* (46) *have ceilings* $\mathsf{S}_{k+1}, \mathsf{P}_{k+1}, \mathsf{D}_{k+1}$ *respectively. Every ordered nonnegative finite pair in those endpoint classes is realized by fixed integer onsite and nearest-neighbour interactions in dimension* $k + 1$*. The local dimension and the finite program may depend on the pair and on* k*.*

Necessity. The finite spectra have uniform computable names. Applying the D_m limit rule inductively gives a uniform $0^{(k-1)}$-Cauchy name for $Q(M)$. Positivity permits the logarithm in the same oracle. The final lower, upper or existing limit has the asserted ceiling by Proposition 10.1. □

Here is a finite local-rule specification completing realization for every fixed $k \geq 3$. The cases $k = 1, 2$ are the fast finite-size engine and Theorem 12.1 respectively. Let A be computable in $0^{(k-1)}$. By the iterated limit lemma there is a total computable binary function

$$A(j) = \lim_{s_1} \cdots \lim_{s_{k-1}} R(j, s_1, \ldots, s_{k-1}),$$

with every intermediate bit limit existing. Set $R = 1$ at $j = 0, 1, 2$. Use coordinates

$$(x, y, w_1, \ldots, w_{k-2}, z), \qquad \text{side lengths } (L_{k-1}, L_{k-1}, L_1, \ldots, L_{k-2}, M).$$

The following fields and predicates are finite for fixed k.

1. Give every axis two exact face flags, with the flag identity of (28) in dimension $k + 1$. For each $a \in \{w_1, \ldots, w_{k-2}, z\}$ introduce its own (b_a, c_a, f_a) counter on every (x, a) slice. Initialize $b_a = f_a = 0$ at $a = 0$, set $c_a = 1$ at $x = 0$, and use the same carry and vertical recursion. No copying along the other axes is needed: each slice is uniquely determined from its own boundary initialization.

2. The only computation plane is the conjunction $P = Z_0 \bigwedge_i W_i^1$. On P use the deterministic machine compiler of Section 5; off P fix every tape, head, output and arrival field to a unique dummy. Machine bonds require P at both endpoints and the two missing-neighbour arrival conditions are imposed.

3. On P, expose the read-only pair of bits $(b_{w_i}, b_{w_i} \text{ OR } f_{w_i})$ for each i. At the far face the second track marks exactly the prefix of length bitlength$(L_i - 1)$. Scanning to its first zero gives a delimiter, and the first track then specifies $L_i - 1$ in least-significant-first binary order. Zero has its delimiter at the first cell. Merely supplying an undelimited infinite string of zero-padded bits would not let a finite-time machine recognize the input's end.

4. After parsing the inputs, the ordinary machine computes $R(j, L_1 - 1, \ldots, L_{k-2} - 1, s)$ for $j = 0, \ldots, s$ at stage s and writes the answers on its persistent output track. Every finite computation finishes. A small box not exposing a delimiter still has a unique finite computation prefix; completed output is only asserted for sufficiently large innermost squares.

5. On $P_w = \bigwedge_i W_i^1$ transport the output from Z_0 along z, and fix the transport field to zero off P_w. The only quantum segment is $P_w Y_1 Z_1 (b_z \text{ OR } f_z)$. Its radix uses the transported output; every other quantum site is frozen. Use the same region guards, endpoint tests, penalties and weighted transitions as the film. For eventual boxes the segment has an x neighbour outside the region, so wrap is still two-site.

All tests are onsite or on an oriented nearest-neighbour bond. Flags, independent counters, machine and transport are determined in that order, which proves uniqueness of the correct classical configuration. Any other classical block costs at least one; its dynamic part is nonnegative. Thus the sector proof is unchanged in higher dimension.

Fix M and every outer size. For the finitely many read positions $j < \ell(M)$, the innermost bit limit exists, and each bit is eventually constant. The entire radix vector is therefore eventually constant in L_{k-1}. Its gap is less than one once the first three bits are read, and the innermost gap limit is precisely that full counter gap. Let the next size grow and repeat: each next intermediate bit limit again stabilizes a finite vector. Induction in the declared order gives

$$Q(M) = \Delta^+(K[2^{\beta A(j)} : j < \ell(M)]).$$

This uses eventual constancy of finite discrete vectors, not continuity of an eigenvalue through degeneracy or a uniform settling modulus. The counter bracket gives $z_M = 2\beta\rho_{\ell(M)}(A) + o(1)$. Choose the target density pair with Lemma 10.2 and an integer $\beta \geq \max(1, b/2)$. This completes realization and the proof of Theorem 15.1 at the finite-alphabet compiler level.

The dimension $k + 1$ is a sufficient construction dimension. Minimality is not proved. Nor does the theorem supply one local dimension for all levels or all prescribed targets. A nondecreasing unbounded sampling function that attains all sufficiently large integers through unit steps preserves these density endpoints; arbitrary sparse sampling need not do so.

16 Ceilings, analytic obstructions and effective collapse

Proposition 16.1 (Coupling-exponent ceilings)**.** *Suppose finite-volume gaps at rational g are uniformly computable, defined eventually in volume, and their inner limits exist and are positive at every sufficiently small positive rational g. For finite endpoints in* (1)*, the lower, upper and existing ceilings are $\mathsf{S}_3, \mathsf{P}_3, \mathsf{D}_3$. If the inner limit has a computable error modulus, or a computable eventual-exactness threshold, these reduce respectively to $\mathsf{S}_2, \mathsf{P}_2, \mathsf{D}_2$. If instead the positive inner gap is uniformly right-c.e., the ceilings are $\mathsf{S}_3, \mathsf{P}_2, \mathsf{P}_2$.*

Proof. An existing inner limit has a uniform $0'$-Cauchy name. Promised positivity lets that oracle find a rational interval bounded away from zero and compute its logarithm. The normalization is then uniformly D_2. Tail infima and suprema over rational couplings give the same relativized lower/upper cut arguments as Proposition 10.1: use a rational buffer and quantify over all rational g in a sufficiently small punctured neighbourhood. An existing outer limit is computable in the next jump by a tail-Cauchy search. This proves the first ceilings. Effective inner convergence computes the gap without the first jump, proving the second. A positive right-c.e. gap has a *left*-c.e. inverse logarithm, so the S_1 row proves the third. □

An infimum over an effectively enumerated adequate family of computable excitation tests is right-c.e. This observation applies only after the variational representation is established for the particular gap definition. The certified upper approximations of Xu et al. [47] concern their operator-algebraic bulk gap; they do not identify it with an open-box, film or commensurate-torus limit. The one-sided ceilings above are bounds, without an attainment claim for intrinsic bulk dispersion.

Proposition 16.2 (Finite analytic obstruction)**.** *If a finite-dimensional Hermitian matrix depends real-analytically on g near g_c, and its gap above the entire ground space is defined to the right, that gap has a nonnegative integer exponent.*

Proof. Rellich's theorem gives analytic eigenvalue branches [27, Chapter II, Section 6]. Two branches either agree identically or have isolated intersections. Finitely many branches therefore have a fixed ordering of distinct values on some $(g_c, g_c + \epsilon)$. The gap there is the difference of two analytic branches, nonzero on that interval. Its Taylor series starts with $c(g - g_c)^m$, $c > 0$ and integer $m \geq 0$. Hence its logarithmic exponent is m. □

A noninteger envelope cannot equal the gap of one fixed finite analytic system throughout a right neighbourhood. In the ruler it is the increasing cell size that evades this obstruction. This argument forces infinitely many effective finite pieces in that representation; it does not force first-order steps in every possible infinite-volume construction.

16.1 Locality and first thermodynamic derivatives

Cutting a large box into side-L boxes removes interactions of norm at most a computable constant times the number of cut bonds. The ground-energy variational inequality yields

$$|e - e_L| \le C/L. \tag{47}$$

The same argument gives a pressure modulus. For Hermitian A, B, Weyl's eigenvalue bounds imply

$$|\log \operatorname{Tr} e^{A+B} - \log \operatorname{Tr} e^{A}| \le \|B\|.$$

The partition function factors across a cut tensor product. Tiling and passing to the limit thus give $|p - p_L| \le C(\beta, h)/L$ uniformly on bounded computable parameter sets. These are two-sided brackets; an unshifted finite-box value need not itself be a lower bound. The modulus describes a box side required for accuracy, not the algorithm's running time.

For a computable convex pressure as a function of an appropriate linear field variable, at a computable interior point u,

$$p'_+(u) = \inf_{t \in \mathbb{Q}_{>0}} \frac{p(u+t) - p(u)}{t}, \qquad p'_-(u) = \sup_{t \in \mathbb{Q}_{>0}} \frac{p(u) - p(u-t)}{t},$$

with t restricted to the domain. The right derivative is right-c.e. and the left derivative left-c.e. At a promised differentiability point they agree; search for upper and lower chord bounds within the requested accuracy. This computes the derivative, uniformly over computable sequences of promised differentiability points, without a given modulus of differentiability.

Corollary 16.3 (Restricted positive-temperature ceiling)**.** *One further normalized limit formed from pressure values or differentiable first thermodynamic densities, at computable sampling and reference data, has lower/upper/existing ceilings* $\mathsf{S}_2, \mathsf{P}_2, \mathsf{D}_2$*. A positive right derivative at a cusp, with inverse-log normalization, instead has ceilings* $\mathsf{S}_3, \mathsf{P}_2, \mathsf{P}_2$*.*

Proof. The locality and convexity arguments compute the first type of finite-scale data. The second type is right-c.e.; its inverse logarithm reverses the side. Apply Proposition 10.1. All logarithms and denominators must be defined with the advertised positivity, and subtracted reference data must have the stated computability. □

These are ceilings, not thermal attainment theorems. They do not cover every correlation, higher response or selected Gibbs-state observable. A noncomputable critical point is additional input and is not included in a theorem for computable reference data.

Proposition 16.4 (Zero-temperature tangent-deficit exponent)**.** *For an affine finite-range family let* g_c *be computable and put*

$$s(g) = e'_+(g_c)(g - g_c) - (e(g) - e(g_c)).$$

Assume $s(g) > 0$ *at the computable sampling points tending down to* g_c *and that* $2 - \alpha = \lim_{g \downarrow g_c} \log s(g) / \log(g - g_c)$ *exists and is finite in the stated rational-coupling sense. Then* $2 - \alpha \in \mathsf{S}_2$ *and* $\alpha \in \mathsf{P}_2$*.*

Proof. Equation (47) computes e, and the interaction norm gives a volume-uniform Lipschitz bound in g, including $e(g_c)$. Ground energy is an infimum of affine functions and is concave. Its right derivative is the supremum of forward chord slopes, so it is left-c.e. Concavity gives $s \ge 0$, and s is uniformly left-c.e. Under promised strict positivity, $\log s$ is left-c.e. Division by the negative computable $\log(g - g_c)$ reverses the side, giving right-c.e. normalized data. Their existing limit is in S_2; subtracting from two gives P_2. □

This defines a particular singular-energy exponent. It matches the usual $e_{\mathrm{sing}} \sim |g - g_c|^{2-\alpha}$ interpretation where that singular term dominates the analytic remainder, for example $1 < 2-\alpha < 2$. It is not an unrestricted theorem about every definition of a specific-heat exponent. Weyl's gap continuity bound has a volume factor $O(L^d)$ and supplies no analogous uniform modulus for the gap. A volume-uniform modulus of dependence on g alone, without an effective approximation mechanism in volume, must not be substituted for an effective inner-limit modulus. For example computable rational approximants $a_L \to \Omega_U$ give $F_L(g) = a_L$ with common Lipschitz constant zero, yet their inner limit is an uncomputable constant. This is a data-level counterexample; the energy-density argument has the additional effective size bound (47).

16.2 Other effective mechanisms and their limits

Finite-range Lieb–Robinson estimates, with a computable initial state and effective constants, give a computable infinite-volume local expectation at every finite computable time [36]. A late-time exponent against a computable reference has the one-limit ceiling. If the reference is itself an untamed limiting value and is used inside the outer normalization, a further dependent limit can restore the next ceiling.

For a nonempty effectively closed subset K of an effectively compact space and computable continuous f, $\min_K f$ is left-c.e.: certify that $K \subset \{f > q\}$ using effective compactness. Effective overtness also semidecides $K \cap \{f < q\} \neq \varnothing$, giving the other side and computability. Ordinary compactness alone does not suffice. These and locality, convexity and variational extrema are useful sufficient mechanisms, not an exhaustive list of all ways physical definitions can simplify.

Higher derivatives require separate control. The convex functions

$$p_n(u) = u^2 + \frac{1 - \cos(nu)}{n^2}$$

converge effectively and uniformly to u^2, but $p_n''(0) = 3$ whereas the limit has second derivative two. Thus effective pressure convergence alone does not justify interchanging second derivatives. Under the ordinary hypotheses giving a second-order expansion, an existing second derivative of a computable function is itself a limit of computable central differences and has a D_2 ceiling. An additional response exponent can then have a D_3 ceiling; no higher level follows merely by writing the derivative as a difference of one-sided values.

If a film uses $\liminf_L$ instead of an existing inner limit, the inner gap is S_2; its inverse-log normalization is P_2. The lower/upper/existing ceilings are then $\mathsf{S}_3, \mathsf{P}_4, \mathsf{S}_3$. No physical sharpness for this alternative convention follows from Theorem 12.1. Range bounds likewise restrict an interval without proving either class preservation or attainment.

16.3 Frustration-free criteria require their hypotheses

Gosset and Mozgunov give local gap thresholds and, in their gapless one-dimensional periodic setting, an $O(n^{-2})$ open-chain upper bound [26]. Lemm and Mozgunov give finite-size criteria with boundaries [30]; their two-dimensional criterion uses a minimum over rhomboidal patches, and general boundaries require the corresponding intersected patch shapes. A square-box exponent alone does not provide those patch or edge estimates. Masaoka, Soejima and Watanabe obtain $z \geq 2$ under their frustration-free correlation hypotheses, including the extended correlations treated there [32]. Lemm and Lucia prove inverse-square critical scaling, up to logarithmic factors, without a ground-state correlation assumption [29, Theorem 1.4 and Remark 1.5]. Their result controls an infimum over specified finite-domain families for one fixed positive finite-range frustration-free

interaction on graphs with the required Euclidean embedding bounds. A square-box estimate alone does not give that domain-family infimum, and periodic boundary conditions require the additional boundary information stated there. These results do not imply an unconditional exclusion interval for the square-box exponents of the present setting.

The specified uniform local decomposition here is frustrated: an onsite flag reward competes with interior bond penalties. For a particular box, adding $4LI$ lets one regroup the flag energy into positive interior and boundary terms. That boundary-dependent regrouping does not identify the original interaction with a uniformly frustration-free family. No general necessity of frustration for every small exponent is proved here.

17 Fine tuning and an explicit instability

A small perturbation being larger than a closing gap does not by itself prove instability. Here a concrete perturbation has a quantitative lower bound in every sector. For each active site's legal digits $\{0, \dots, r_j - 1\}$ let P_j be the complete-graph Laplacian, acting as the identity on its side/head symbol. Put

$$V_\epsilon = \epsilon \sum_j P_j, \qquad 0 < \epsilon \le 1,$$

and let it vanish off the region. Conditioning on the local classical radix label makes this one fixed translation-invariant onsite term. Its per-site norm is at most $\epsilon 2^\beta$, so its norm can be made smaller than any prescribed positive tolerance.

Theorem 17.1 (Onsite perturbation destroys the programmed exponent)**.** *For the two-dimensional bulk ruler and the wetting ruler, this perturbation preserves the classical selection and the ground energy. In the positive-coupling phase its thermodynamic gap satisfies, for constants independent of the selected segment length ℓ,*

$$\min\left(c_{\mathrm{cl}}, 2\epsilon, \frac{2\epsilon\, 2^{-\beta}}{5\ell^2}\right) \le \Delta'_\infty(g) \le \frac{2\pi^2}{\ell^2}. \tag{48}$$

Here c_{cl} is the positive classical-sector separation in the relevant thermodynamic construction. Hence the perturbed gap still closes, but its coupling exponent is zero.

Proof. The perturbation is positive and annihilates zero-head states with uniform legal-digit superpositions. Thus every classical block still has a zero-energy dynamic state, and its classical offset is unchanged. Legal digits and legal one-head side strings form an invariant subspace under every counter transition and the new digit edges. Its complement is entirely penalized and lies at least one up; exterior errors and multiple heads also cost at least one.

Free zero-head sectors are products of complete graphs, with gap at least 2ϵ whenever they have a positive eigenvalue. On the free one-head space, the digit flips connect all legal digit vectors within each of the 2ℓ head classes. Write the counter Laplacian as L_c, digit Laplacian as L_d, and let Q project to functions constant on each class's digit vectors. Then $L_d \ge 2$ on $Q^\perp$. The compressed quadratic form QL_cQ contains the cycle

$$\mathsf{INC}_0, \dots, \mathsf{INC}_{\ell-1}, \mathsf{RET}_{\ell-1}, \dots, \mathsf{RET}_0, \mathsf{INC}_0$$

with every edge weight at least $2^{-\beta}$. Carry and return conditions hold on a fraction $1/r_j$ of digit vectors; wrap and restart close the two ends. Consequently its gap is at least $\gamma = 4\, 2^{-\beta}/\ell^2$. Extra increment edges are nonnegative. This also proves that the enlarged free one-head graph is connected.

For a normalized vector orthogonal to constants write $\psi = \psi_0 + \psi_1$, with $\psi_0 = Q\psi$, and let $x = \|\psi_0\|, y = \|\psi_1\|$. The counter's weighted degree is at most two, so $\|L_c\| \le 4$. The triangle inequality for $L_c^{1/2}$ gives

$$\langle \psi, (L_c + \epsilon L_d)\psi \rangle \ge \max\{2\epsilon y^2, (\sqrt{\gamma}x - 2y)_+^2\}.$$

If $y \le \sqrt{\gamma}x/4$, the second term is at least $\gamma/8$. Otherwise $y^2 \ge \gamma/20$, and the first is at least $\epsilon\gamma/10$. This proves the one-head lower bound $2\epsilon 2^{-\beta}/(5\ell^2)$. Combining the other sectors and the classical separation gives the left side of (48).

For the upper bound take the head-only function $\cos(\pi(i + 1/2)/\ell)$, equal on both rails and constant in digits. It is orthogonal to constants, digit flips have zero energy, and the rung edges have zero energy. The remaining path Rayleigh quotient is at most $2\pi^2/\ell^2$. Finally $\ell = \Theta(\log(1/g))$ for either ruler. The normalized logarithm of both bounds tends to zero. Ties do not alter this estimate when the relevant thermodynamic gap is defined. □

This is an existence theorem for destabilizing perturbations, not a claim that every perturbation changes the exponent. A proposed symmetry-protected persistence of the wall geometry under diagonal wall perturbations requires a new charging estimate at run ends. It remains a sketch and is not used in any theorem. Making the corner cost size-dependent to program a structural exponent also requires a new minimizer and defect analysis; it is not an established realization of an uncomputable ν.

18 The third level in two dimensions by separate registers

Spatial coordinates need not be reserved for different logical tasks. The construction here places the frame, ruler, computation, masks and broadcast on separate finite registers at the same sites of $(\mathbb{Z}/L\mathbb{Z})^2$. Throughout this section $L = n!$, volume precedes coupling, and the gap is the next distinct energy above the entire ground space. The finite alphabet depends on the prescribed program and a fixed integer β, never on L or g.

18.1 A shared geometry with ordinary nearest-neighbour rules

Use the frame of Section 13, now in one xy plane. Its minimum is four; every other configuration costs at least five. This is precisely the periodic-to-open frame of [11, Section 6.3, equation (133), Lemma 55]. Its complement is an open $(L-1)$-square. The four adjacent-interior flags are X_0, X_1, Y_0, Y_1. Put the unchanged five-symbol ruler of Section 2 on an independent register, with its horizontal rules along x and vertical rules along y. Write

$$f_g(p) = -g/p + r/p^2, \quad r = 1/16, \quad f_* = \min(0, \min_{q \ge 1} f_g(q)), \quad \mathcal{P}(g) = \{p \ge 2 : f_g(p) = f_*\}.$$

For $0 < g < 11/480$, the minimizing pitches are at least six. There is one minimizing pitch except at $g_p = r(2p+1)/(p(p+1))$, where $p, p+1$ tie, and $p = 2r/g + O(1)$. The planar estimate (32) applies without summing over a third axis:

$$E_{\text{wall}} - L^2 f_* \ge \sum_s p_s(f_g(p_s) - f_*) + \tfrac{5}{8}V_{\text{err}}. \tag{49}$$

The nonnegative contribution of W-free columns is retained in the underlying identity (31). In particular a defective ruler cannot gain an extensive negative energy by evading clean-stripe costs.

A bit v, constant on y bonds and equal to [ruler $= V$] off W, aligns the divider columns in every stripe. Require $v = 1$ on W sites in the frame column. Thus a zero-penalty striped ruler has a common pitch, and the frame column is one divider column. Every uniform pitch dividing L has such a filling. At a tie, mixed pitches fail alignment. A phase bit I has unit equality penalties on both axes. The inert phase $I = 0$ restricts the ruler to A, D and fixes the auxiliary classical fields to dummy values; it does not remove any bare ruler energy. The active phase $I = 1$ enables the following rules. Frame and ruler fields remain independent.

Let F_y, T_y mark respectively the first and last non-W rows in a stripe. Define the cell-first flag by the actual bond after a V column. All these flags have two-site defining truth tables. Use two independent copies of the thickness counter (34). In its own horizontal interval each has Boolean fields b, c, f, initialized by $b = f = 0$ at F_y and $c_0 = 1$ at the interval's first column, with internal recurrences

$$c_{j+1} = b_j c_j, \qquad b'_j = b_j \text{ XOR } c_j, \qquad f'_j = f_j \text{ OR } b_j.$$

The local copy restarts after every V and is dummy on V, W. The global copy restarts only at X_0, ignores interior dividers, and is dummy on W and the frame column. Increment is checked only between non-W rows, and carry only inside its specified interval. The frame row does not interrupt either counter. At stripe time t, b_j is the jth binary bit of t and $b_j \vee f_j = [2^j \le t]$. At T_y this selects

$$\ell(p) = \text{bitlength}(p - 2).$$

Copy the top indicators into $P_{\text{cell}}, P_{\text{global}}$ on *every* y bond, including W and the frame row. Set the local mask to zero on V and the global mask to zero on the frame column. The masks mark, respectively, the first ℓ sites after each divider and the first ℓ sites of the whole framed tape. Common pitch makes all stripe-top demands agree. Since $\ell(p) < p - 1$ for $p \ge 6$, every reader has an adjacent nonreader.

Fix $A \le_T 0'$ containing $0, 1, 2$ and a total computable approximation $a(j, s) \to A(j)$. Run one initialized functional tableau, as in Section 5, throughout the whole frame interior, using x for tape and y for time. Its work and persistent output tracks are separate; both edge-arrival fields are blank, and inactive fields are dummy. The machine fairly finishes stages $a(0, s), \ldots, a(s, s)$. Its output $o_t(j)$ settles after a finite time t_j. The ruler's W rows and V columns do not cut this tableau. The head moves at most one site per step, so the included times $0, \ldots, L - 2$ fit in width $L - 1$.

Broadcast bits B, C, D live on every site, including all frame and W sites. Their ordinary bond and onsite predicates are

$$\begin{aligned} B(x, y + 1) &= B(x, y), & D(x, y) &= C(x - 1, y), \\ C(x, y + 1) &= D(x, y), & C(x, y) &= B(x, y) \quad \text{on } W. \end{aligned}$$

Starting at a wall row y_0, these imply $B(x, y) = b(x)$ and $C(x, y_0 + t) = b(x - t)$. The next wall imposes $b(x - p) = b(x)$. Conversely every p-periodic word uniquely fills C, D when $p \mid L$. This is the planar version of Lemma 13.1; the apparent diagonal shift is two ordinary bonds. Impose the sole output source at $Y_1 \wedge P_{\text{global}}$:

$$B = o_{L-2}(j), \qquad j < \ell(p), \tag{50}$$

and impose $B = 0$ wherever $P_{\text{cell}} = 0$. There is no F_y condition on (50). The source may lie on a wall, stripe floor, stripe top or intervening row; the registers it tests are still present there.

Lemma 18.1 (Simultaneous planar completion)**.** *For each minimum frame and aligned minimizing pitch dividing L, all active auxiliary penalties can vanish. Every such filling has the same repeated reader word $[o_{L-2}(j) : j < \ell(p)]$.*

Proof. Fill the functional counters, copy their top masks through all rows, and fill the initialized tableau. Repeat its final prefix followed by zeros with period p, and use the preceding formulas for C, D. Every constraint vanishes. Conversely, counters and masks fix the source positions, the functional tableau fixes their values, periodicity copies them to every cell, and the zero mask fixes all remaining residues. The Robinson labels enter none of these tests. Every predicate reads one site or a single bond, with finite stored flags checked on their defining bonds. Thus this is a fixed finite onsite/nearest-neighbour rational interaction. □

The source condition is essential. For example, at $L = 12, p = 6$ with walls $y = 0, 6$, stripe floors are $1, 7$. A final source row $y = 11$ would disappear if it also required F_y. An erroneous all-zero broadcast would then escape comparison with a nonzero final prefix. Condition (50) tests that prefix at its actual final row. Earlier machine guesses are not retained as extra readers.

18.2 The full spectrum, exponent and critical point

On each stripe top activate the guarded full counter of Lemma 6.1 on the $\ell(p)$ local-mask sites, with radix $2^{\beta B}$. Its first guard uses the cell-first flag; its last guard uses the actual bond to the next nonreader. Freeze all exterior dynamic fields at unit cost. Every controlled edge Laplacian remains nonnegative even in inconsistent classical blocks. Add a qubit at each site, penalized by $|1\rangle\langle 1|$ when active. On inert y bonds use $(|01\rangle - |10\rangle)(\langle 01| - \langle 10|)$. This completes rational affine coefficient matrices.

Lemma 18.2 (Planar full-gap identity)**.** *For each fixed $0 < g < 11/480$, eventually along $L = n!$,*

$$\Delta(L, g) = \min_{p \in \mathcal{P}(g)} d_p, \quad d_p = \Delta^+ K[2^{\beta A(j)} : j < \ell(p)]. \tag{51}$$

For $g \le 0$ the inner gap limit is zero.

Proof. The direct-sum decomposition by classical labels is $H = \bigoplus_\sigma (E_{\mathrm{cl}}(\sigma)\mathbf{1} + K_\sigma)$ with $K_\sigma \ge 0$. Completion attains the common lower bound $E_* = 4 + L^2 f_*$. A nonminimum frame costs at least one. A ruler violation costs at least $5/8$ by (49). A clean wrong pitch costs at least $L\delta(g)$, with the same positive $\delta(g)$ as in Section 13; a W-free or wholly inert ruler costs $L^2|f_*|$. These exceed $5/8$ eventually. Every wrong auxiliary predicate or nonconstant phase contributes an additional integer penalty. Independent bare-layer lower bounds prevent trading these costs against a lower ruler or frame energy.

In a minimizing active block there are $(L/p)^2$ identical disjoint counters and unit-gap exterior factors. Tensor sums with nonempty kernels have first positive gap equal to the least factor gap, regardless of zero-mode multiplicities. For $L - 2 \ge \max_{j<\ell(p)} t_j$ the reader word is settled. Put $k_p = \beta|A \cap [0, \ell(p))|$. The full-sector bounds are

$$\frac{4^{-k_p}}{64\ell(p)^2} \le d_p \le 4\pi^2 4^{-k_p} \le \frac{\pi^2}{16} < \frac{5}{8}, \qquad d_p \le \frac{\pi^2}{\ell(p)^2}. \tag{52}$$

Here $k_p \ge 3$. All other classical blocks start above this gap. At a tie, both pitches eventually divide $n!$; alignment leaves their two uniform blocks and their least positive gap is the minimum in (51). We use the full counter bound, including free paths and all head sectors, without equating it to a main orbit.

For $g \le 0$ an inert all-A or all-D ruler with dummy auxiliaries and a minimum frame attains energy four. Its qubit rings have a positive single-excitation eigenvalue $2 - 2\cos(2\pi/L)$, which bounds the full gap from above and tends to zero. This suffices without an exact formula for the ring's gap. □

Taking logarithms of (52) gives

$$\frac{\log(1/\Delta_\infty(g))}{\log(1/g)} = 2\beta\rho_{\ell(p)}(A) + O\Big(\frac{\log \ell(p)}{\ell(p)}\Big). \tag{53}$$

At a tie take the larger inverse logarithm; neighboring bitlengths differ by at most one, so an existing density gives the same limit. The explicit relativized mechanical-word construction in Section 13 supplies a $0'$-computable set of any prescribed $0''$-computable density $\rho \in [0,1]$. For completeness, use $0'$-computable rationals $u_s \to \rho$, $N_0 = 1$, $N_{s+1} = (s+2)N_s$, and mechanical bits of slope u_s on $[N_s, N_{s+1})$. Endpoint densities differ from u_s by at most $1/(s+2) + 1/N_{s+1}$. Intermediate densities are, up to $1/n$, convex combinations of the previous endpoint density and u_s. Hence all densities converge. Set the first three bits to one and apply the ordinary limit lemma to obtain the finite ordinary program. For a finite $\alpha \geq 0$ in D_3, choose $\beta \geq \max(1, \alpha/2)$ and $\rho = \alpha/(2\beta)$. Equation (53) gives exponent α. The nested-thirds diagonal in Corollary 13.4 also supplies planar exponents outside $\mathsf{S}_2 \cup \mathsf{P}_2$.

Even when $\alpha = 0$, the last bound in (52) closes the right gap. The energy density is zero on the left and $f_* = -g^2/(4r) + O(g^4)$ on the right, since the frame density $4/L^2$ vanishes. Thus zero is a nonanalytic critical point. The volume limit is eventually exact at fixed g; no effective bound on the finite prefix settling time is assumed. Each reader excitation can be placed independently at any of the $(L/p)^2$ cells. The product-ground-vector argument of (42) gives local spectral weight per area at least $1/p^2$ at fixed g. Receding the frame seams and taking a local subsequential limit gives the selected product representation with gap d_p, by the same local spectral-measure argument in Section 13. This does not classify all ground states or establish local nondegeneracy. These observations prove Theorem 1.4.

18.3 Limit depth, finite data and restricted dimension bounds

Proposition 18.3 (Finite-data and dependent-limit ceilings)**.** *Algebraic local entries give uniformly computable next-distinct finite-volume gaps. A fixed finite tuple of computable local entries also gives computable gaps, nonuniformly in that tuple. An existing one-size exponent therefore lies in* D_2*. With k dependent existing limits of uniformly computable data, and existence of every required intermediate limit, the resulting real lies in* D_{k+1}*.*

Proof. In the algebraic case root isolation decides multiplicities and selects the first two distinct roots. For a fixed computable real tuple $c = (c_1, \ldots, c_m)$, let $I_c = \{P \in \mathbb{Q}[X_1, \ldots, X_m] : P(c) = 0\}$. This ideal has a finite rational Gröbner basis. Hard-code it as finite advice. It decides which polynomial expressions vanish; for a nonzero expression approximation eventually decides its sign. Real-closed-field root and multiplicity algorithms relative to this ordered coefficient field compute each finite spectrum. Complex entries are replaced by their real and imaginary parts. The advice need not be recoverable from programs for arbitrary coefficient names. This is a pointwise assertion, not a uniform degeneracy-decider. Each existing limit of a computable rational array has a Cauchy name in $0'$ by tail-Cauchy search. Relativize this step successively. Positivity allows the logarithms used here. Effective moduli can remove jumps; independent parallel limits do not add their depths. □

In particular, Proposition 16.1 is sharp already in two dimensions for the present volume-then-coupling convention. Spatial dimension alone does not count these dependent limits. It still constrains an explicitly narrower architecture. Consider classical-forced designs whose nested boxes grow along pairwise disjoint axis blocks, whose innermost computation is deterministic on a finite transverse strip, and whose final quantum reader has a fixed finite state space independent of that strip's length. In this class attaining a proper D_{k+1} value with k nested box-size limits requires

$d \geq k+1$. Indeed, if the innermost block used only one axis, its finite computable strip-state transition would eventually cycle. An existing finite-reader limit must have a constant value on that cycle and is computable by evaluating any cycle state; no real-equality decision is needed. This removes one jump. To avoid that collapse the first block needs two axes, and each other disjoint block needs one. Growing readers, quantum strips and nondeterministic tableaux are outside this statement. The planar construction above shares axes and uses a coupling limit, so is outside that architecture.

Proposition 18.4 (Elementary one-dimensional benchmarks)**.** *For rational classical finite-range open chains, every existing size-gap exponent is zero. Rational quantum nearest-neighbour open chains realize size-gap exponents zero, one and two.*

Proof. For the classical chain choose a common denominator D of the local energies. Distinct total energies differ by at least $1/D$. From a ground product configuration, follow single-site changes to an excited configuration. The first exit from the ground set changes only boundedly many bounded local terms, so the gap is at most a constant independent of length. Onsite projectors attain zero. For spin-1/2 chains put $n = |1\rangle\langle 1|$ and $t = |01\rangle\langle 10| + |10\rangle\langle 01|$. The families $\sum_j n_j$, $\sum_j t_{j,j+1}$, and $2\sum_j n_j - \sum_j t_{j,j+1}$ have the stated exponents. The Jordan–Wigner transform gives one-particle levels respectively 1, $2\cos(k\pi/(L+1))$, and $2-2\cos(k\pi/(L+1))$. The full fermionic spectrum consists of subset sums. For hopping alone fill all negative levels; each excitation costs a sum of absolute nonzero levels. Its gap is $2\sin(\pi/(2(L+1)))$ for even L and $2\sin(\pi/(L+1))$ for odd $L \geq 3$. Zero modes enlarge the kernel. The third family has positive levels and vacuum ground state, with gap $2-2\cos(\pi/(L+1))$. These are full spectra, without an imposed particle-number sector. □

These benchmarks do not determine the full quantum-chain attainment class. General one-limit ceilings and this restricted strip argument must not be interpreted as an unrestricted spatial lower bound.

19 Where uncomputable content can survive

The fragile ruler is not the only way a computable microscopic law can carry noncomputable asymptotic information. Two distinct homes are a noncomputable selection among several states, and a growing statistical readout whose expectation has a noneffective limit. This section makes that distinction precise. All alphabets, ranges, local tables, initial data, boundary rules and readouts called computable have uniform algorithms. Computability of a measure means uniform computability of its finite cylinder probabilities. We discuss finite-alphabet classical Gibbs measures and probabilistic cellular automata (PCA); no quantum Gibbs uniqueness theorem is implicit.

19.1 Effective state classes and local consensus

Lemma 19.1 (Constant observables on an effective compact class)**.** *Let $\mathcal{M}$ be the probability measures on a finite-alphabet configuration space $\Sigma^{\mathbb{Z}^d}$. A nonempty subset S specified by uniformly computable continuous equalities is effectively closed in an effectively compact space. If a computable continuous local functional L is constant on S, that common value is computable. In particular, a singleton S is a computable measure.*

Proof. For increasing finite boxes B_n use the metric $\sum_{n\geq 0} 2^{-n-1}\|\mu_{B_n} - \nu_{B_n}\|_{\mathrm{TV}}$. Given an accuracy, choose N making the tail small. If B_N has M patterns, approximate its probability vector by a rational simplex mesh of denominator D with M/D below the remaining accuracy. Extend

each finite pattern by a fixed exterior, and use finite mixtures of these extensions as a finite net. Consistency of marginals gives completeness and the same finite-box tests semidecide a finite open cover. Thus the space is effectively compact. The complement of a computable zero set is effectively open, as is the countable union for the specified equalities.

For effectively open V, the inclusion $S \subset V$ is semidecidable: search for a finite subcover of $\mathcal{M}$ from V and the enumerated complement of S. Dovetail rational intervals of width below the requested accuracy and test $S \subset \{L \in I\}$. If L is constant, an interval containing its common value in its interior is eventually certified. For a singleton apply this to each cylinder probability, uniformly. □

This is the effective compactness mechanism behind isolated-zero results of Galatolo–Hoyrup–Rojas [22, Theorem 2.9 and Corollary 3]. The simplex mesh must have denominator depending on the number of patterns; a fixed coarse denominator would not give the claimed net.

Theorem 19.2 (Unique states and consensus)**.** *A computable finite-range PCA with a unique stationary probability measure has a computable stationary measure. A computable bounded finite-range classical interaction at computable finite temperature, with a unique full DLR Gibbs measure, has a computable Gibbs measure. More generally, any computable local expectation constant on a nonempty effective stationary or DLR class is computable. If all states unanimously place each specified computable local expectation strictly on the same side of a computable threshold, the resulting sequence of bits is computable.*

Proof. For a PCA with Markov operator P, stationarity is given by the countable cylinder equalities $\mu(f) = \mu(Pf)$. The finite-cone expression for Pf is computable and continuous. A stationary measure exists by compactness of Cesàro averages; their stationarity residual is at most $2/T$ on a norm-one test. For Gibbs measures use all DLR identities $\mu(f) = \mu(\gamma_\Lambda f)$, including tests formed by products with exterior cylinders. Finite-range kernels are computable local functions: their finite Boltzmann sums have strictly positive partition functions. Compactness supplies Gibbs measures by taking finite-box limits. These identities characterize the full DLR class; one-site mean identities alone do not. Apply Lemma 19.1. Additional effectively closed restrictions are allowed only when specified and nonemptiness is promised. For the threshold bits dovetail the two inclusion tests in the strict sublevel and superlevel sets. Compactness gives a positive gap for each test separately, without a uniform gap in the test index. □

The PCA uniqueness theorem is known: Marcovici–Sablik–Taati [31, Proposition 2.2] proves precisely this result. The effective compactness and isolated-fixed-point methods of [22] give the general framework. The effective description of translation-invariant equilibrium-state sets is also developed by Gayral–Sablik–Taati [23]. Certified extremal local-expectation bounds of Fawzi–Fawzi–Scalet [17] give a broader antecedent for the consensus conclusion when the extrema coincide. The compact-cover argument here is a direct classical specialization. Uniqueness restricted to translation-invariant measures computes that restricted class; it must not be substituted for uniqueness among all Gibbs measures in the following boundary result.

Proposition 19.3 (What uniqueness makes effective)**.** *Full Gibbs uniqueness gives a computable worst-boundary influence modulus for each local observable, and hence an effective spatial covariance bound for it. PCA uniqueness gives an effective uniform* Cesàro *convergence modulus for each local observable.*

Proof. For a norm-one f supported well inside Λ_L, compute the minimum l_L and maximum u_L of $\gamma_{\Lambda_L} f$ over its finite exterior collar. They enclose every infinite Gibbs expectation. These intervals are

nested by conditioning in nested boxes. If their width failed to tend to zero, compact subsequential limits of extremizing finite-box measures would give two Gibbs states with different f expectations. Search with rational enclosures until $u_L - l_L < \eta$. This terminates by uniqueness. For h supported outside that box, conditioning on the exterior gives $|\operatorname{Cov}_\mu(f,h)| \le (u_L - l_L)\|h\|_\infty$.

For the PCA put $g_T(x) = T^{-1}\sum_{t<T} P^t f(x)$ and compute its finite-cone minimum and maximum. Their diameter tends to zero: otherwise subsequential limits of extremizing empirical measures are stationary and disagree on f. The unique stationary expectation lies in every such interval. Find T with diameter below $\eta/2$. For any initial law split a length-N average into T-blocks and a remainder. Every complete block differs from the common value by less than $\eta/2$, and the remainder costs at most $2T/N$ for signed norm-one f. Taking $N \ge 4T/\eta$ gives the required uniform modulus for all subsequent averages. □

The conclusion for PCA does not assert ordinary unaveraged mixing. These algorithms can be extremely slow. An effective decay modulus for values will also not, by itself, compute a scaling exponent.

19.2 Noncomputable selection with computable finite preparations

Proposition 19.4 (An Ising selector)**.** *There is a computable sequence of randomized finite-volume boundary preparations for a computable low-temperature zero-field Ising interaction whose limiting Gibbs measure has a noncomputable local magnetization.*

Proof. Let K be the halting set and put

$$r_s = \sum_{e \le s} 4^{-e-1}[e \text{ halts within } s \text{ steps}], \qquad r = \sum_{e \in K} 4^{-e-1}.$$

The limit $r \in [0, 1/3]$ is noncomputable: the tail after a digit is at most one third of that digit's weight, so a Cauchy name recovers the halting bits successively. Choose computable β, J_x, J_y with $\sinh(2\beta J_x)\sinh(2\beta J_y) > 1$. The plus and minus finite-box measures $\nu_L^{\pm}$ converge to Gibbs measures $\mu^{\pm}$. Ferromagnetic stochastic monotonicity gives the limits of increasing cylinder events, and finite inclusion-exclusion gives all cylinder limits. Their magnetizations are $\pm m$, where the Onsager–Yang formula is

$$m = \left[1 - \{\sinh(2\beta J_x)\sinh(2\beta J_y)\}^{-2}\right]^{1/8} > 0.$$

For the formula and its anisotropic form see [5, equations (2.1)–(2.2)] and [48]. The positive number m is computable. Choose plus boundary with the computable rational probability r_L and minus otherwise. The resulting finite preparation is computable and tends to $\mu_r = r\mu^+ + (1-r)\mu^-$, a DLR measure. Its local magnetization $(2r-1)m$ computes r, so is noncomputable. □

The randomization is an explicit part of each finite preparation. No deterministic computable infinite boundary, unique Gibbs state, or unique selection by the Hamiltonian is claimed. The same selector works throughout the zero-field anisotropic low-temperature family; this is not a full interaction ball allowing a longitudinal field.

19.3 Expectation limits and an independent-spin benchmark

Proposition 19.5 (The expectation-limit ceiling)**.** *If μ is computable and O_s are uniformly computable finite cylinder observables, any existing finite limit $\lim_s \mu(O_s)$ is in D_2. A computable convergence modulus makes it computable. A unique-state product model already allows every*

limit-computable bit set as a deterministic almost-sure growing readout limit, when the program is in the decoder.

Proof. Approximate $\mu(O_s)$ by rationals within 2^{-s}. Their existing limit is computable in $0'$ by tail-Cauchy search, or computable without an oracle given a modulus. For the example take the radius-zero PCA that independently resets every spin to Bernoulli(p), $0 < p < 1/2$ computable. After one step every input law becomes the same product law, so it is the unique stationary law. For $a(j, s) \to A(j)$ computable, take $2s + 1$ distinct spins, form their majority, and XOR it with $a(j, s)$. The error probability is at most $\exp[-2(2s + 1)(1/2 - p)^2]$, a summable sequence. The first Borel–Cantelli lemma gives eventual agreement with the current guess for every fixed j, simultaneously for countably many j. Thus the readouts converge almost surely and in expectation to $A(j)$. The bias law itself contains no such logical computation; the decoder supplies it. □

The distinction between expectations and individual samples is necessary. Under the fair computable product law, $R_n(x) = \sum_{k=1}^{n} 2^{-k} x_k$ has the deterministic uniform tail bound $|R - R_n| \leq 2^{-n}$, yet its sampled limit is a uniform random real and hence noncomputable almost surely. Its expectation is $1/2$. This random sample value is not a deterministic noncomputable expectation. If bounded readouts converge almost surely to a fixed constant, dominated convergence does give the same expectation limit; an effective L^1 rate would compute it.

19.4 A fault-tolerant revising computation

The next construction puts the logical revisions in a fixed physical update rule. The reliability theorem is due to Toom and Gács–Reif [43, 21]; we use its arbitrary-trajectory form as stated by Taati–Miękisz [42, Theorem 3.1]. The growing-readout conclusion is an elementary corollary of that imported theorem.

Theorem 19.6 (Reliable growing readouts)**.** *There is one fixed finite-alphabet three-dimensional stacked PCA rule such that, for every limit-computable set A, a computable cloned initialization and computable growing majority readouts $O_{j,s}$ satisfy*

$$O_{j,s} \longrightarrow A(j) \quad \textit{almost surely}, \qquad \mu(O_{j,s}) \longrightarrow A(j)$$

below a noise threshold. The limits persist on a relatively open region of positive fixed-neighborhood product-update kernels. At every computable parameter point the causal history law μ is computable. A restricted constant-normalizer thermal family gives the same conclusion for a four-dimensional half-space DLR history with a fixed computable bottom row.

Proof. Fix a universal one-dimensional CA simulating an ordinary finite program for $a(j, s) \to A(j)$. Its finite initial input is computable. Stage completion times T_s and register locations $i_{j,s}$ are computable by finite simulation until the completion marker; every stage terminates. The same rule handles every program. An original radius-r rule can be made one-sided: compose with a shift by r, then group $2r$ sites into a symbol. The blocked rule uses only its own block and the next block. At time T the original register at i is recovered at drifting coordinate $i - rT$ before blocking. All coordinates remain computable.

Clone this one-dimensional initial tape along two redundant axes. At each step take the full-symbol north-east-center majority in each redundant plane (with a fixed choice for three distinct symbols), then apply the one-dimensional rule. This is the stacking rule of [42, Section 3.1]. For any $\delta < 1/2$, its reliability theorem provides $\varepsilon_0 > 0$ such that joint-fault bounds $\varepsilon_0^{|F|}$ imply full-symbol

error probability at most δ at every site and time, relative to the arbitrary noiseless trajectory. Thus revisions, rather than only monotone outputs, are covered by the import.

For product local updates, use fresh independent coins at every site and time. Let R bound the redundant-coordinate speed. At time T_s read $m_s = 2s + 1$ copies of the register, separated by $2RT_s + 1$ along one redundant axis. Their finite backward cones have disjoint update variables. Deterministic initial values do not create a stochastic dependence. A projected bit error entails a full-symbol error, so these independent indicators have means at most δ. Their majority obeys

$$\Pr\{O_{j,s} \neq a(j,s)\} \leq e^{-2m_s(1/2-\delta)^2}. \tag{54}$$

For example the exponential-moment bound for centered Bernoulli variables is $\mathbb{E}e^{\lambda(Y-\mathbb{E}Y)} \leq e^{\lambda^2/8}$; multiply over copies and optimize λ. This proves (54) without identical distributions. Summability and Borel–Cantelli give almost-sure eventual agreement with each current logical guess. Countable intersection covers every j. Expectation error is at most the probability in (54) plus $|a(j,s) - A(j)|$. Likewise $Q_s = \sum_{j \leq s} 4^{-j-1} O_{j,s}$ converges almost surely and in expectation to $\sum_{j \in A} 4^{-j-1}$, by truncating a fixed prefix and bounding the geometric tail.

A finite spacetime cylinder has a finite ancestor graph down to the computable initial row. Summing products of computable kernel entries over this graph computes its probability. Thus the whole causal law is computable. Positivity and $K(G(r) \mid r) > 1 - \varepsilon_0$ are finitely many strict conditions, giving the claimed open kernel region. Conditioning first on the past of the latest fault layer and iterating backwards bounds each finite fault event by $\varepsilon_0^{|F|}$, even for probabilities depending on predecessor patterns.

For the thermal version assign unit energy to a wrong update. With q symbols the kernel is

$$K_\beta(a \mid r) = \frac{e^{-\beta[a \neq G(r)]}}{1 + (q-1)e^{-\beta}}, \qquad p_\beta = \frac{(q-1)e^{-\beta}}{1 + (q-1)e^{-\beta}}.$$

For $p_\beta < \varepsilon_0$ the same readout works. On a finite ancestor-closed directed graph the path probability is the product of its update factors. To condition a finite resampling region Λ, exhaust the half-space by such graphs containing Λ and all its immediate children. In conditional probability ratios, precisely the factors whose support meets Λ remain, including factors centered at future children outside Λ. The expression stabilizes once these children are present. Martingale convergence proves the conditional identity for the full exterior. Since the normalizer is independent of predecessors, this is exactly the DLR kernel of the unit-fault interaction. This proof uses the fixed bottom row and asserts no DLR uniqueness or stationarity.

More generally, fix for each predecessor pattern a permutation π_r with $\pi_r(0) = G(r)$, and give $\pi_r(k)$ the common energy J_k, $J_0 = 0$, $J_k > 0$ for $k > 0$. The row normalizer $Z = \sum_k e^{-\beta J_k}$ is still constant. The strict condition $\sum_{k>0} e^{-\beta J_k}/Z < \varepsilon_0$ gives an open region in this temperature/energy family. At computable points all laws above are computable. A general energy perturbation has predecessor-dependent normalizers, which cannot be discarded in DLR conditioning, so arbitrary Gibbs perturbations are outside the claim. □

The probability-one event may depend on the kernel. Computable laws are claimed at computable parameter points, whereas reliability is a pointwise statement throughout the region. The effective physical error bound controls agreement with the current guess, not the last logical revision. The construction is a thermal history with growing readouts; it supplies neither a noncomputable local number in a unique equilibrium phase nor an intrinsic equilibrium critical exponent.

The history/Gibbs correspondence itself has the prior formulations of Domany–Kinzel and Goldstein–Kuik–Lebowitz–Maes [13, 25], and the temperature-matched trajectory construction in [42]. The direct fixed-bottom conditional proof specifies the half-space used here.

19.5 Scaling envelopes and the precise two-homes conclusion

Proposition 19.7 (Decay versus logarithmic scaling)**.** *Effective decay of numerical correlation data does not compute its logarithmic order. If positive computable data eventually satisfy $cr^{-\alpha} \le C(r) \le C_0 r^{-\alpha}$ for fixed $0 < c \le C_0 < \infty$, the exponent α is computable pointwise. Uniform computation requires the envelope witnesses.*

Proof. For computable rationals $\alpha_n \to \alpha \in [1, 2]$, the data $C(2^n) = 2^{-n\alpha_n} \le 2^{-n}$ have an effective value-decay bound but logarithmic order α, possibly noncomputable. No positive-definiteness or physical correlation realization is asserted for this numerical example. Conversely the envelope gives

$$\alpha - \frac{\log_2 C_0}{n} \le -\frac{\log_2 C(2^n)}{n} \le \alpha - \frac{\log_2 c}{n}.$$

Hard-code an integer above the logarithmic constants and an eventual cutoff, choose n effectively large, and evaluate the positive sample's logarithm. Fixed finite logarithmic powers add only an effective $O(\log n/n)$ error. Uniform extraction without witnesses fails: let $C_e(2^n) = 2^{-n}$ until machine e halts by time n, and 2^{-2n} thereafter. Every sequence is eventually a pure power, but a uniform exponent algorithm would decide halting. □

Here is the precise exclusion underlying the two homes. Let S be one of the nonempty effective stationary or DLR classes above, possibly with explicitly supplied effectively closed restrictions, and let $\mu \in S$. If a computable local observable has a noncomputable expectation in μ, its expectation is not constant on S, by Lemma 19.1. If instead μ is computable, existing expectation limits of uniformly computable growing cylinder observables have the ceiling and modulus condition of Proposition 19.5. This is a statement about specified state classes and statistical expectations, not an exhaustive classification of arbitrary sample-dependent observables. State selection and fault-tolerant revision therefore coexist with the local uniqueness obstruction. Uniform strict contraction of a PCA instead supplies geometric coupling decay; bounded finite-range one-dimensional finite-temperature Gibbs chains have a uniform finite-block Doeblin contraction. In these restricted neighborhoods the ordinary spatial correlation length cannot diverge at an interior parameter point. Neither argument decides the corresponding higher-dimensional equilibrium exponent question.

20 Restricted computability and an exact ruler criterion

The onsite instability in Section 17 concerns the particular programmed rulers, not every possible robust exponent. Several restricted positive results clarify what additional certificates would suffice. The response observable must be fixed before making any stability claim.

For a concave ground-energy density e at a transition g_c, put

$$s_{\mathrm{RT}}(t) = e(g_c) + e'_+(g_c)t - e(g_c + t), \qquad \rho_{\mathrm{RT}}(t) = s'_{\mathrm{RT}}(t+), \quad t > 0.$$

After subtracting a vacuum bulk energy, another convention is $\rho_{\mathrm{vac}}(t) = -e'_+(g_c + t)$. They agree when $e'_+(g_c) = 0$, but not at a jump: if $e(g_c + t) = -mt$, then $\rho_{\mathrm{vac}} = m$ has order zero while $s_{\mathrm{RT}} = \rho_{\mathrm{RT}} = 0$ has no positive logarithmic order. Moving the critical point does not remove this distinction. For a convex thermal pressure use $\rho_{\mathrm{RT}}(t) = P'(g_c + t) - P'_+(g_c)$, so that the excess is relative to the right tangent. These are density and correlation claims when specified below, not substitutions for the spectral gap.

20.1 Classical chains and bounded thermal susceptibility

Proposition 20.1 (Finite-range classical chains)**.** *The zero-temperature energy density of a finite-alphabet translation-invariant classical chain with fixed finite range and affine coupling is the minimum of finitely many affine cycle means. Immediately to the right of every crossing its right-tangent deficit vanishes. Fixed-width classical strips have the same property. At finite temperature, bounded finite local energies give an analytic pressure at every finite coupling.*

Proof. For range R, form the de Bruijn graph of $V = q^R$ words. An edge appends one symbol and has its corresponding local energy. Removing loops from a finite walk leaves a simple path of at most $V - 1$ edges. If edge magnitudes are at most W and m is the least simple-cycle mean, a walk of n edges has energy at least $nm - 2W(V - 1)$. Repetition of a minimizing cycle gives the reverse thermodynamic bound. There are finitely many simple cycles; affine local weights give affine means. A finite minimum is piecewise affine, which proves the tangent assertion. Blocking a fixed-width strip only changes the finite graph. At finite temperature the weighted de Bruijn matrix is primitive: some common power has every entry positive. Its Perron root is simple and positive, hence analytic locally in each finite coupling. Its logarithm gives the pressure. □

This familiar cycle mechanism does not exclude first-order crossings or every aperiodic minimizing configuration. The range-three chain paper of 1973 is by Bundaru, Angelescu and Nenciu [8]; Morita's general finite-range ordering work is separate [35]. The direct finite-graph proof supplies the arbitrary-range assertion here. Diverging strip width, hard infinite energies and zero-temperature transfer singularities are outside the finite-temperature assertion.

Proposition 20.2 (Explicit classical thermal curvature)**.** *Consider bounded finite-range classical interactions on $\mathbb{Z}^d$, alphabet size q, inverse temperature $\beta > 0$, and field $-gN$ with $N = \sum_x n(\sigma_x)$ for a nonconstant onsite charge n. On a compact field interval suppose one-site conditional energies have oscillation at most B, and two charge values differ by at least $\delta_n > 0$. If supports have sup-norm diameter at most R, the pressure $P = \lim(\beta|\Lambda|)^{-1} \log Z_\Lambda$ is strongly convex with constant*

$$c_0 = \beta(R+1)^{-d} q^{-2} e^{-2\beta B} \delta_n^2 > 0. \tag{55}$$

If on a right interval it is twice differentiable and $P'' \le C$, then $c_0 t \le \rho_{\mathrm{RT}}(t) \le Ct$ and its exponent is one. Without the upper bound, any existing finite positive-onset order lies in $[0, 1]$.

Proof. In an open box select one residue class modulo $R + 1$. No interaction meets two selected sites. Given the complement, the selected spins are independent and every symbol has conditional probability at least $p_* = e^{-\beta B}/q$. The identity $\mathrm{Var}(n) = \frac{1}{2} \sum_{a,b} p_a p_b (n_a - n_b)^2$ gives conditional variance at least $p_*^2 \delta_n^2$. Total variance therefore bounds $\mathrm{Var}(N)$ below by this number times the selected-site count. The finite pressure has $P''_\Lambda = \beta \,\mathrm{Var}(N)/|\Lambda|$. Subtract the corresponding quadratic and pass its convexity inequality to the thermodynamic limit. Selected density tends to $(R + 1)^{-d}$, proving (55) without interchanging second derivatives and limits. The two curvature bounds imply $c_0 t \le P'(g_c + t) - P'_+(g_c) \le Ct$ by the mean value theorem and a right limit. Integration gives tangent excess between $c_0 t^2/2$ and $Ct^2/2$. Dividing logarithms gives order one. With only the lower bound, division by $\log t < 0$ gives order at most one. Right continuity of the bounded convex derivative gives $\rho_{\mathrm{RT}}(t) \to 0$, so its finite order is nonnegative. □

Strong convexity and this conditional-variance/sparse-sublattice method are established by Dobrushin–Nakhapetian [12]. Equation (55) is an explicit elementary finite-range specialization. If $0 < c \le c_0$ and $C_1 \ge C$ are supplied effective bounds,

$$\left| \frac{\log \rho_{\mathrm{RT}}(t)}{\log t} - 1 \right| \le \frac{\max(|\log c|, |\log C_1|)}{|\log t|}$$

is a computable dyadic modulus on a supplied right interval. It does not require evaluating an unknown critical point; evaluation in recentered coordinates would require a name for that point. A response t^α, $0 < \alpha < 1$, shows why a divergent susceptibility can evade the exponent-one conclusion.

A conditional upper-bound certificate consists of the identity $P'' = \beta \sum_x \mathrm{Cov}(n_0, n_x)$ and the estimate $|\mathrm{Cov}(n_0, n_x)| \le Ae^{-m|x|_1}$ uniformly on the interval. Then

$$C = \beta A \left(\frac{1 + e^{-m}}{1 - e^{-m}} \right)^d .$$

For a periodic pure phase average the response over a period cell, using the bound at every origin; mixing translated phases can add nondecaying covariances. Stable Peierls-phase mixing from contour theory [18] requires its own finite-reference-phase and low-temperature hypotheses. It does not supply uniform differentiated critical-neighborhood constants for an arbitrary hierarchical ruler. The observable qualification also matters: $P(t) = \max(0, t + ct^2/2)$ has total-density jump one but right-tangent excess ct.

20.2 Algebraic transfer matrices and isolated effective fixed points

Proposition 20.3 (A fixed-dimensional algebraic transfer class)**.** *Suppose an exact injective matrix product state has fixed bond dimension, tensor entries with effective algebraic-function branch descriptions, and an algebraic critical coupling. If its simple positive leading transfer eigenvalue is λ_0, its normalized subleading radius satisfies $0 < r(t) < 1$, $r(t) \to 1$, and the mode couples to the chosen finite observable, then $\xi(t) = -1/\log r(t) = t^{-\gamma + o(1)}$ for an effectively recoverable positive rational γ.*

Proof. Elimination makes all eigenvalue branches algebraic over the algebraic-number rational function field. Convergent Puiseux expansions and their effective equality tests [7] apply. Conjugation, modulus, division and finite maxima preserve algebraic branches. A nonzero branch difference has a fixed leading sign near zero, so the maximizing branch stabilizes. Thus $1 - r(t) = ct^\gamma(1 + o(1))$ with $c > 0$ and positive rational γ, and the first nonzero term can be computed from the branch data. Finally $-\log r \sim 1 - r$. □

The overlap hypothesis prevents attributing a measured length to a dark transfer mode. For an explicit example take $A_0 = I, A_1 = tX, A_2 = tY, A_3 = tZ$ with Pauli matrices. The channel has eigenvalues $1 + 3t^2$ on I and $1 - t^2$ on each Pauli matrix, so $\xi \sim 1/(4t^2)$. The physical observable $O = |0\rangle\langle 1| + |1\rangle\langle 0|$ has connected correlation

$$\langle O_0 O_k \rangle_c = \frac{4t^2}{(1 + 3t^2)^2} \left(\frac{1 - t^2}{1 + 3t^2} \right)^{k-1}, \quad k \ge 1.$$

Indeed the inserted channel takes I to $2tX$ and X to $2tI$; the left fixed functional is $\mathrm{Tr}\,/2$. The tensors span M_2 for $t > 0$. The two-site parent projection therefore gives a fixed-range parent with the injective uniqueness property of [38, Section 4.1.2]. Its entries are rational functions obtained by the finite Gram inverse. Since the projection is bounded, the singularity at zero is removable. This family has zero frustration-free energy throughout, not an energy singularity or an asserted affine spectral-gap law. Approximate MPS, growing bond dimension and unrestricted computable parameter dependence are outside the proposition.

Proposition 20.4 (An isolated RG certificate)**.** *Let R and DR be computable near a specified rational compact finite-dimensional box K, and suppose $R(x) - x$ has exactly one zero x_* in K.*

Then x_* and $DR(x_*)$ are computable. A specified isolated real eigenvalue $\lambda > 1$ and a physical identity $\nu = \log b/\log\lambda$, integer $b \geq 2$, compute ν.

Proof. The norm $\|R(x) - x\|$ is computable. Refine rational boxes and discard a box only after certifying strict positivity throughout it. Never discard the zero. Outside any neighborhood of the unique zero, compactness gives a positive minimum, eventually certified by effective continuity. Dovetail subdivision until the unexcluded boxes have an enclosing box of the requested small diameter. This computes x_*, including a boundary zero, and then $DR(x_*)$. Equivalently apply [22, Theorem 2.9]. Computable polynomial-root approximation gives the spectrum as a multiset, and the supplied isolator selects λ. Positivity of $\log\lambda$ permits the stated division. □

An ordinary C^1 promise does not supply an effective derivative. Neither stability alone nor an arbitrary finite RG truncation supplies this exact reduction, compact isolator or physical identity. Computable RG steps can have uncomputable flows [46] without contradicting this isolated readout.

20.3 Why soft thermodynamic data do not suffice

Proposition 20.5 (A volume-grid imitation)**.** *For each limit-computable $\zeta \in [1,2]$ there is a computable nondecreasing density ρ, zero on $(-\infty, 0]$ and one on $[1,\infty)$, of logarithmic order ζ at zero. Put $s(u) = \int_0^u \rho(v)\,dv$ and $E(A,B;g) = -As((g-B)/A)$, $A > 0$. For $N = L^d$, uniformly computable rational polyhedral energies E_L with $N+1$ affine branches have volume-grid slopes and satisfy*

$$E - A/N \leq E_L \leq E + (2A + |g - B|)/N.$$

They are concave and positively homogeneous in (A,B,g), and their limiting density order is the constant ζ on the open domain $A > 0$, $B \in \mathbb{R}$.

Proof. Take computable rationals $\zeta_j \to \zeta$ in $[1,2]$. At $t_n = 2^{-n}$ prescribe $\rho(t_n) = 2^{-Q_n}$, $Q_n = \sum_{j\leq n}\zeta_j$, with $Q_0 = 0$, and interpolate linearly. The ratios of successive values are in $[1/4, 1/2]$, $\rho(t) \leq t$, and the interpolation has a known bounded Lipschitz constant. The tail integral below 2^{-n} is at most $2^{-2n}/2$, so s is computable without a modulus for ζ_j. Cesàro convergence gives $Q_n/n \to \zeta$; bounded ratios extend the order from nodes to all t. Differentiation of the perspective gives the claimed density order at $g_c = B$.

On $0 \leq r \leq 1$ let $c(r) = \max_{0\leq u\leq 1}(ru - s(u))$. The compact computable optimization has a known Lipschitz bound, so c is computable, $0 \leq c \leq 1$, and $|c(r) - c(r')| \leq |r - r'|$. Convex duality, or the tangent of slope $\rho(u)$ with the endpoint slopes on the linear extensions, gives $-s(u) = \min_{0\leq r\leq 1}(c(r) - ur)$ for all real u. For $r_j = j/N$, approximate $c(r_j)$ within $1/(4N)$, round to the nearest $1/N$ grid and clamp to $[0,1]$, obtaining rational a_j with error at most $1/N$. Set

$$E_L = \min_{0\leq j\leq N}\{Aa_j + (B-g)j/N\}.$$

Rounding lowers no branch by more than A/N. Moving a dual minimizer to the grid costs at most $(A + |g - B|)/N$, and rounding then costs at most another A/N. This proves both bounds. □

Thus computable finite energies, rational volume-grid slopes, concavity, homogeneity, a boundary-to-volume rate and constancy of an exponent on an open parameter domain do not force a computable exponent. The missing microscopic requirement is one common fixed finite local table whose configurations obey consistent restriction and gluing rules. The independently rounded branches do not supply that table. Thermal leading onsite responses with $\zeta > 1$ would also contradict Proposition 20.2; adding an analytic response changes the leading observable.

One simple sufficient transfer principle is still useful. If an exponent is constant on an open set and one member has a valid finite certificate for a computable value, equality transfers that value to every member. Uniform computation needs the certificate or an effective guaranteed search. Density of rational interaction tables does not give density of certified computable exponents. A more explicit certificate consists of positive computable $X(t)$, effective $c, C, t_0 > 0$ and $m(k)$ such that

$$ct^{\alpha+2^{-k}} \le X(t) \le Ct^{\alpha-2^{-k}} \quad (0 < t \le \min(t_0, 2^{-m(k)})).$$

Then the normalized logarithm differs from α by at most $2^{-k} + \max(|\log c|, |\log C|)/|\log t|$. Choose a small dyadic t effectively and evaluate its logarithm, finding a positive lower enclosure first. This computes α. It assumes neither a computable derivative of every computable pressure nor that arbitrary continuous critical recentering is effective.

20.4 An exact calibrated ruler and its perturbations

Fix a finite-dimensional local-potential basis and subtract its vacuum bulk and linear wall contributions. Assume that the *full* thermodynamic energy, for every perturbation in the neighborhood in question, is exactly

$$e_v(g) = \min\left(0, \inf_{p\in\mathcal{P}}\left[-\frac{g}{p} + \frac{C(p) + v\cdot R(p)}{p^2}\right]\right). \tag{56}$$

Here $C(p) > 0$, $\log C(p)/\log p \to a \in (0,1)$, $\|R(p)\|_1 = o(p)$, and the unbounded pitch set is logarithmically dense: the least available $p \ge x$ has $\log p/\log x \to 1$. In a pattern-count interpretation the complete vector is $\alpha p^2 + \beta p + R(p)$; its bulk term has been subtracted and the physical field is $g + v\cdot\beta$. Equation (56) is an assumption about all competing configurations, not just clean cells. Establishing it for one finite local rule is a separate obligation.

Lemma 20.6 (Logarithmically dense ruler optimization)**.** *If $D(p) > 0$ has logarithmic order $a < 1$ on this pitch set, then*

$$s(t) = \sup_p\{t/p - D(p)/p^2\} = t^{q+o(1)}, \quad q = 1 + \frac{1}{1-a}, \qquad s'(t+) = t^{1/(1-a)+o(1)}.$$

Proof. For small $\delta > 0$ and all sufficiently large p, $p^{a-\delta} \le D(p) \le p^{a+\delta}$. Each fixed finite pitch has negative profit for small t. Maximizing $t/p - p^{a-\delta-2}$ over all positive real p bounds s above by a constant times $t^{1+1/(1-a+\delta)}$. For a lower bound choose $x = (2/t)^{1/(1-a-\delta)}$ and the least available $p \ge x$. Then $D(p)/p \le t/2$ and $\log p/\log x \to 1$, so $s(t) \ge t/(2p)$ has order $1+1/(1-a-\delta)$. Squeeze logarithmic orders and let $\delta \downarrow 0$. Convexity and $s(0) = 0$ give $s(t)/t \le s'(t+) \le s(2t)/t$, proving the derivative order. The supremum is nonnegative because the branches tend to zero. □

Theorem 20.7 (Residual-to-cost criterion)**.** *Within* (56)*, a sufficiently small full ball of perturbations preserves the positive density-onset exponent $1/(1-a)$ at the actual calibrated transition if and only if*

$$\sup_{p\in\mathcal{P}} \frac{\|R(p)\|_1}{C(p)} < \infty. \tag{57}$$

If the ratio is unbounded, every perturbation ball contains one fixed signed-coordinate perturbation giving a finite-pitch first-order onset. Its vacuum-relative order is zero and its right-tangent positive order is undefined.

Proof. If the bound is K, choose $\|v\|_\infty < \eta$ with $\eta K < 1/2$. Then $D_v = C + v \cdot R$ lies between $C/2$ and $3C/2$. It has order a, is positive and obeys $D_v/p \to 0$. The vacuum wins for $g \le 0$, and every $g > 0$ beats it at large enough pitch. Thus $g_c = 0$, and Lemma 20.6 gives the common exponent. The right derivative at zero is zero, so both response conventions agree. If $K = 0$ the conclusion is immediate.

If the ratio is unbounded, finite dimensionality supplies a fixed coordinate and sign σ with $\sigma R_i/C$ unbounded above along a subsequence. Choose any sufficiently small fixed rational $\varepsilon > 0$ and $v = -\varepsilon\sigma e_i$. Then D_v is negative somewhere. The thresholds $h(p) = D_v(p)/p$ tend to zero, so their negative infimum h_* is attained among finitely many pitches. Let p_* be the smallest minimizer. For $g = h_* + t$ and $p \ge p_*$,

$$\frac{h(p) - h_* - t}{p} \ge -t/p \ge -t/p_*.$$

For the finitely many $p < p_*$ the same comparison holds whenever $t \le (h(p) - h_*)/(1 - p/p_*)$. A positive common right interval therefore exists. Below h_* the vacuum wins, while on that interval $e_v(h_* + t) = -t/p_*$. This is the actual shifted onset, with jump $1/p_*$. Subtracting its right tangent leaves zero. □

Pattern-discrepancy control relative to defect costs has an important antecedent in Miękisz's strict-boundary stability analysis [34]. The recentered finite-pitch conclusion above additionally uses the exact envelope and its tail behavior. It is not a proof that a full robust lattice realization exists.

An integer-consistent example makes the failure explicit. Let $p_n = n^4$, $C(p_n) = n$, and formal pattern counts

$$(n^8 - n^2 - n, n^2, n), \qquad R(p_n) = (-n^2 - n, n^2, n), \quad n \ge 2.$$

They are nonnegative and sum to p_n^2, while $\|R\|_1/C = 2n + 2$. The unperturbed density order is $4/3$. Biasing the second pattern by $-\varepsilon$, with $\varepsilon = 3/(2N)$ and integer $N \ge 3$, gives $h(n) = n^{-3} - \varepsilon n^{-2}$ and the exact identity

$$h(n) - h(N) = \frac{(n - N)^2(n + 2N)}{2N^3 n^3} \ge 0.$$

Hence $g_c = -1/(2N^3)$, $p_* = N^4$ and the density jump is N^{-4}. The perturbation norm tends to zero as $N \to \infty$, but each fixed perturbation preempts the positive onset. These counts do not constitute a common finite local rule.

20.5 Static obstructions to shortcuts from noisy computation

The following exact checks separate the reliable trajectory theorem from an equilibrium energy estimate. First, on a trajectory cloned across a transverse $W \times W$ torus, every full-symbol majority agrees with the base symbol. The stacked prediction is the base prediction, even for erroneous base histories. Therefore both update-error counts and any stipulated replicated additive marks are multiplied by exactly W^2, including differences between two histories. Transverse cloning alone does not improve their energy ratio.

Second, start a majority history with an all-one plane. Change the entire next plane to zero and retain zero thereafter. This costs exactly JW^2 in update errors. A positive mark cost r on ones charges the honest history rW^2T, so the competitor wins for $T > J/r$. There is no uniform estimate bounding all time-distributed marks by a constant times these faults. This organized deterministic plane error does not contradict stochastic low-noise reliability.

Third, the blank/nonblank trajectory model of [42, equation (13), Sections 3.2 and 4] has, after a constant subtraction, update cost JD with $J = \log(16(1 - \varepsilon_0)/\varepsilon_0) > 0$. It has an all-blank zero-error

trajectory and fully nonblank zero-error trajectories from valid tilings. If Q counts nonblank sites, adding blank and nonblank fields and subtracting the vacuum constant gives $H = JD + (\varepsilon - g)Q$. Since $D \geq 0$ and $0 \leq Q \leq |\Lambda|$, its energy density is bounded below by $\min(0, \varepsilon - g)$. The two zero-error endpoints attain the bound; restricting to open boxes costs only boundary terms. Thus

$$e(g) = \min(0, \varepsilon - g).$$

This is a density jump of one, not the positive tangent onset sought above. At $g = 0$, every translation-invariant ground-state measure has zero expected $JD + \varepsilon Q$ and hence is all blank. This measure assertion is not uniqueness of every configuration under finite modifications.

Finally the desired arithmetic can be scheduled without thereby solving the static construction. For computable rationals $a_k \to a \in (0, 1)$, set $s_0 = 0$, $d_k = [s_{k-1} < ka_k]$, $s_k = s_{k-1} + d_k$, $p_k = N^k$, $m_k = N^{s_k}$ for fixed $N \geq 2$. For any eventual band $0 < L < a_k < U < 1$, induction gives $Lk - C \leq s_k \leq Uk + C$: a zero step certifies the lower bound, and a one step certifies the upper bound; the opposite bounds propagate. Hence $s_k/k \to a$ and $\log m_k / \log p_k \to a$. Choosing N children or one child at stage k gives this count. For a slow rational generator, simulate it for k steps and retain the latest completed rational. Completed indices still tend to infinity and finite-string feedback comparisons have polynomial overhead in k. This supplies a counting schedule, not protected calibrated minimum costs, an interaction compiler or the full energy identity (56).

These results complement the explicit instability of the programmed rulers and the instability theorem of Castilla-Castellano–Lucia [9]. Neither instability result proves a general equivalence between robustness and computability. The restricted certificates above identify sufficient mechanisms and the exact ruler theorem identifies its own necessary cost control; the general equilibrium question remains separate.

20.6 Anchored robust density and counting laws

The reliable computation of Section 19.4 also supports a fixed scale readout with an exact arithmetic range. The anchors are essential: a computable bottom configuration, a distinguished program origin, infinite cloning in two directions, and a prescribed time-then-window limit. This is a theorem about anchored causal histories, not an intrinsic equilibrium gap exponent.

Fix a finite alphabet Σ, a finite neighborhood and a positive product-update kernel K. Its simplex has one probability vector for each predecessor pattern. A law is robust on a nonempty relatively open set U when, for each fixed initial configuration under consideration, it exists and is constant as K varies in U. Different programs may have different robust laws. At computable kernel entries and a computable initial configuration, every finite spacetime cylinder probability is computable by its finite ancestor graph. All supports, site maps and cylinder functions below are uniformly computable from their integer indices: one algorithm returns each finite support and computes its cylinder function to any requested rational accuracy. This includes the canonical geometry and projection, which are fixed independently of the program and kernel. A fixed but noncomputable choice of sites would instead supply extra advice.

At even $t \geq 2$ choose the odd, computable schedule

$$m_t = 2\lceil \log_2 t \rceil^2 + 1.$$

The ceiling here is computed by integer bit length. This rounded logarithmic-square schedule has the same summability and growth-versus-mixing properties used below. For a fixed bit projection b and computable site map $z(n, t)$, let $O(n, t)$ be the majority of $b(X_t)$ at $z(n, t) + \ell(2t + 1)e_1$,

$0 \le \ell < m_t$. The redundant-coordinate speed of the stack is at most one, so these endpoints have disjoint depth-t noise cones. Put

$$\rho_N(t) = N^{-1} \sum_{n<N} O(n,t), \qquad c_N(t) = \sum_{n<N} O(n,t),$$
$$\Phi_{\mathrm{dens}} = \lim_{N\to\infty} \lim_{t\to\infty} \mathbb{E}\rho_N(t), \qquad \Phi_{\exp} = \lim_{N\to\infty} \frac{\log(1 + \lim_{t\to\infty} \mathbb{E}c_N(t))}{\log N}. \tag{58}$$

Use $N \ge 2$ in the exponent, and restrict to existing indicated limits. In particular 1+count, rather than the count itself, is strictly positive. The sample analogues use the same prescribed order. They are not assertions that one trajectory executes the outer limit in time. A depth-d readout means d specified nested existing limits of expectations of uniformly computable finite cylinder functions, with any stated computable continuous operations. This broad decoder class can itself carry a target program: a constant cylinder function $N^{-1}\sum_{n<N} a(n,t)$ can ignore the world and have any D_3 density as its two-limit value when the decoder varies with the target. The result below instead fixes one counting readout before choosing the finite program in the dynamics. It does not classify every computable decoder by a memory mechanism.

Proposition 20.8 (Robust depth ceiling)**.** *At computable data and kernel, a depth-d value is in D_{d+1}. If it is robust on a nonempty open kernel set, the same absolute ceiling holds throughout that set, including its noncomputable kernels.*

Proof. Replace each finite expectation by rational approximations with effective vanishing errors. One existing limit is computable in $0'$ by Cauchy-tail search; iterate relative to successive jumps. For (58), the nonnegative inner count has a $0'$-name, which computes the logarithm of one plus it. A further existing limit gives $0''$. Every nonempty relatively open region of the finite positive-rate simplex contains a computable kernel. A robust value equals its value there; this is witness transfer, without a uniform algorithm locating a witness in an arbitrary open set presentation. □

Theorem 20.9 (Anchored robust third level)**.** *One fixed computable universal three-dimensional stacked rule G, one nonempty open positive-rate kernel region U, and one fixed computable projection and site map in* (58) *have the following property. Starting from the computable bottom configuration obtained by infinitely cloning a finite program at a distinguished origin, the ranges over programs of the existing density laws and of the existing counting-exponent laws are each exactly $\mathsf{D}_3 \cap [0,1]$, for every $K \in U$. Each law is independent of K, and its expectation and sample iterated limits agree almost surely. All these limits retain the prescribed time-then-window order and the stated anchors.*

Proof. Fix a universal ordinary interpreter with a product tape alphabet: a protected output-bit track, scratch tracks and head/state fields. Every ordinary instruction copies the output track. Only a designated write instruction changes a selected output bit, including when a head passes through its cell. Initialize all output bits to zero, the finite program and head near the origin, and other scratch cells blank. A fixed radius-r local CA simulation preserves this track invariant. At stage s the interpreter computes a total binary approximation $a(0,s),\dots,a(s,s)$ and writes register n at original coordinate $-2r(n+1)$. If $a(n,s) \to A(n)$, infinitely many completed stages visit n and all sufficiently late writes repeat $A(n)$. Between writes the output stays fixed. Thus its logical bit $L_n(t)$ stabilizes, without an effective settling bound.

For shift $(Sx)_i = x_{i+1}$, set $H = S^r F$ and group $2r$ cells. Then $H^t(x)_i = F^t(x)_{i+rt}$ and the blocked neighborhood is $\{0,1\}$. At even times the register is at blocked coordinate $-(n+1) - t/2$, internal offset zero. Hence

$$z(n,t) = (0, 0, -(n+1) - t/2)$$

and one fixed bit projection work for every program. Stack this blocked rule using full-symbol north-east-center majority. The Toom/Gács–Reif reliability theorem, in the arbitrary-trajectory form [42, Theorem 3.1], gives a positive rational ε_0 for site error target $1/4$. Define

$$U = \{K : K(a \mid r) > 0 \text{ for all } a, r, \quad K(G(r) \mid r) > 1 - \varepsilon_0 \text{ for all } r\}.$$

The rule, region and readout are fixed before choosing the program. Product updates give the required finite joint-fault bounds by conditioning backwards over time layers. The cloned bottom is computable, even though its redundancy is infinite.

The m_t endpoint symbols use disjoint independent noise variables. A bit error requires a symbol error, of probability at most $1/4$. Hoeffding therefore gives

$$\Pr\{O(n,t) \neq L_n(t)\} \leq e^{-m_t/8} =: e_t, \qquad |\mathbb{E}O(n,t) - L_n(t)| \leq e_t. \tag{59}$$

The bound is summable over even t, since m_t grows as $(\log t)^2$. Borel–Cantelli gives eventual equality to the current logical bit for each fixed n. A countable intersection covers all registers. Hence the inner limits of every fixed window, both in expectation and almost surely, are exactly its count in A. The probability-one event can depend on K.

For any $\beta \in \mathsf{D}_3 \cap [0,1]$, take uniformly $0'$-computable rationals $u_s \to \beta$ in $[0,1]$. With $N_0 = 1$, $A(0) = 0$, $N_{s+1} = (s+2)N_s$, put the mechanical bits $A(N_s + j) = \lfloor (j+1)u_s \rfloor - \lfloor j u_s \rfloor$ on the successive blocks. If $d_s = |A \cap [0, N_s)|/N_s$, then for $N = N_s + j$,

$$\left| \frac{|A \cap [0,N)|}{N} - \beta \right| \leq \frac{N_s}{N}|d_s - \beta| + \frac{j}{N}|u_s - \beta| + \frac{1}{N}. \tag{60}$$

At the next endpoint the first weight is $1/(s+2)$, so endpoint errors tend to zero. Equation (60) then proves convergence at every intermediate prefix, including $\beta = 0, 1$. The set is $0'$-computable. The ordinary limit lemma supplies the total computable approximation executed by the interpreter. This density correspondence is classical [16, 15]; it is not a new density theorem.

For the counting exponent $\gamma \in \mathsf{D}_3 \cap [0,1]$ use $0'$-computable rational $u_k \to \gamma$ and

$$A = \bigcup_{k \geq 1} [2^k, 2^k + M_k), \qquad M_k = \lfloor 2^{k u_k} \rfloor.$$

For $u_k = a/b$, M_k is the largest integer m with $m^b \leq 2^{ka}$, computed by exact integer comparison. Thus a floor at integer equality does not require a new real-equality oracle. If $2^K \leq N < 2^{K+1}$, the full preceding block gives lower bound M_{K-1} and the sum through block K gives an upper bound. Eventual $|u_k - \gamma| < \eta$ yields

$$(K-1)(\gamma - \eta) - O(1) \leq \log_2(1 + |A \cap [0,N)|) \leq K(\gamma + \eta) + O(\log K) + O(1),$$

with zero as the lower bound if $\gamma < \eta$. Divide by $\log_2 N \in [K, K+1)$ and let $\eta \downarrow 0$. This includes both endpoints. Compile this set's approximation in the same way. Conversely, suppose all inner prefix expectation limits exist at any $K \in U$, including a noncomputable kernel. The amplification error in (59) tends to zero, so each deterministic integer prefix count $\sum_{n<N} L_n(t)$ converges along the sampled even times and is eventually constant. Differences of consecutive prefix counts make the corresponding logical output bits eventually constant along those times. If the exponent starts at $N = 2$, choose the first two bits to have that prefix's eventual count; this finite choice changes neither asymptotic statistic. The resulting set is limit-computable, independently of K. Every stipulated fixed-window expectation and sample limit is its count, uniformly throughout U. Its existing density or counting order is in D_3 by the same relativized limit argument. Thus necessity does not assume robustness for an arbitrary program at a noncomputable kernel. Proposition 20.8 supplies the corresponding ceiling at a computable witness. □

The programs realizing a prescribed density and a prescribed counting exponent may differ. A positive-density set has counting exponent one; the theorem does not assert arbitrary independent pairs for one program. The fixed-rule synthesis uses the prior reliable-computation technology [43, 21, 42] and the asymptotic-density literature [16, 15]. At fixed finite time, only finitely many logical output registers can have been reached from the program. The remaining far-blank region has one common noisy-majority expectation $q_t(K) \le e_t$, rather than an exactly zero physical value. Thus $\lim_t \lim_N \mathbb{E}\rho_N(t) = 0$, whereas the prescribed opposite order is β. For $\beta \neq 0$ the two orders differ.

20.7 Prescribed higher depths and restricted memory mechanisms

Corollary 20.10 (Anchored higher-depth readouts)**.** *For each fixed $d \ge 2$, the same protected interpreter, kernel region and anchored initialization realize exactly $\mathsf{D}_{d+1} \cap [0,1]$ by a fixed depth-d readout with $d-2$ tuple-index limits outside time and window. A separate fixed one-limit weighted readout realizes $\mathsf{D}_2 \cap [0,1]$. The computable bottom boundary, distinguished program origin, infinite cloning and prescribed outer limits are retained.*

Proof. Iterated relative limit approximation expresses a target as $d-2$ existing limits of uniformly $0''$-computable real names $c(k_1, \dots, k_{d-2}) \in [0,1]$. The conversions must use names, not uniformly decidable strict rational cuts. Explicitly, for $0''$-computable rational approximants $q(k,j)$ with error 2^{-j}, the relative limit lemma gives $0'$-computable guesses $q_s(k,j)$ eventually equal to each rational code. At stage s choose the largest $m \le s$ for which the first m closed rational error intervals meet $[0,1]$, and output a rational in that intersection; use a default if necessary. Each fixed finite prefix is eventually correct, so $m \to \infty$ and the outputs converge to $c(k)$. This avoids an unjustified diagonal approximation to the jump. Apply the mechanical blocks uniformly in the tuple, then the uniform ordinary limit lemma to obtain $a(k,n,s)$. Fix a computable encoding of tuples into register indices and service the first s encoded registers at stage s. Every register is visited infinitely often. The prescribed outer limits now give the target. Conversely, at any kernel where the stipulated limits exist, the consecutive-prefix argument above applies uniformly to every tuple. It gives a limit-computable logical table whose inner values agree throughout U; the depth ceiling then gives necessity. The readout is fixed separately for each d; changing depth changes the definition, not the physical rule. For $d = 1$, use $\sum_{n<t} 2^{-n-1} O(n,t)$. A $0'$-computable binary expansion of a target has a computable stabilizing bit approximation. At dyadic targets choose a fixed computable expansion; for target one use the all-ones fractional expansion. Finite-prefix stabilization and the geometric tail give expectation and almost-sure convergence. This is a one-limit readout, not the two-limit density renamed. □

Three restricted comparisons are useful, without being exhaustive. For a kernel with Dobrushin coefficient $\kappa = \sum_v c_v < 1$, sitewise coupling gives marginal distance at most κ^t from its unique stationary law. Let its projected bit marginal be π. Disjoint-cone endpoint samples from deterministic data remain independent. If $\pi \neq 1/2$, an eventual fixed bias and majority amplification give expectation limit zero or one. If $\pi = 1/2$, couple the m_t bits to independent fair bits to obtain error at most $m_t \kappa^t \to 0$. The density expectation law is therefore in $\{0, 1/2, 1\}$ and its counting exponent in $\{0, 1\}$. At a fixed kernel the initial-data image is a singleton; the three possibilities are across kernels. No almost-sure convergence of a Boolean readout to $1/2$ is asserted. For a general mixing error δ_t, the stronger condition $m_t \delta_t \to 0$ suffices; the sharper condition $\sqrt{m_t}\delta_t \to 0$ also follows from binomial comparison. Arbitrarily growing majorities need not forget.

For static Toom memory, use the stationary site map $z_{\mathrm{id}}(n,t) = (0, 0, -(n+1))$ for an unshifted binary identity rule, with the same transverse copy spacing. Alternatively shift and block both

the identity dynamics and the readout together. Using the moving universal-computation site map with an unshifted identity would sample different tape bits. With the matching coordinates, every computable tape is recovered with error e_t, so its existing density lies in D_2; computable mechanical tapes attain every such value. In the same universal rule as Theorem 20.9, programs with a computable logical settling bound $S(n)$ likewise give exactly D_2 densities: $A(n) = L_n(S(n))$ is computable, and write-once total programs realize all computable tapes. Here $S(n)$ is a bound in physical CA time; a bound given in interpreter stages is converted by deterministic simulation until the corresponding commits finish. Choose computable even $T(N)$ beyond the first N settling bounds with $Ne_{T(N)} \le 2^{-N}$. The observed mean then converges to its density almost surely by a union bound and Borel–Cantelli. This is a data-reading realization, with the anchors and growing support retained.

At a computable kernel and computable data, a supplied computable inner modulus $S(n, j)$ makes the inner expectations uniformly computable; one remaining existing limit is in D_2. Such effectiveness is pointwise unless uniform modulus data are supplied. A name of a computable kernel does not uniformly decide whether $\pi = 1/2$. These are sufficient mechanisms, not an exhaustive classification or a necessary condition for a particular law to be computable. For example, put halting-set guesses only at register indices 2^j, all other bits zero. The eventual set has density zero but no computable uniform settling bound. Robust equality transfers the ceiling from a computable kernel to the whole region. The Cesàro and full-Gibbs-boundary moduli of Proposition 19.3 give additional restricted effective-inner-limit mechanisms. For equilibrium responses, Section 16 computes pressure and first derivatives at computable points of promised differentiability. With a computable critical point and tangent, a strictly positive computable tangent deficit has an existing logarithmic order in D_2. A positive response $f(t) = t^{a+o(1)}$, $a > -1$, has integral of order $a + 1$: bound it by $t^{a\pm\eta}$ and integrate, taking $\eta < a+1$. This transfers that conditional deficit-order ceiling to the corresponding response order. The critical data and positivity are substantive hypotheses. The known unique-measure mechanism is credited in Theorem 19.2.

20.8 Rates and robust hidden settling scales

Proposition 20.11 (Rates with computable reference scales)**.** *Let uniformly computable $q_s \to r$. If r is noncomputable, every positive computable $h(s) \to 0$ is exceeded by $|q_s - r|$ infinitely often. Against a positive computable reference $h_0(s) \to 0$, any existing finite exponent $\lim_s \log|q_s - r| / \log h_0(s)$ is zero. If r is computable and the errors are eventually nonzero, that existing finite exponent is in D_2.*

Proof. An eventual error bound $h(s)$, with a finite cutoff hard-coded, would compute r by searching for sufficiently small $h(s)$ and approximating q_s. A positive rate exponent would supply such a bound $h_0(s)^a$ for a positive rational a. Since the error tends to zero, the defined quotient is eventually nonnegative. For computable r, positive errors and their logarithms are computable on the promised domain, so the exponent is an existing limit of computable values. Zeros need a separate convention and are not passed silently through a real logarithm. □

Fix a program in Theorem 20.9 with a density or counting exponent $\beta \notin \mathsf{D}_2$. Let $\tau(N)$ be the least time after which its first N logical bits never change. This deterministic quantity is identical for every $K \in U$. For every total computable B, infinitely many N have $\tau(N) > B(N)$: an eventual bound, with the finite cutoff hard-coded, would compute every final bit and make the existing law D_2. Consequently the usual power, exponential and fixed-height iterated-exponential growth comparisons have no finite upper exponent; an extended value $+\infty$ is not excluded.

After $t \ge \tau(N)$, expectation error of the count is at most Ne_t, and the probability of any erroneous bit in the window is at most Ne_t. A sufficient accuracy time is the maximum of $\tau(N)$

and a computable amplification margin. This is not an equality of settling times or a deterministic last-fault bound. At each $K \in U$, even a physical guarantee of window-density error below $1/(4N)$ for every $t \geq B(N)$ cannot have a computable eventual upper bound. Otherwise choose computable even $t \geq B(N)$ with $e_t < 1/(4N)$. The current logical prefix density is within $1/(2N)$ of the final one. Both lie on the $1/N$ grid, so their counts are equal. This computes all final prefix counts and hence the noncomputable set.

At *computable* K, any computable even diagonal $\sigma(N)$ gives computable finite expectations. An existing diagonal limit is D_2, hence cannot be β. The corresponding sample diagonal, using rational approximations for the logarithmic statistic when necessary, is an effective inquiry; Theorem 21.1 makes its fixed-target convergence probability zero. The physical law itself is still the specified iterated limit.

For the density there is also a pointwise statement at every $K \in U$ along any computable unbounded diagonal. First suppose $\sigma(N) \to \infty$ and set $x_N = N^{-1} \sum_{n<N} L_n(\sigma(N))$, a computable rational sequence. The expectation differs from x_N by at most $e_{\sigma(N)} \to 0$, so it cannot converge to $\beta \notin \mathsf{D}_2$. For samples, the expected fraction of incorrect registers is at most $e_{\sigma(N)}$; Markov's inequality gives $\Pr\{|\rho_N(\sigma(N)) - x_N| > a\} \leq e_{\sigma(N)}/a$. Some deterministic subsequence stays more than $2a$ from β, since x_N cannot converge to it. On a sample-convergence event the displayed deviations exceed a eventually on that subsequence, an event of probability zero by the bound. A computable unbounded σ has a computably chosen subsequence tending to infinity, so the same exclusion applies. Bounded-time paths retain the computable-kernel qualification. An additive density error does not control logarithms of sparse counts, so the counting-exponent diagonal statement also retains that qualification. In the constant-normalizer thermal family of Section 19, the same statement is a four-dimensional causal half-space DLR construction with computable fixed bottom boundary; time is height. It does not imply unique equilibrium or arbitrary-interaction stability.

20.9 Null and meager, but dense, exceptional kernels

Here an inquiry is an external computable controller allowed to inspect successively larger finite spacetime cylinders of the anchored history, including time-one observations. This access model is essential for the density conclusion below. It does not assert an autonomous noisy computer that internally acquires raw-fault data.

Theorem 20.12 (Kernel information and fixed-target inquiry)**.** *Fix an anchored program and a law $\beta \in \mathsf{D}_3 \setminus \mathsf{D}_2$ in Theorem 20.9. At every computable $K \in U$, no computable inquiry in this model converges to β with positive probability. More generally, every fixed target reached with positive probability at K is computable in the kernel oracle's jump K'. The kernels in U permitting inquiry to β form a null, meager but dense subset, in intrinsic simplex measure and topology.*

Proof. Finite cylinder laws are computable relative to a name of K. Relativize the proof of Theorem 21.1: on success a target name is computable in $(X \oplus K)'$. For almost every tape X, relative generalized lowness bounds this by $X \oplus K'$. The relativized positive-measure cone argument then places the fixed target below K'. This proves the computable-kernel conclusion.

For measure and category, use $q - 1$ free probabilities per row; the last entry is recovered by subtraction. Cover the positive simplex by rational boxes in these intrinsic coordinates. Rescaling each box to a cube and interleaving binary digits preserves degrees off the null, meager dyadic hyperplanes. For almost every code Y, $Y' \equiv_T Y \oplus 0'$. A positive-measure set on which $\beta \leq_T K'$ would thus give a positive-measure cone relative to $0'$, contradicting $\beta \not\leq_T 0'$.

For category, 1-generic codes are comeager and also satisfy $Y' \equiv_T Y \oplus 0'$. Indeed a generic oracle either meets a c.e. set of finite halting witnesses or has a prefix with no extension in it; $Y \oplus 0'$ can

find which alternative applies. If a fixed functional computes a non-Z-computable set from $Y \oplus Z$ on a nonmeager Borel set, its success set is comeager in some cylinder after restriction. Search relative to Z for any finite extension on which it halts on the requested input. Such an extension exists, and its open cylinder meets the comeager success set, so every found answer is correct. This would compute the set from Z. Countably many functionals give the category cone theorem. Use $Z = 0'$ and a cut/name of β. Thus learnable kernels lie in a null and meager set, using free coordinates rather than a fictitious fair-random join of constrained row entries.

For density, in any open patch choose one blank-row probability p in a rational interval (u, v) whose closure stays inside the patch. Choose rational $u < u_0 < v_0 < v$ and set

$$p = u_0 + (v_0 - u_0) \sum_{j \geq 0} 2\, \chi_{0'}(j) 4^{-j-1}.$$

The separated base-four digits give $0' \leq_T p$. In particular p is noncomputable, hence irrational and non-dyadic. Compensate in a second entry in the same row; the patch's positive margins allow this arbitrarily small change. Other free coordinates can be rational. The sampled entry itself therefore satisfies $\beta \leq_T 0'' \leq_T p'$. At time one, fresh known blank predecessor neighborhoods away from the finite program give independent samples of that output, with probability p. Let f_n be the frequency and choose the rational radius $r_n = 1/\lceil n^{1/3} \rceil$, computed by integer comparisons. Hoeffding makes failure of $[f_n - r_n, f_n + r_n]$ to contain p summable. Eventually these intervals contain p, shrink to it and determine shared binary prefixes of unbounded length, since p is irrational. Choose a total p-computable rational sequence $\Theta^p(m) \to \beta$ by the relative limit lemma. Only p is the oracle: a prefix estimated from one entry is not a name of the whole kernel. At stage n simulate the first n computations for n steps using the current finite prefix as a partial oracle, and output the largest completed index not querying beyond it, or a default. Eventually the prefixes are correct, and each fixed computation eventually fits and finishes; the selected indices tend to infinity. Thus this external finite-stage inquiry converges to β almost surely. Every open patch contains such a kernel. □

The reach set is targetwise existential. This does not prove that one common learner identifies every unknown law, or that every K'-computable target can be read from every kernel without an access theorem. Robustness of the law is pointwise on U; unlearnability is not, because the exceptional kernels are dense.

20.10 A restricted statement about active records

Strict positive rates give a minimum transition probability $\varepsilon_{\min} > 0$. For any fixed finite set V in a current layer, every pattern on it has conditional probability at least $\varepsilon_{\min}^{|V|}$ given the past. Conditional Borel–Cantelli implies every such pattern recurs infinitely often. Thus a fixed nonconstant function of that fixed window cannot converge almost surely. The function and its finite support are fixed, and it is observed at every integer time. This does not imply that every convergent numerical estimator has growing support: the one-site function $X_t(0)/(t+1)$ for a binary spin converges to zero despite being nonconstant at each time. Constants, expectations and repeated access to a fixed past record also give no such conclusion. The separated majorities above have growing support by their definition; that property is asserted for those readouts only.

21 What an effective inquiry can learn

An effective adaptive protocol has finitely encoded histories, a computable choice of its next finite experiment and rational output, and uniformly computable finite conditional outcome laws on valid

positive-probability histories. Every successful stage finishes in finite time; no computable cost or convergence bound is assumed. Finite systems with algebraic Hamiltonians and measurements, and evolution for computable finite times, satisfy these hypotheses: matrix exponentiation is computable, although its entries need not remain algebraic. The controller may use independent fair coins. Noncomputable control parameters, oracle states and an infinite target-advice tape are outside this model. The conclusion is conditional on this effective experimental description.

Theorem 21.1 (Fixed-target randomized reach)**.** *If such a protocol's rational outputs converge to a fixed real α with positive probability, then $\alpha \in \mathsf{D}_2$. Conversely every D_2 real is the almost-sure limit of some such protocol. Consequently a fixed exponent outside D_2 in Theorem 1.4 has zero convergence probability for every effective protocol in this class.*

Proof. Couple all outcomes to a single fair binary tape X. Recursively partition a unit interval into subintervals in the conditional probabilities prescribed after each history. Every finite-history endpoint is a computable real. Outside their countable null set, strict comparisons are decided after finitely many tape bits and probability approximations. Interval lengths reproduce all finite joint laws, including adaptive settings. Direct controller coins can be included as further steps. Thus outputs q_k are partial X-computable functions defined at every successful stage outside that null set.

On the measurable event $q_k \to \alpha$, the jump X' computes a Cauchy name of α by deciding whether a proposed rational tail contains a deviation beyond a requested tolerance. Almost every tape is 2-random and satisfies $X' \leq_T X \oplus 0'$. This generalized-lowness theorem is due to Sacks and Stillwell; see [37, Corollary 3.12] and its attribution through Kautz's thesis. Intersecting with the success event preserves its positive measure. If α is irrational its rational lower cut is computable from its Cauchy name; if it is rational it is already computable.

We use the following relativized cone argument. If a fixed set A is computable from $X \oplus Z$ on a positive-measure set of tapes, some fixed functional does so on a positive-measure subset. The Lebesgue density theorem supplies a finite cylinder on which this subset has relative measure greater than $1/2$. For each input, enumerate relative to Z the halting cylinders yielding answer zero and those yielding answer one. Their measures are lower semicomputable and disjoint; only the correct answer reaches a certified measure greater than $1/2$ in the chosen cylinder. Waiting for it computes A from Z. The cylinder and functional are finite nonuniform constants. This is the usual positive-measure cone mechanism; see also [6]. Apply it with $Z = 0'$ and the cut of α. Thus $\alpha \leq_T 0'$.

Conversely choose a computable rational approximation to a given D_2 real. A protocol that prints it and ignores any dummy experiment succeeds on every run. Finally, a positive success probability for an exponent outside D_2 would contradict the first conclusion. □

The deterministic converse is the classical limit-computability phenomenon of Gold and Putnam [24, 40]. The randomized obstruction combines generalized lowness for almost every oracle with the relativized positive-measure cone theorem. Our use here is their fixed-target formulation for effective adaptive experiments and its application to the constructed gap exponents. This target-by-target existential statement is distinct from identification of an unknown system by one learner, as studied in formal learning theory [28].

The target is fixed independently of the outcome tape. A computable protocol can output successive binary prefixes of a random real, whose realized limit is almost surely outside D_2; each fixed target then has probability zero. Likewise fixing an arbitrary noncomputable tape gives an oracle computation, not an ordinary deterministic algorithm. The converse above is existential and may ignore the data; it is not a common learner identifying unknown systems. Every nonnegative

finite D_2 target also has a planar material realization here, but no spectroscopy assumption is needed for that existential statement. Only the exponents *outside* D_2, not all of D_3, have the stated zero-reach obstruction. Countably many effective protocols still give a null union of successes for any one such fixed target.

Theorem 21.2 (Hidden finite-size scale)**.** *Suppose finite gaps $\Delta(L, g)$ are uniformly computable on permitted sizes and rational $0 < g < g_0$. Suppose $D(g) = \lim_L \Delta(L, g) > 0$ and the existing rational-coupling exponent $\alpha = \lim_{g\downarrow 0} \log(1/D(g))/\log(1/g)$ is finite. Assume permitted sizes have a computable cofinal rounding-up selector $s(n) \geq n$. Let $L_*(g)$ be the least permitted size such that every later permitted size satisfies*

$$D(g)/2 \leq \Delta(L, g) \leq 2D(g).$$

If $L_*(g)$ has a total computable eventual upper bound on rational couplings near zero, then $\alpha \in \mathsf{D}_2$.

Proof. Hard-code rational $0 < r < \min(g_0, 1)$ inside the eventual domain and take $g_k = r2^{-k-1}$. Given a bound B, choose $L_k = s(\max(1, \lceil B(g_k)\rceil))$. The factor-two inequality gives normalized-logarithm error at most $\log 2/\log(1/g_k) \to 0$. Compute the finite normalized gap to rational error 2^{-k}; promised positivity permits a terminating search for a positive lower enclosure before evaluating its logarithm. These computable rationals converge to α, proving the claim. For factorial sides compute successive factorials until the requested threshold is reached. Locality is not used in this implication. □

For a beyond-D_2 exponent the contrapositive is precise: for every total computable B and rational $\eta > 0$ there is a rational $0 < g < \min(\eta, g_0)$ with $L_*(g) > B(g)$. More strongly, along any computable rational $g_k \to 0$, every computable bound $b(k)$ has infinitely many violations; otherwise the same proof would compute a convergent rational approximation. This is absence of an eventual computable upper bound, not eventual domination at every coupling or an algorithm locating violations. The least relative-error tail threshold is also not automatically an optimal experimental size or a certified stopping rule. The construction's structural pitch is computable and proportional to $1/g$; that fact neither bounds the settling threshold nor identifies the pitch with a correlation length.

Corollary 21.3 (Incompatibility with a uniform single-length tail)**.** *The beyond-D_2 realizations admit no factor-two thermodynamic tail bound valid at all permitted $L \geq Cg^{-N}$ for all small rational g, with finite constants C, N. The same holds with any computable raw threshold, including a fixed finite-parameter exponential threshold after enlarging its constants to rational or integer bounds.*

Proof. Round that raw threshold up by s. It is then a computable upper bound for the least *permitted* L_*, contrary to Theorem 21.2. For example a raw threshold 25 rounds to factorial side 120, so the rounding must not be omitted. □

In particular consider the explicit ansatz

$$\Delta(L, g) = L^{-z}\Phi(L/\xi(g))(1 + r(L, g)), \qquad \frac{\Phi(x)}{\Phi_\infty x^z} \longrightarrow 1, \quad D(g) = \Phi_\infty \xi(g)^{-z}.$$

To imply the excluded bound it needs uniform control of r on $L/\xi(g) \geq X$ for all sufficiently small g. Choose both relative factors between $3/4$ and $5/4$ there. Their product is within $[1/2, 2]$. If $\xi(g) \leq Ag^{-\nu}$ for finite constants, a larger integer power and rational coefficient give the required computable threshold; an exponential envelope works in the same way. A pointwise scaling ansatz

without a uniform thermodynamic tail does not imply this statement. The finite-size scaling hypotheses of Fisher–Barber [19] are therefore not being disproved in their ordinary applications. The result excludes extending this specified uniform ansatz to these constructed gaps.

Size-driven transitions [3] already exhibit enormous hidden scales in two-dimensional systems. Their constant-gap changes of ground-state character across a family differ from the relative gap-settling obstruction for one fixed family as coupling varies here. Neither finite checks of particular couplings nor that prior phenomenon replace the arithmetic proof of the present obstruction.

22 Open problems and physical scope

Theorem 1.4 realizes third-level bulk exponents already in two dimensions on factorial tori, with the gap above the entire ground space. Theorem 1.2 supplies a second route. Thus bulk attainment beyond $\mathsf{S}_2 \cup \mathsf{P}_2$ in this convention is settled. What remains open is attainment under stronger definitions: local nondegeneracy, an extremum over all KMS ground states, a unique ground state, all-integer-size convergence, or robustness. The internal frame still anchors the classical computation; removing such anchors is a separate question. Section 13.7 gives the definition separation, and Proposition 16.1 gives the distinct one-sided ceiling when a right-c.e. gap representation has actually been established. No minimum-dimension conclusion follows.

Conjecture 22.1 (Stable exponents). A gap exponent constant on a norm-open set of fixed-local-dimension translation-invariant affine families, with a continuously varying critical point and a fixed gap convention, is computable.

The explicit instability above is evidence for a distinction between these programmed examples and robust exponents; it does not prove this conjecture. An effectively isolated fixed point of a finite-dimensional computable renormalisation map on an effectively compact rational domain is computable: eliminate closed subregions certified to contain no zero until the remaining region has the requested diameter. With computable derivative data its linearization eigenvalues are also computable. Any exponent formula using these eigenvalues needs its stated nonzero and positivity conditions. Stability has not been shown to supply such a fixed point, isolation certificate or modulus. Uncomputably complex renormalisation flows [46] make that missing implication substantive.

Conjecture 22.2 (Anchoring and dimension). The general need for an anchor to attain two-sided inner limits, and the necessity of dimension at least $k + 1$ for k nested box-size limits in classical-forced designs, remain open.

The restricted disjoint-axis, deterministic finite-reader result in Section 18 gives a dimension bound for that architecture. A fixed-width classical strip reduces to a finite state graph. Reachable far-end labels as a function of the remaining length form a unary regular-language problem and are effectively ultimately periodic. A finite-valued existing limit of these labels is eventually constant. This restricted observation does not prove a dimension lower bound for quantum strips or all unanchored models.

Theorem 1.5 settles the robust third level for the anchored causal-history class: computable bottom boundary, program origin, infinite cloning and prescribed outer limits. Whether an unanchored translation-invariant, Bernoulli-initialized construction has the corresponding robust scale-law range remains open. The central equilibrium question is whether a computable finite-range interaction can have a unique equilibrium phase with an intrinsic uncomputable critical exponent. Theorem 19.2 computes its local numbers but does not settle an additional asymptotic exponent. Fault-tolerant growing readouts and noncomputable state selection are distinct realizations, as Section 19 shows. Other questions include a genuine uncomputable correlation-length exponent

ν and a fixed local dimension independent of the prescribed pair. A proposed structural-length construction with size-dependent cell costs remains incomplete. Ordinary negative finite exponents, infinite endpoints, alternative orders of limits and all-size torus conventions are not classified by the nonnegative attainment theorems. Changing these definitions requires new existence and sector arguments.

How this work was done

The authors developed the argument with AI models from more than one developer and take responsibility for its mathematics, attribution and presentation. The paper specifies finite classical local rules, deterministic two-site computation tableaux, guarded reversible counter moves, all-sector spectral estimates and arithmetic limit arguments. For any fixed finite program these rules compile to fixed finite matrices. No numerical diagonalization of an arbitrary program's full many-body Hamiltonian is supplied.

The ancillary directory contains a sequential runner, readable finite checks, recorded outputs and SHA-256 checksums. Its checks cover penalty localization, complete finite register product spaces, classical flag and carry rules, word interpolation identities, wall-layer transfer matrices, digit-mixing perturbations, finite film transport and nested-input delimiters. The bulk broadcast checks also cover translated equality graphs, complete source fillings, the two prefix masks, defective rulers, tie alignment, a gapless-ring witness, reader counts and tensor sums with degenerate kernels. The independent component analysis retains closed free paths as well as the main counter orbit. Floating-point eigenvalues are consistency evidence, not interval-certified proofs. The general proofs, arbitrary slow stabilization, noncomputable limits, and the higher-dimensional compiler are mathematical arguments, not numerically certified limits. The ancillary README states exactly what each check covers.

The planar-register checks additionally cover translated source phases, mask domains, literal broadcast recurrence fillings, full tensor/direct-sum gaps and open-chain spectra. The state checks cover finite causal cones, majority amplification, full DLR conditionals including future children, and the constant-normalizer restriction. Restricted-response checks cover cycle bounds, conditional variance, finite MPS contractions, dual grids, calibrated ruler preemption and arithmetic tracking. The two inquiry checkers verify adaptive finite inverse-CDF coupling, random-limit versus fixed-target quantifiers, the factor-two logarithmic estimate, factorial rounding and exact ruler ties. These are finite consistency checks, not proofs of randomized reach, noncomputability or the absence of a computable settling bound. Robust-law checks exercise protected outputs, drifting coordinates, all-prefix density bounds, exact rational-power block lengths and independent finite noise cones. These computations do not prove eternal reliability, noncomputability, or a general robust local interaction; the imported reliability theorem and the analytic arguments have their stated scopes.

The sequential psutil runner monitors memory across the child process tree. Individual checkers omit platform-specific peak-memory fields when those APIs are unavailable. The numerical checks do not depend on Windows memory APIs. Recorded process-tree memory is sampled RSS, not an operating-system historical peak. The finite noisy examples do not certify membership in an all-time reliability region or a universal interpreter implementation.

References

[1] Per Bak. Commensurate phases, incommensurate phases and the devil's staircase. *Reports on Progress in Physics*, 45(6):587–629, 1982. `https://doi.org/10.1088/0034-4885/45/6/001`.

[2] Paul Balister, Béla Bollobás, Robert Morris, and Paul Smith. Universality for monotone cellular automata, 2022. Version 3, introduction and footnote 4. https://arxiv.org/abs/2203.13806v3.

[3] Johannes Bausch, Toby S. Cubitt, Angelo Lucia, David Pérez-García, and Michael M. Wolf. Size-driven quantum phase transitions. *Proceedings of the National Academy of Sciences*, 115(1):19–23, 2018. https://doi.org/10.1073/pnas.1705042114.

[4] Johannes Bausch, Toby S. Cubitt, and James D. Watson. Uncomputability of phase diagrams. *Nature Communications*, 12:452, 2021. https://doi.org/10.1038/s41467-020-20504-6.

[5] Rodney J. Baxter. Onsager and Kaufman's calculation of the spontaneous magnetization of the Ising model, 2011. Version 2. https://arxiv.org/abs/1103.3347v2.

[6] Vasco Brattka, Guido Gherardi, and Rupert Hölzl. Probabilistic computability and choice, 2013. https://arxiv.org/abs/1312.7305.

[7] Manfred Buchacher. The Newton–Puiseux algorithm and effective algebraic series, 2025. Version 5. https://arxiv.org/abs/2209.00875v5.

[8] M. Bundaru, N. Angelescu, and G. Nenciu. On the ground state of Ising chains with finite range interactions. *Physics Letters A*, 43:5–6, 1973. https://doi.org/10.1016/0375-9601(73)90518-5.

[9] Laura Castilla-Castellano and Angelo Lucia. Instability of the undecidable behavior of the spectral gap in 1D, 2026. https://arxiv.org/abs/2607.08686.

[10] Toby S. Cubitt, David Pérez-García, and Michael M. Wolf. Undecidability of the spectral gap. *Nature*, 528:207–211, 2015. https://doi.org/10.1038/nature16059.

[11] Toby S. Cubitt, David Pérez-García, and Michael M. Wolf. Undecidability of the spectral gap. *Forum of Mathematics, Pi*, 10:e14, 2022. https://doi.org/10.1017/fmp.2021.15.

[12] R. L. Dobrushin and B. S. Nakhapetian. Strong convexity of the pressure for lattice systems of classical statistical physics. *Teoreticheskaya i Matematicheskaya Fizika*, 20(2):223–234, 1974. https://geodesic.mathdoc.fr/item/TMF_1974_20_2_a8/.

[13] Eytan Domany and Wolfgang Kinzel. Equivalence of cellular automata to Ising models and directed percolation. *Physical Review Letters*, 53:311–314, 1984. https://doi.org/10.1103/PhysRevLett.53.311.

[14] Seth Douglas and Nidhal Mghirbi. The uncomputable price of precision, 2026. Zenodo. https://doi.org/10.5281/zenodo.23736188.

[15] Rodney G. Downey, Carl G. Jockusch, Jr., Timothy H. McNicholl, and Paul E. Schupp. Asymptotic density and the Ershov hierarchy, 2014. Version 2. https://arxiv.org/abs/1309.0137v2.

[16] Rodney G. Downey, Carl G. Jockusch, Jr., and Paul E. Schupp. Asymptotic density and computably enumerable sets, 2013. https://arxiv.org/abs/1307.0040.

[17] Hamza Fawzi, Omar Fawzi, and Samuel O. Scalet. Certified algorithms for equilibrium states of local quantum hamiltonians, 2025. Version 3. https://arxiv.org/abs/2311.18706v3.

[18] Roberto Fernández. Contour ensembles and the description of Gibbsian probability distributions at low temperature, 1998. Corrected lecture notes. https://webspace.science.uu.nl/~ferna107/papers/cont.pdf.

[19] Michael E. Fisher and Michael N. Barber. Scaling theory for finite-size effects in the critical region. *Physical Review Letters*, 28:1516–1519, 1972. https://doi.org/10.1103/PhysRevLett.28.1516.

[20] Michael E. Fisher and Walter Selke. Infinitely many commensurate phases in a simple Ising model. *Physical Review Letters*, 44:1502–1505, 1980. https://doi.org/10.1103/PhysRevLett.44.1502.

[21] Péter Gács and John Reif. A simple three-dimensional real-time reliable cellular array. *Journal of Computer and System Sciences*, 36(2):125–147, 1988. https://doi.org/10.1016/0022-0000(88)90024-4.

[22] Stefano Galatolo, Mathieu Hoyrup, and Cristóbal Rojas. Dynamics and abstract computability: Computing invariant measures, 2011. Author manuscript. https://members.loria.fr/MHoyrup/computinvmeas.pdf.

[23] Léo Gayral, Mathieu Sablik, and Siamak Taati. Characterisation of the set of ground states of uniformly chaotic finite-range lattice models. *Nonlinearity*, 38:075011, 2025. https://doi.org/10.1088/1361-6544/addbb8.

[24] E. Mark Gold. Limiting recursion. *The Journal of Symbolic Logic*, 30(1):28–48, 1965. https://doi.org/10.2307/2270580.

[25] Sheldon Goldstein, R. Kuik, Joel L. Lebowitz, and Christian Maes. From PCA's to equilibrium systems and back. *Communications in Mathematical Physics*, 125:71–79, 1989. https://doi.org/10.1007/BF01217769.

[26] David Gosset and Evgeny Mozgunov. Local gap threshold for frustration-free spin systems, 2016. https://doi.org/10.1063/1.4962337.

[27] Tosio Kato. *Perturbation Theory for Linear Operators*. Springer, 1995. Reprint of the second edition. https://doi.org/10.1007/978-3-642-66282-9.

[28] Kevin T. Kelly. Learning theory and descriptive set theory. *Journal of Logic and Computation*, 3(1):27–45, 1993. https://doi.org/10.1093/logcom/3.1.27.

[29] Marius Lemm and Angelo Lucia. On the critical finite-size gap scaling for frustration-free Hamiltonians. *Reviews in Mathematical Physics*, 37(7):2550015, 2025. https://doi.org/10.1142/S0129055X25500151. Version 2: https://arxiv.org/abs/2409.09685v2.

[30] Marius Lemm and Evgeny Mozgunov. Spectral gaps of frustration-free spin systems with boundary, 2019. https://doi.org/10.1063/1.5089773.

[31] Irène Marcovici, Mathieu Sablik, and Siamak Taati. Ergodicity of some classes of cellular automata subject to noise. *Electronic Journal of Probability*, 24(41):1–44, 2019. https://doi.org/10.1214/19-EJP297.

[32] Rintaro Masaoka, Tomohiro Soejima, and Haruki Watanabe. Rigorous lower bound on dynamical exponents in gapless frustration-free systems. *Physical Review X*, 15:041050, 2025. https://doi.org/10.1103/d4c4-5p2r.

[33] Tom Meyerovitch. Growth-type invariants for $\mathbb{Z}^d$ subshifts of finite type and arithmetical classes of real numbers. *Inventiones Mathematicae*, 184:567–589, 2011. https://doi.org/10.1007/s00222-010-0296-1.

[34] Jacek Miękisz. Stable quasicrystalline ground states, 1998. Version 2. https://arxiv.org/abs/cond-mat/9301017v2.

[35] T. Morita. Orderings of one-dimensional Ising systems with an arbitrary interaction of finite range. *Journal of Physics A: Mathematical, Nuclear and General*, 7:289–300, 1974. https://doi.org/10.1088/0305-4470/7/2/014.

[36] Bruno Nachtergaele and Robert Sims. Lieb–Robinson bounds and the exponential clustering theorem. *Communications in Mathematical Physics*, 265:119–130, 2006. https://doi.org/10.1007/s00220-006-1556-1.

[37] André Nies, Frank Stephan, and Sebastiaan A. Terwijn. Randomness, relativization and Turing degrees. *The Journal of Symbolic Logic*, 70(2):515–535, 2005. https://www.math.ru.nl/~terwijn/publications/jslrevision.pdf.

[38] David Pérez-García, Frank Verstraete, Michael M. Wolf, and J. Ignacio Cirac. Matrix product state representations. *Quantum Information and Computation*, 7:401–430, 2007. https://arxiv.org/abs/quant-ph/0608197v2.

[39] James Purcell, Zhi Li, and Toby Cubitt. Chaitin phase transition, 2024. https://arxiv.org/abs/2410.02600.

[40] Hilary Putnam. Trial and error predicates and the solution to a problem of Mostowski. *The Journal of Symbolic Logic*, 30(1):49–57, 1965. https://doi.org/10.2307/2270581.

[41] Christof Schmidhuber. Chaitin's Omega and an algorithmic phase transition, 2021. Version 3. https://arxiv.org/abs/1909.09231v3.

[42] Siamak Taati and Jacek Miękisz. Quasicrystalline Gibbs states in 4-dimensional lattice-gas models with finite-range interactions, 2025. Version 1. https://arxiv.org/abs/2512.24436v1.

[43] Andrei L. Toom. Stable and attractive trajectories in multicomponent systems. In R. L. Dobrushin and Ya. G. Sinai, editors, *Multicomponent Random Systems*, pages 549–575. Marcel Dekker, 1980.

[44] James D. Watson and Johannes Bausch. The complexity of approximating critical points of quantum phase transitions, 2021. https://arxiv.org/abs/2105.13350.

[45] James D. Watson and Toby S. Cubitt. Computational complexity of the ground state energy density problem, 2021. https://arxiv.org/abs/2107.05060.

[46] James D. Watson, Emilio Onorati, and Toby S. Cubitt. Uncomputably complex renormalisation group flows. *Nature Communications*, 13:7618, 2022. https://doi.org/10.1038/s41467-022-35179-4.

[47] Xiangling Xu, Matthias Schötz, Jie Wang, Victor Magron, Igor Klep, Omar Fawzi, and Marc-Olivier Renou. The bulk spectral gap is certifiable from above but uncomputable from below, 2026. Version 4, 15 October 2026. https://arxiv.org/abs/2606.03836v4.

[48] Chen Ning Yang. The spontaneous magnetization of a two-dimensional Ising model. *Physical Review*, 85:808–816, 1952. https://doi.org/10.1103/PhysRev.85.808.

[49] Xizhong Zheng and Klaus Weihrauch. The arithmetical hierarchy of real numbers. *Mathematical Logic Quarterly*, 47(1):51–65, 2001.