Which critical temperatures are computable? Ising pair ferromagnets and ideal Bose gases

Seth Douglas Nidhal Mghirbi

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Abstract

Which critical inverse temperatures can a computable interaction have? For translation-invariant Ising pair ferromagnets on the integer line, with an effective summable tail, the finite values are exactly the positive left-c.e. reals. This answers Spandl's 2008 question. For ideal Bose gases with the same effective hopping data, positive coordinate-neighbour hopping and fixed positive rational total hopping and density, saturation points in dimensions one and two are exactly the left-c.e. reals above an explicit floor, and each value is realized by a hopping whose canonical condensation onset is that value. In dimensions at least three, saturation is computable. For finite-range periodic Ising pair ferromagnets with positive rational couplings in dimension at least two, a terminating algorithm computes the critical point; its lower search uses a proof, given here, of the quasi-transitive sharpness extension stated by Duminil-Copin, Raoufi and Tassion. A finite first moment gives a one-spin corollary. At inverse-square decay we prove saturation of the magnetization jump inequality, an open problem of Aizenman, Chayes, Chayes and Newman first claimed in a 2026 preprint by Bhattacharya. We also prove that the half-line and full-line critical points coincide there, answering a question of Berger, Johansson and Öberg. For couplings $1/r^2$, a computer-assisted bracket, verified by two independently written programs, is 0.631 to 1.598. Lower tail certificates approaching the asymptotic coefficient make the critical point computable, while every c.e. degree can occur in the approach to a fixed coefficient. For long-range hopping, the infinite-volume limit of a canonical empty-site probability can have the degree of the halting problem. For one fixed three-dimensional nearest-neighbour gas, computable nested slabs realize every finite limit-computable zero-mode onset at or above the cubic onset.

Claude Opus 5.5, GPT-6.1 Sol, GPT-6.2 Sol and GPT-6.2 Astra assisted with this work; the authors take responsibility for every claim (see "How this work was done").

Archive and programs: https://github.com/Apsiape/critical-temperatures.

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1 Introduction

A computable interaction determines every finite partition function to arbitrary accuracy. It need not determine its infinite-volume critical point with a computable error bound. We identify exactly which critical inverse temperatures occur in two elementary classes, and examine the extra information that makes them computable.

For one spin at each integer, translation-invariant ferromagnetic Ising pair interactions with an effective summable tail realize exactly the positive left-c.e. critical inverse temperatures (Theorem A). At fixed positive rational row sum T, the range is the left-c.e. reals above $1/T$. In particular a critical point can have the degree of the halting problem. This answers the question posed by Spandl [1, p. 893; see also p. 894 after Proposition 21] in precisely this class of interactions.

For the ideal-gas hopping class in Section 2, the finite excited-state saturation points in dimensions one and two are exactly the left-c.e. reals above $T^{-1}\log(1+1/\rho)$ (Theorem B). Each value is realized by a hopping whose onset of single-mode condensation, in the canonical ensemble on periodized cubic lattices, is that saturation point. Positive nearest-neighbour hopping in dimension at least three supplies effective control of the singularity and makes saturation computable. The one-dimensional examples can have completely monotone, strictly positive long-range hopping. At a computable inverse temperature, the infinite-volume limit of one canonical empty-site probability can have degree $0'$ (Theorem C).

The distinction is one of error bounds in an infinite limit. For the computable realizations, every fixed finite-volume quantity is computable. The information in the exceptional Ising realizations is placed at successively larger interaction ranges; in the Bose realizations it is placed in successively softer momentum structure. The results give no procedure for extracting a noncomputable real from finitely many observations.

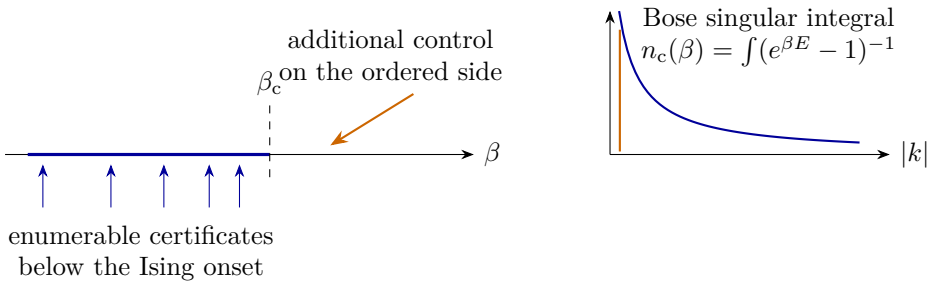

Figure 1: Two sources of one-sided approximation. Ising sharpness provides finite subcritical witnesses. The Bose saturation integral is approximated from below. Effective control of the complementary side is an additional mathematical property. The curves are schematic.

Finite range changes the Ising question. Theorem J gives a terminating algorithm for every finite periodic decoration with rational ferromagnetic pair couplings. Its lower search uses subcritical sharpness on weighted quasi-transitive graphs (Proposition 7.1). Duminil-Copin, Raoufi and Tassion state this extension of their Theorem 1.2 without proof [4, p. 80], and Appendix A writes one. The ordered-side construction of Theorem J, combined with transitive sharpness, gives the first-moment corollary for one spin per site.

Theorem J extends the known computability of critical points: Häggström proved it for Bernoulli percolation on $\mathbb{Z}^d$ [15]; the planar periodic Ising case follows from Li [16] and Cimasoni–Duminil-Copin [17]. For one-spin finite-range models in $d \geq 3$ the conclusion is folklore-level, following the slab arguments of Bodineau and Severo together with sharpness. Our ordered-side proof adapts Lebowitz–Pfister, Bodineau and Severo [18, 19, 20]; it treats inequivalent sites and nonplanar rank-two components explicitly.

Class and supplied data	Finite critical inverse temperatures
Ising chain, effective ℓ^1 tail, rational row $T > 0$	Exactly left-c.e. $a > 1/T$
Ising chain, $r^2 J_r \to B > 0$, lower tail certificates approaching B	Computable; exact jump $2aBm(a)^2 = 1$
Same asymptote, rational T, B, without effective approach	Every c.e. degree occurs
Ising chain, $J_r \asymp r^{-p}$, $1 < p < 2$, a lower power bound	Computable; $m(a) = 0$
Finite-range periodic rational Ising pair model	Computable by Theorem J; rank decides finiteness
One-spin Ising on $\mathbb{Z}^d$, $d \geq 2$, finite first moment and effective ℓ^1 tail	Computable on the finite-onset promise
Ideal gas, effective ℓ^1 hopping, row T, density ρ, positive neighbours	In $d = 1, 2$, exactly left-c.e. $a > T^{-1} \log(1 + 1/\rho)$; in $d \geq 3$, exactly computable values above that floor
Fixed nearest-neighbour gas in $d = 3$, computable slabs	Zero-mode onsets: exactly limit-computable $a' \geq a_0$

Table 1: The effective classifications. Ising entries concern pair ferromagnets. The ideal-gas entry classifies saturation; canonical condensation at saturation is proved for the constructed realizations and for every gas in $d \geq 3$. Slab onsets refer to macroscopic occupation of the zero mode.

The inverse-square boundary is more delicate. We prove the equality asked for by Aizenman–Chayes–Chayes–Newman [10, p. 7] and first claimed in Bhattacharya's preprint [11, Theorem S1.1]. The deletion response and boundary-record method follow that preprint. A correlation-weighted record bound and an exact hard-cutoff counting identity replace its interface estimates and its Imbrie–Newman input. The proof below uses only refereed external inputs. One named homogeneous-free-block lemma carries both the inverse-square results and the comparable powers below two. Knowing the numerical value of $B = \lim r^2 J_r$ is weaker than having effective lower certificates approaching it: at fixed rational row and coefficient, every c.e. degree occurs in the critical point. The same free-block analysis shows that the half-line and full-line critical points coincide at inverse-square decay (Proposition 5.5). This answers Problem 1 of Berger–Johansson–Öberg [6] and proves, for this class, the equality of uniqueness thresholds conjectured by Johansson–Öberg–Pollicott [7, Remark 1]; we found no earlier solution. For $J_r = r^{-2}$, Appendix B proves the bracket $631/1000 < \beta_c < 11544/7225$; it is computer-assisted, verified by two independently written programs.

Purcell–Li–Cubitt's quantum family with a transition at a halting probability [21] and Bausch–Cubitt–Watson's uncomputable quantum phase diagrams [22] are the nearest uncomputability precedents. Here the parameter is the inverse temperature of one fixed classical Ising interaction, or the saturation inverse temperature of noninteracting bosons. The specially arranged tails are essential to the realizations. A further choice is the volume sequence: for one fixed nearest-neighbour gas in three dimensions, computable nested slabs can have any finite limit-computable zero-mode onset at or above the cubic one. The information then lies in the aspect sequence.

Readers interested in the classifications can read Sections 3 and 4. Section 5 isolates the inverse-square argument. Sections 7–8 give the finite-range and first-moment results; their statistical-mechanical inputs are stated in Section 2. Section 9 explains which canonical limits depend on the volume sequence. Appendix A proves Proposition 7.1, and Appendix C gives exact finite checks and their reproduction commands.

2 Models and effective data

Definition 2.1. A real is computable if a program gives rational error-2^{-n} approximations on input n. It is left-c.e. if it is the limit of a computable increasing rational sequence, and limit-computable if it is the limit of a computable rational sequence with no prescribed rate. A nonnegative kernel is computable in ℓ^1 if its entries are uniformly computable and a computable function $R(n)$ bounds its omitted row sum by 2^{-n}. A statement uniform in the kernel takes both these programs as input.

On the Ising chain the energy is $-\sum_{\{x,y\}} J(|x-y|)\sigma_x\sigma_y$, each unordered pair counted once. The dimensionless field in the exponent is $h\sum_x \sigma_x$. We set

$$m(\beta) = \lim_{h\downarrow 0}\lim_{\Lambda\uparrow\mathbb{Z}} \langle\sigma_0\rangle_{\Lambda,\beta,h} = \mu^+_\beta(\sigma_0), \qquad \beta_{\mathrm{c}} = \inf\{\beta > 0 : m(\beta) > 0\}.$$

Here $\langle\cdot\rangle_{\Lambda,\beta,h}$ has free boundary conditions, μ^+_β is the zero-field plus-boundary state, and the second equality is Lemma 2.3 below. The row sum is $T = 2\sum_{r\ge 1} J(r)$. On $\mathbb{Z}^d$ the corresponding row is $\sum_z J(z)$, with $J(z) = J(-z)$ and $J(0) = 0$.

Proposition 2.2 (Refereed sharpness import)**.** *For a nonnegative summable translation-invariant pair kernel on $\mathbb{Z}^d$,*

$$\varphi^J_\beta(S) = \sum_{x\in S} \langle\sigma_0\sigma_x\rangle^{\mathrm{f}}_{S,\beta,0} \sum_{y\notin S} \tanh(\beta J(y-x)), \tag{2.1}$$

$$\beta_{\mathrm{c}} = \sup\{\beta : \exists\, S \ni 0 \text{ finite},\ \varphi^J_\beta(S) < 1\}. \tag{2.2}$$

The plus susceptibility is finite for $\beta < \beta_{\mathrm{c}}$. For $\beta > \beta_{\mathrm{c}}$, $m(\beta) \ge \sqrt{1-(\beta_{\mathrm{c}}/\beta)^2}$.

This is Duminil-Copin–Tassion [2, Theorem 2.1(1),(2), and the identity following (2.1) in Section 2.1], with its 2018 correction [3]. The underlying graph is the locally finite transitive lattice; couplings may join arbitrary pairs. Summability, implicit in their estimates, is supplied here. Their current weights $(\beta J_{xy})^n/n!$ on unordered pairs and the factor $\tanh(\beta J)$ fix our normalization despite the ambiguous ordered-pair notation in their displayed Hamiltonian. The finite-range restriction in their item (3) is unnecessary for the statements imported above.

We use GKS, FKG and GHS for pair ferromagnets with nonnegative fields [8, (1.5)–(1.8)]. In particular spin-product covariances are nonnegative, and third field derivatives of the pressure are nonpositive. We also use the duplicate-spin inequality [9, Section 2, (5)–(7c)]. If two finite systems have coefficients $K_A \ge |K'_A|$ and moments u_A, v_A, then

$$u_{A\triangle B} - v_{A\triangle B} \ge |u_A v_B - v_A u_B|. \tag{2.3}$$

It includes arbitrary signed smaller fields and hard pins by limits. One verification conditions two independent copies on $z_i = \sigma_i\sigma'_i$: the remaining coefficients are $K_A + K'_A z_A \ge 0$, and one averages $1 - z_A z_B \ge \pm(z_B - z_A)$ times a nonnegative spin moment.

Lemma 2.3 (Field limit and plus state)**.** *Let $J \ge 0$ be a summable translation-invariant pair kernel on $\mathbb{Z}^d$ and $\beta \ge 0$. For $h \ge 0$ let $\mu^{\mathrm{f},h}$ and $\mu^{+,h}$ be the limits of the free and plus-boundary finite-volume states with dimensionless field h. Then $\mu^{\mathrm{f},h}(\sigma_0) = \mu^{+,h}(\sigma_0) = p'(h)$ at every $h > 0$ where the pressure p is differentiable, which is all but countably many h, and*

$$\lim_{h\downarrow 0} \mu^{\mathrm{f},h}(\sigma_0) = \lim_{h\downarrow 0} \mu^{+,h}(\sigma_0) = \mu^+_\beta(\sigma_0).$$

Proof. A plus box keeps every exterior spin at $+1$; summability makes the induced exterior fields finite. By GKS and FKG with nonnegative fields, free-box means of σ_x increase with the

box and plus-box means decrease. Hence both limits exist, do not depend on the exhausting sequence, are translation invariant, and satisfy $\mu^{\mathrm{f},h}(\sigma_0) \le \mu^{+,h}(\sigma_0)$.

Let $\Lambda_n = \{0,\dots,n-1\}^d$ and $p_n^{\mathrm{f}}(h) = |\Lambda_n|^{-1} \log Z_{\Lambda_n}^{\mathrm{f}}(h)$, and define p_n^+ in the same way. Removing the couplings between the 2^d translates of Λ_n in Λ_{2n} lowers $\log Z^{\mathrm{f}}$, since its coupling derivatives are nonnegative pair means. Thus $p_{2^k}^{\mathrm{f}}(h)$ is nondecreasing in k; it is at most $\log 2 + h + \beta \sum_z J(z)$, so it has a limit $p(h)$. Moreover

$$|\log Z_{\Lambda_n}^{+}(h) - \log Z_{\Lambda_n}^{\mathrm{f}}(h)| \le \beta \sum_{x\in\Lambda_n,\, y\notin\Lambda_n} J(y-x) \le \beta|\Lambda_n| \sum_z J(z) \min(1, d|z|_\infty/n),$$

because at most $d|z|_\infty n^{d-1}$ sites $x \in \Lambda_n$ have $x + z \notin \Lambda_n$. Dominated convergence makes the right side $o(|\Lambda_n|)$, so $p_{2^k}^+ \to p$ as well.

All these functions are convex in h, and their derivatives are the spatial averages $M_n^{\mathrm{f}}(h)$ and $M_n^+(h)$ of the box means. At every h where the convex limit p is differentiable, both derivative sequences converge to $p'(h)$ along $n = 2^k$. Each free-box mean is at most $\mu^{\mathrm{f},h}(\sigma_0)$ and each plus-box mean is at least $\mu^{+,h}(\sigma_0)$, so

$$\mu^{+,h}(\sigma_0) \le p'(h) \le \mu^{\mathrm{f},h}(\sigma_0) \le \mu^{+,h}(\sigma_0).$$

This proves equality at every differentiability point; a convex function has at most countably many others. Both sides are nondecreasing in h, so their limits as $h \downarrow 0$ coincide. Finally $\mu^{+,h}(\sigma_0) \ge \mu^{+,0}(\sigma_0)$ for $h \ge 0$, while $\mu^{+,h}(\sigma_0)$ is at most the mean of any fixed plus box at field h, which is continuous in h. Letting $h \downarrow 0$ and then taking the infimum over boxes gives $\lim_{h\downarrow 0} \mu^{+,h}(\sigma_0) = \mu^{+,0}(\sigma_0) = \mu_\beta^+(\sigma_0)$. □

The same proof applies to periodic graphs with finitely many vertex types, using cell cubes: the averaged identity and the typewise inequalities $\mu^{\mathrm{f},h} \le \mu^{+,h}$ give equality for each type.

The FK representation used below has cluster weight 2 and edge parameter $p_e = 1 - e^{-2\beta J_e}$. Expanding the pair Boltzmann factors and assigning independent fair signs to free clusters gives the free two-point function as a connection probability; wiring to a plus ghost gives the magnetization. The conditional opening probability is p_e or $p_e/(2 - p_e)$ according as the endpoints are already joined or separate. Coupled edge updates prove comparison in parameters and boundary partitions. Conditioning off a subgraph induces a partition of its boundary; free interiors are dominated by every such partition. These finite facts also give the monotone infinite-volume limits.

For bosons there is one orbital at each lattice site. Fix rational $T, \rho > 0$ and an inversion-symmetric, nonnegative, ℓ^1-computable hopping J on $\mathbb{Z}^d \setminus \{0\}$ with row T and $J(\pm e_i) > 0$. The one-particle Laplacian has band

$$E(k) = \sum_x J(x)(1 - \cos(k \cdot x)), \qquad n_{\mathrm{c}}(\beta) = \int_{[-\pi,\pi]^d} \frac{1}{e^{\beta E(k)} - 1} \frac{\mathrm{d}k}{(2\pi)^d}. \tag{2.4}$$

Its sole zero on the torus is 0. A positive rational lower bound on the finitely many neighbour hoppings is found by search. It gives an effective quadratic lower bound for E. Saturation is the unique root a of $n_{\mathrm{c}}(a) = \rho$, or $a = \infty$ if this integral is always infinite. Finite rings and boxes use periodized hopping, hence sample this same band at their momenta. Whenever a canonical observable is stated, the particle number is $N = \lfloor \rho V \rfloor$, with V the number of sites. A mode occupation is denoted N_j and a site occupation n_x.

3 The exact spectrum on the Ising line

The finite-set criterion provides the upper restriction, and widely spread weak bonds give all its possible values.

Theorem A (Ising critical inverse temperatures)**.** *For translation-invariant Ising pair ferromagnets with one spin per integer and $J \geq 0$ computable in ℓ^1, the finite critical inverse temperatures are exactly the positive left-c.e. reals. At a prescribed positive rational row sum $T = 2\sum_r J(r)$, they are exactly the left-c.e. reals $a > 1/T$. A realizing interaction and its tail modulus can be constructed from a lower name of a; every coupling can be a positive rational, with J strictly decreasing and discretely convex.*

3.1 Necessity and finite-range computations

Each test (2.1) is computable. Its internal moments are finite sums and its omitted exterior weight is bounded by β times an effective row tail. Strict inequalities can therefore be enumerated over rational β and finite S. By (2.2) their running maxima give a lower name of every finite β_c. For $T > 0$,

$$\varphi^J_{1/T}(\{0\}) = 2\sum_r \tanh(J(r)/T) < 1.$$

The strictness follows from any nonzero summand and $\tanh x < x$. Effective summability gives continuity in β, so $\beta_c > 1/T$. The zero kernel has infinite onset.

For a finite-range kernel K let $\chi^K_F(b) = \sum_{x\in F} \langle \sigma_0\sigma_x\rangle^{\mathrm{f}}_{F,b}$ on centered intervals, and let $\chi^K(b)$ be its increasing limit. This is finite and computable, uniformly in the finite kernel and $b > 0$. Indeed, let d bound the range and $p_* < 1$ bound the edge parameters. The FK graph is dominated by independent edges of probability p_*. A cut has at most $d(d+1)/2$ crossing edges. Cuts spaced $2d+1$ apart involve disjoint edges, giving an effective geometric arm bound with empty-cut probability at least $(1-p_*)^{d(d+1)/2}$. The same argument bounds boundary influence: reveal the upper configuration from the boundary strips, couple ordered free and wired laws, and draw identically on any region isolated by closed upper cuts. Thus finite inner correlations converge effectively and their two-point tails are effectively summable. This computes χ^K and also proves $\varphi^K_b([-n,n]) \to 0$ effectively. A finite transfer matrix gives the same algorithm.

Lemma 3.1 (Amplification by distant weak bonds)**.** *Let K have range d, let $b, \lambda > 0$, and suppose $b\lambda\chi^K(b) > 1$. On $d < r \leq M$ put*

$$L_M(r) = \frac{\lambda(M+1-r)}{(M-d)(M-d+1)}, \qquad 2\sum_r L_M(r) = \lambda. \tag{3.1}$$

For every $\chi_0 < \infty$ and all sufficiently large M, some finite free interval has susceptibility greater than χ_0 for $K + L_M$ at b.

Proof. Choose a fixed radius r_0 for which the mean size $\bar{s}$ of the seed's free K-cluster in its radius-r_0 window satisfies $b\lambda\bar{s} > 1$. The $K + L_M$ FK law dominates independent free K interiors together with independent added edges of probabilities $p_\ell = \tanh(bL_M(|\ell|))$. The bound is the minimum FK insertion probability, applied to a second parallel edge and then combining the two couplings. Write $u_M = \max p_\ell \to 0$ and $t_M = \sum_\ell p_\ell \to b\lambda$, the sum taking both signs.

Explore an ideal branching process: each particle samples a fresh radius-r_0 free cluster, and each of its sites proposes children at all allowed displacements with probabilities p_ℓ. Its offspring mean is $\bar{s}t_M > 1$ eventually. For fixed depth, the total number T_* of particles has a second moment bounded uniformly in M, since the cluster size is bounded and a sum of independent Bernoulli trials has bounded first and second moments here. Choose a depth whose expected generation size exceeds $\chi_0 + 3$, then a fixed cap U with $\mathbb{E}[T_*\mathbf{1}_{T_*>U}] < 1$.

Call a proposal a collision if its center is within $4r_0+2$ of an old or simultaneously proposed center. Before the cap, its probability is $O_{U,r_0,b\lambda}(u_M)$. For old centers, the union of forbidden

neighborhoods has bounded size and each trial has probability at most u_M. For simultaneous proposals, sum one trial's probability and bound the sum of trials landing within a bounded neighborhood of its target by a constant times u_M; the first sum is bounded by $(2r_0+1)t_M$. Repeat this over at most U windows.

On collision-free histories the windows are disjoint. Reveal the actual internal FK edges in the next window; conditional on everything already revealed, its law is a mixture of partitions and dominates a fresh free sample. Use that lower sample for the next proposals. The added Bernoulli variables are on fresh unordered pairs: any repeated pair would target an earlier window and would have been a collision. Dummy independent trials may be used for these discarded proposals. Hence the ideal marginal is preserved and its collision-free children are distinct vertices connected in the actual graph. The whole capped exploration fits in a deterministic interval of radius $(k+1)(M+2r_0+2)$, where k is its depth. Removing capped histories costs less than one in expected size, and collisions cost at most $U\,O_U(u_M) < 1$. The actual mean cluster size exceeds χ_0. All comparisons concern finite graphs. □

3.2 An induction with permanent disorder witnesses

First allow the row sum to vary. From the lower name choose rationals $0 < l < z_1 < z_2 < \cdots < a$ tending to a. Such a strictly increasing sequence is obtained by damping running lower approximations toward a fixed smaller bound. Start with $K_0(1) = 1$ and a rational budget $B_1 > 2/l$. At stage s maintain

$$z_{s+1} B_s \chi^{K_{s-1}}(z_{s+1}) > 1, \tag{3.2}$$

where B_s bounds all future row mass. With $K = K_{s-1}$ choose rational

$$\frac{1}{z_{s+1}\chi^K(z_{s+1})} < \lambda < \min\left(B_s, \frac{1}{z_s \chi^K(z_s)}\right).$$

This open interval is nonempty by (3.2) and monotonicity. Choose $0 < B_{s+1} \le 2^{-s}$ with $\lambda + B_{s+1} < B_s$ and $z_s(\lambda + B_{s+1})\chi^K(z_s) < 1$. Then choose a centered S_s so that

$$\varphi^K_{z_s}(S_s) + z_s(\lambda + B_{s+1})\chi^K(z_s) < 1.$$

These are effective strict searches by the finite-range computation.

Append (3.1) beyond the current range, choosing M and a centered free interval F_s with

$$\varphi^{K+L_M}_{z_s}(S_s) + z_s B_{s+1} \chi^{K+L_M}_{S_s}(z_s) < 1, \tag{3.3}$$

$$\chi^{K+L_M}_{F_s}(z_{s+1}) > \max\{s, 2/(lB_{s+1})\}. \tag{3.4}$$

For fixed S_s, internal new coefficients tend to zero, and the added outgoing tanh weight per site is at most $z_s\lambda$. Thus the first inequality holds for all sufficiently large M. Lemma 3.1 gives the second. Dovetail finite tests to find both. Require also $M > \operatorname{diam} S_s + d + 1$, the first new value at most half the old last value, and the new decrement at most half that old value. Set $K_s = K + L_M$ and $d_s = M$. Equation (3.4) establishes (3.2) at the next stage.

Every distance is assigned a rational value once and for all. The row remaining beyond d_s is at most B_{s+1}, so the limit J is computable in ℓ^1. Each triangular band decreases strictly and ends at a value equal to its decrement. At a join the new first value is at most half the old last value c; the join decrement lies between $c/2$ and c, while the preceding decrement is c and the new decrement is at most $c/2$. Thus the whole sequence is positive, strictly decreasing and convex.

All later bands are outside S_s. Its internal correlations never change, and their additional outgoing weight is at most $z_s B_{s+1}$. Hence (3.3) is a permanent witness for $\beta_c \ge z_s$, giving $\beta_c \ge a$. At a, (3.4) and monotonicity make finite free susceptibilities unbounded. They are bounded by the infinite plus susceptibility. Proposition 2.2 therefore rules out $a < \beta_c$. This proves $\beta_c = a$.

3.3 Spending exactly a prescribed row sum

For rational $T > 0$ and $a > 1/T$ choose $z_1 > 1/T$, a small rational $c > 0$ with $z_1(T - 2c) > 1$, and $K_0(1) = c$, $B_1 = T - 2c$. Now B_s is the exact unspent row. A preliminary band makes the two budgets compatible with exact spending.

At entry to stage s, append a triangular band of row mass $\gamma \in [0, B_s]$ on a fixed interval $(d, d']$, calling the result $K(\gamma)$. Choose d' large enough that even mass B_s satisfies the convex-join bounds. For fixed finite range $\chi^{K(\gamma)}(t)$ is continuous and computable in γ, uniformly on this compact interval, by the uniform empty-cut estimate. Define

$$g_t(u) = tu\chi^{K(B_s-u)}(t), \qquad 0 \le u \le B_s.$$

The functions are continuous, vanish at zero, and $g_{z_s}(u) < g_{z_{s+1}}(u)$ for $u > 0$. By the invariant $g_{z_{s+1}}(B_s) > 1$. At the infimum of the open set where the upper function exceeds one, its value is one and the lower function is strictly smaller. Nearby points therefore give a rational $\lambda \in (0, B_s)$ with $g_{z_s}(\lambda) < 1 < g_{z_{s+1}}(\lambda)$. Choose rational $0 < B_{s+1} \le 2^{-s}$ so small that $\gamma = B_s - \lambda - B_{s+1} > 0$ and

$$z_s(\lambda + B_{s+1})\chi^{K(\gamma)}(z_s) < 1 < z_{s+1}\lambda\chi^{K(\gamma)}(z_{s+1}).$$

Use $K(\gamma)$ as the base of the preceding construction and append the diffuse band of mass λ. Together the two bands spend exactly $B_s - B_{s+1}$. Previous witnesses retain their reserved future mass and have no new internal bonds. The limit has row exactly T, with the same effective tail and shape conclusions. This proves the fixed-row assertion and completes Theorem A. □

For example a rational shift and scaling of $\sum_{e\in H} 4^{-e-1}$ gives a positive left-c.e. target of the degree of any c.e. set H; separated digits allow membership to be recovered successively. The construction uses an enumeration of H, never its membership oracle. Its distant bands are a special feature of the interaction, and the conclusion concerns its infinite-volume onset.

4 Saturation of the ideal Bose gas

Here the same one-sided spectrum comes from an improper integral.

Theorem B (Bose saturation points)**.** *In the model (2.4), with fixed rational $\rho, T > 0$ and positive coordinate-neighbour hopping, the finite saturation inverse temperatures in $d = 1, 2$ are exactly the left-c.e. reals*

$$a > T^{-1}\log(1 + 1/\rho).$$

In $d = 1$ every value has a strictly positive completely monotone hopping and a band increasing on $[0, \pi]$. The realizing hoppings constructed below for $d = 1, 2$ have canonical zero-mode density $(\rho - n_c(\beta))_+$ on periodized cubic lattices of volume L^d at $N = \lfloor \rho L^d \rfloor$, in probability and mean.

For $d \ge 3$, saturation is finite and computable uniformly from the ℓ^1 hopping data. Its possible values are exactly the computable reals in the same open interval, already with finite-range hopping. On cubic boxes these gases have the same canonical zero-mode conclusion.

4.1 Separating the singular and regular parts

Write brackets for normalized torus integration and put

$$I = \langle 1/E \rangle, \quad f(x) = \frac{1}{e^x - 1}, \quad H(x) = f(x) - \frac{1}{x}, \quad H(0) = -\frac{1}{2}. \tag{4.1}$$

The Taylor series at zero makes H computable and continuous on every bounded nonnegative interval, with $-1/2 \le H \le 0$. Consequently

$$n_{\mathrm{c}}(\beta) = I/\beta + \langle H(\beta E)\rangle\,. \tag{4.2}$$

The second term is computable from the band. The nonnegative integral I is left-c.e., using the continuous truncations $\min(M, 1/E)$, with value M at zero. If $I < \infty$, n_{c} is continuous and strictly decreasing from infinity to zero by domination on compact positive β intervals and monotone convergence at the endpoints. The test $n_{\mathrm{c}}(q) > \rho$ enumerates its strict lower cut, so a is left-c.e. An oracle for I computes a by strict root brackets; conversely

$$I = a\,[\rho - \langle H(aE)\rangle]. \tag{4.3}$$

Thus $a \equiv_T I$. Strict Jensen applied to $f(\beta E)$, since $\langle E\rangle = T$ and the band is nonconstant, gives $n_{\mathrm{c}}(\beta) > f(\beta T)$ and the stated strict floor.

4.2 One-dimensional realization

For $0 < v < 1$ let

$$P_v(k) = \frac{1 - v^2}{(1-v)^2 + 2v(1-\cos k)}, \quad v_i = 1 - 2^{-i}, \quad G_0(k) = \sum_{i\ge 1} \frac{P_{v_i}(k)}{i(i+1)}.$$

The Poisson series gives $\langle P_v\rangle = 1$. Each atom is positive and decreases on $(0, \pi]$. The series converges effectively off zero. If $2^{-m-1} < |k| \le 2^{-m}$, its mth term gives

$$G_0(k) \ge \frac{c}{|k|(1 + \log(1/|k|))^2}.$$

Also $P_v(k) \le C/|k|$ uniformly for $v \ge 1/2$, by the quadratic lower bound on $1 - \cos k$ and $2xy \le x^2 + y^2$.

At a finite stage use rational positive data in

$$G = 1 + \epsilon G_0 + \sum_{j=1}^{s} d_j P_{v_{i_j}}, \quad A = \langle G\rangle = 1 + \epsilon + \sum_j d_j, \quad W = G^{-1}, \quad M = \langle W\rangle\,, \quad E = TW/M, \tag{4.4}$$

with $W(0) = 0$. The bound on G_0 computes W uniformly near zero; off zero the series is effective. Thus $M > 0$ and E are computable, $\langle E\rangle = T$, and $I = MA/T$ is computable at each stage.

The Fourier signs have a useful elementary proof. Truncate the mixtures and set $x = 2(1 - \cos k)$. Their denominator has the form $F(x) = 1 + \sum c_j/(x + t_j)$ with $c_j, t_j > 0$. Its simple zeros are negative and interlace its negative poles. Since $F' < 0$ between poles,

$$F(x)^{-1} = 1 - \sum_{\ell} \frac{a_\ell}{x + s_\ell}, \qquad a_\ell, s_\ell > 0.$$

The Fourier coefficients of $(s + 2 - 2\cos k)^{-1}$ are positive multiples of $v^{|r|}$, where $v + v^{-1} = s + 2$. Hence $J(r) = -\widehat{E}(r) \ge 0$ and every alternating finite difference is nonnegative. Bounded convergence passes these conclusions to the mixtures. Poisson integration at zero and positivity give $2\sum_{r\ge 1} J(r) = \widehat{E}(0) = T$. Their values are computed by compact integration, and searching partial sums against the known total supplies an ℓ^1 tail modulus.

The final finite-mass mixture obeys

$$c'|k| \le E(k) \le C'|k|(1 + \log(1/|k|))^2. \tag{4.5}$$

The lower bound uses $G \le C''/|k|$ and the upper bound uses G_0. In particular the hopping has infinite range. Complete monotonicity then makes every $J(r)$ strictly positive: a zero coefficient would make all later coefficients zero; a vanishing first decrease would make all later decreases zero. Either possibility would give finite range, whose band is $O(k^2)$, contradicting (4.5).

Let $z_s \uparrow a$ be strictly increasing rationals above the floor. Initially choose $\epsilon > 0$ so the crossing in (4.4) lies below z_1. As $\epsilon \downarrow 0$, $M \to 1$, $A \to 1$, and the regular-part identity gives $n_c(z_1) \to f(z_1 T) < \rho$, so this strict search terminates. Fix rational $m_0 > 0$ with $2m_0 < M$ initially.

At stage s add mass $d \in [0, D]$ on one sufficiently narrow new atom. Choose rational D with $m_0(A + D)/(Tz_{s+1}) - 1/2 > \rho$. Its index can be made so large that the uniform change of W is less than $m_0 2^{-s-1}$ for every $d \le D$: near zero use the old upper bound on W, and off zero use the effective vanishing of DP_{v_i}. Summing tolerances keeps $M \ge m_0$. The crossing at $d = 0$ is below z_s and at $d = D$ above z_{s+1}. It varies continuously, by (4.2), so rational d with crossing $a_s \in (z_s, z_{s+1})$ is found by strict search.

The resulting W_s, M_s, E_s converge effectively and uniformly. Moreover

$$M_s A_s/T = a_s[\rho - \langle H(a_s E_s)\rangle] \le a(\rho + 1/2),$$

so $A_s \uparrow A < \infty$. Monotone convergence identifies A with the final mixture's integral. Passing to the displayed identity by uniform continuity of H gives (4.3) and hence exactly the target crossing a. The construction uses only its lower name. This proves the one-dimensional sufficiency.

4.3 Two and at least three dimensions

For $d = 2$ put $x(k) = 2\sum_{i=1}^{2}(1 - \cos k_i)$ and replace the Poisson atoms by

$$Q_t(k) = \frac{1}{Z(t)(t + x(k))}, \qquad Z(t) = \left\langle (t + x)^{-1} \right\rangle .$$

They have unit integral and $Z(t) \asymp 1 + \log(1/t)$ with effective constants, by comparison with disk integrals of $(t + c|k|^2)^{-1}$. Thus Q_t tends effectively to zero away from zero. With $t_i = 2^{-2^i}$ and weights $1/[i(i+1)]$, the least i with $t_i \le x$ gives

$$G_0(k) \ge \frac{c}{x \log(1/x)[1 + \log\log(1/x)]^2}.$$

This is an effective bound tending to infinity at zero. The construction (4.4) and its uniform estimates therefore apply unchanged, and give the exact target.

For the Fourier signs, partial fractions again reduce to $(s + x)^{-1}$. Its convergent resolvent expansion is a positive sum of characteristic functions of powers of the simple nearest-neighbour walk, so every coefficient is nonnegative. Positivity and Poisson approximation give the known total row and effective tail as before. The coordinate-neighbour hoppings are strictly positive: at fixed k_2 the band strictly increases in $|k_1|$ whereas $\cos k_1$ decreases and has zero mean. Integrating the product of their differences on two independent k_1 samples proves strictly negative covariance, hence a strictly negative band coefficient at e_1. Exchange coordinates for e_2.

For $d \ge 3$, the neighbour lower bound gives $E(k) \ge c|k|^2$. Dyadic shell integration bounds $\int_{|k|<\delta} E^{-1}$ by an effective constant times δ^{d-2}. Ordinary quadrature off zero computes I, and hence a, uniformly. Strict Jensen still excludes the floor.

All computable targets above the floor occur at finite range. To obtain $n_c(b) < \rho$ for a prescribed computable b above it, mix a fixed small neighbour mass ξ with uniform steps

on larger and larger finite cubes. The latter characteristic functions tend to zero almost everywhere. The integrable neighbour majorant first permits that limit, and then $\xi \downarrow 0$ gives $n_c(b) \to f(bT) < \rho$. For the opposite inequality, put almost all mass on $\pm e_1$ and a small positive fraction on every neighbour. As that fraction tends to zero, Fatou's lemma and the nonintegrable one-coordinate quadratic singularity give $n_c(b) \to \infty$. Strict searches find two finite-range examples. Their affine segment has a common positive quadratic lower bound, so $n_c(b)$ is computable and continuous along it. It is convex in the parameter; starting below and ending above ρ gives one upward crossing, since the secant slope from the starting point is positive at every crossing. Rational strict brackets compute its parameter and retain finite range and row T.

4.4 Canonical condensation on the specified grids

For every finite periodic lattice the excited occupations at zero chemical potential are independent geometrics X_j of means $g_j = f(\beta E_j)$, conditioned on $X = \sum_{j\neq 0} X_j \le N$. The remaining $N - X$ particles occupy the unique zero mode. This follows directly from the Fock-basis weights.

In the monotone one-dimensional realizations, right rectangles for the decreasing integrable $g(k)$ control the improper Riemann sums. Also $kg(k) \le \int_0^k g(u)\,\mathrm{d}u \to 0$. Hence

$$\frac{1}{V}\sum_{j\neq 0} g_j \to n_c(\beta), \qquad \max_{j\neq 0} g_j/V \to 0, \qquad \mathrm{Var}(X)/V^2 \to 0. \tag{4.6}$$

In the two-dimensional realizations $g(k) = F(x(k))$ with F decreasing. A grid shell of maximum index m has $O(m)$ points, $x \ge c(m/L)^2$, and its contribution is bounded by a constant times the integral of $rF(cr^2)$ on $((m-1)/L, m/L)$. Integrability gives (4.6). For $d \ge 3$ there are $O(m^{d-1})$ points per shell, each bounded by $C(L/m)^2$; the near-zero sum is $O(\delta^{d-2})$ and the largest mean divided by $V = L^d$ is $O(L^{2-d})$. This proves (4.6) there too.

When $n_c(\beta) < \rho$, conditioning has probability tending to one, so $N_0/V \to \rho - n_c(\beta)$ in probability and mean. When $\rho \le n_c(\beta)$, tilt the geometric law by e^{hX}, $h < 0$, to density $\rho - \delta/2$. Its total divided by V concentrates. Undoing the tilt, configurations with $X \le N - \delta V$ are exponentially smaller than those with $|X/V - (\rho - \delta/2)| < \delta/8$, because e^{-hX} increases with X. Thus $N_0/V \to 0$, including equality. This completes Theorem B. □

Theorem C (A canonical one-site probability)**.** *For the monotone one-dimensional realizing hoppings of Theorem B, at any $\beta > a$ the canonical one-site law on $\mathbb{Z}/L\mathbb{Z}$ with periodized hopping and $N = \lfloor \rho L \rfloor$ converges as $L \to \infty$, along all ring sizes, and its limiting vacancy is*

$$v = \frac{1}{1+n}\exp\!\left(-\frac{\rho - n}{1+n}\right), \qquad n = n_c(\beta). \tag{4.7}$$

At computable $\beta > a$ this real has the Turing degree of a. In particular for $T = \rho = 1$ there is such long-range hopping with $a = 2 + \Omega$, where $\Omega \in (0,1)$ is a universal prefix-free halting probability; at $\beta = 4$ its limiting vacancy has degree $0'$. The grand-canonical ideal vacancy at the same mean density is $1/(1+\rho)$, in particular $1/2$ at density one.

Proof. Write $C = \rho - n$. Under the canonical conditioned law, (4.6) also gives $\max_{j\neq 0} N_j/V \to 0$ in probability, since $\sum \mathbb{E}X_j^2 = o(V^2)$. In a mode number vector the mth site factorial moment is exactly

$$\frac{(m!)^2}{V^m} \sum_{\sum k_j = m} \prod_j \frac{(N_j)_{k_j}}{(k_j!)^2}.$$

It is at most $m!(N/V)^m$. For $k_0 = l$, all terms with a repeated excited index have total bounded by a constant times $\max_{j\neq 0} N_j/V$; choose a repeated pair and sum it using $\sum (N_j/V)^2 \leq \rho \max N_j/V$. The other terms give the limit

$$(m!)^2 \sum_{l=0}^{m} \frac{C^l n^{m-l}}{(l!)^2 (m-l)!}.$$

The uniform factorial bound permits summation of the generating function near $z = 1$, yielding

$$[1 - n(z-1)]^{-1} \exp\!\left(\frac{C(z-1)}{1 - n(z-1)}\right). \tag{4.8}$$

Tightness follows from the fixed site mean N/V. Every limiting probability law has this generating function near one; analytic uniqueness on $|z| < 1$ identifies it everywhere and gives (4.7) at zero. Equivalently the law is Poisson with random intensity $|Y + \sqrt{C}|^2$, for circular Gaussian Y of variance n.

At fixed ρ the logarithmic derivative of the right side of (4.7) with respect to n is $(\rho - n)/(1 + n)^2$, strictly positive for $n < \rho$. Its inverse on the compact range is computable by strict bracketing and continuity. From n recover $I = \beta[n - \langle H(\beta E)\rangle]$ and then a; (4.3) proves the reverse reduction. The halting probability is left-c.e. and has degree $0'$: a precision oracle bounds the remaining Kraft mass and decides halting for programs of any fixed length. Thus Theorem B supplies the stated example. Finally a grand-canonical ideal state is Gaussian; its one-site number law is geometric with its specified mean ρ, giving the stated computable vacancy. □

5 The inverse-square boundary

We first isolate the block statement used throughout this section and the next. Its scale is explicit, so that the argument for powers below two uses the same lemma.

Lemma 5.1 (Homogeneous free blocks)**.** *Let $1 < p \leq 2$ and let J be a summable ferromagnetic chain. If $p = 2$, assume $r^2 J_r \to B \in (0, \infty)$; if $p < 2$, assume $cr^{-p} \leq J_r \leq Cr^{-p}$ for all large r, with $0 < c \leq C$. Fix any ordered β, with $m = m(\beta) > 0$. There are dyadic lengths $N_s \to \infty$ with the following property. Let $\mathbb{E}$ be the expectation under the free-boundary Ising measure on $\{1, \ldots, N_s\}$ at β, and for a positive-length interval $I \subset [0, 1]$ let X_I be the average of σ_x over the sites $x \in N_s I$. Then, for any two intervals I, H of a fixed finite collection,*

$$\mathbb{E}(X_I^2 - m^2)^2 \to 0, \qquad \mathbb{E}(X_I - X_H)^2 \to 0. \tag{5.1}$$

The weighted error in the proof has scale N_s^{2-p}, interpreted as 1 at $p = 2$. The lemma uses only the stated tail hypothesis and applies at an ordered critical point.

Proof. Use normalized spin counting measure, and write $p_J(h)$ for pressure per site. The plus state clusters: it is maximal among Gibbs states, and conditioning a tail event would decompose it into Gibbs states below itself; equality of their increasing moments forces tail triviality. More concretely the conditional mean at the origin given the exterior of a large interval is bounded above by its finite plus mean and has expectation m. Its L^1 distance from m tends to zero, since that upper bound decreases to m. Multiplying by an exterior spin gives $\tau^+(r) = \mu^+(\sigma_0 \sigma_r) \to m^2$.

Set $K_R(r) = J_r \mathbf{1}_{r<R}$, $\lambda_R = \beta \sum_{r \geq R} J_r$, and $\Delta_R = p_J(0) - p_{K_R}(0)$. We claim

$$\Delta_R / \lambda_R \to m^2. \tag{5.2}$$

Deleting the tail changes any conditional field by at most $d_R = 2\lambda_R$. Thus, for real $t \geq 1$, $m \leq M_{K_R}(d_R t) \leq M_J(d_R t) \to m$, where M is the positive-field magnetization. We use the finite Lee–Yang circle property [13], with the following verification for our pair convention. The edge polynomial $1 + t(u + v) + uv$, $t = e^{-2\beta J} \in [0, 1]$, is zero-free for $|u|, |v| < 1$, because $|1 + tu|^2 - |t + u|^2 = (1 - t^2)(1 - |u|^2)$. To contract two half-edge variables replace $A + Bu + Cv + Duv$ by $A + Dw$. For fixed remaining variables in the disk, $A \neq 0$ and the diagonal quadratic has no disk roots, so $|A/D| \geq 1$ when $D \neq 0$. The contracted polynomial is again zero-free in the disk. Repeated contractions at each vertex impose that all its half-edges are chosen or all omitted. Isolated vertices contribute $1 + z$. Spin-flip reciprocal symmetry then puts every root of the uniform field polynomial on the unit circle.

The magnetizations have positive real part in $\Re h > 0$, since each zero contributes $(1 + z/\zeta)/(1 - z/\zeta)$ for $z = e^{-2h}$. Their transforms $(1 - M)/(1 + M)$ form a bounded normal family. Finite-volume limits agree on the dense positive real set of pressure differentiability and hence throughout the half-plane. Apply the same argument to $M_{K_R}(d_R t)$: its only analytic subsequential limit is the constant m, by the identity theorem from $t > 1$. Therefore it converges to m at every real $t > 0$.

It follows by dominated convergence after rescaling the field that $\int_0^{d_R} (M_J - M_{K_R})(h)\, \mathrm{d}h = o(\lambda_R)$. Interpolate the deleted couplings back at field d_R. Association and coupling monotonicity bound every pair mean below by $M_{K_R}(d_R)^2$, giving $p_J(d_R) - p_{K_R}(d_R) \geq \lambda_R M_{K_R}(d_R)^2$. Subtract the field integral to obtain the lower bound in (5.2). Interpolation at zero field gives the upper bound $\beta \sum_{r \geq R} J_r \tau^+(r)$, which has ratio $m^2 + o(1)$ by clustering. This proves the claim.

Let

$$b_K(N) = N p_K(0) - \log Z_N(K, 0), \quad b_K(0) = 0, \qquad S(N) = \sum_r \min(N, r) J_r.$$

The partition increments increase with N: add an independent fair endpoint spin and interpolate its incident couplings, using GKS to compare the old pair means when the opposite end is enlarged. Their Cesàro limit is $p_K(0)$; hence b_K is nonnegative and nondecreasing. Tiling a long interval by length-N blocks counts at most $M \min(N, r)$ crossing pairs of length r for M blocks. Their means are at most $\tau^+(r)$. Interpolating and taking $M \to \infty$ gives

$$0 \leq b_J(N) \leq \beta \sum_r \min(N, r) J_r \tau^+(r) \leq (\beta m^2 + o(1)) S(N). \tag{5.3}$$

Here $S(N) \to \infty$ under either tail assumption.

For $\varepsilon_s \downarrow 0$, take increasingly late dyadic running minima of $b_J(N) - (\beta m^2 + \varepsilon_s) S(N)$. They exist since this expression tends to minus infinity along dyadic lengths. At the selected $N = N_s$, for every earlier dyadic R,

$$b_J(N) - b_J(R) \leq (\beta m^2 + \varepsilon_s)[S(N) - S(R)]. \tag{5.4}$$

Fix dyadic $\delta \in (0, 1)$ and put $R = \delta N$. The hard cutoff makes $F_R = 0$, where $F_n = \log Z_n(J) - \log Z_n(K_R)$. Jensen's inequality on the ratio of partition functions and monotonicity of b_{K_R} give

$$\beta \sum_{r=R}^{N-1} J_r \sum_{i=1}^{N-r} \tau_N(i, i + r) \geq F_N \geq (N - R)\Delta_R - [b_J(N) - b_J(R)]. \tag{5.5}$$

The counting identity

$$(N - R) \sum_{r \geq R} J_r - [S(N) - S(R)] = \sum_{r=R}^{N-1} (N - r) J_r \tag{5.6}$$

is exact, with each pair counted once. Both $N\sum_{r\ge R} J_r$ and $S(N)-S(R)$ are $O_\delta(N^{2-p})$. Equations (5.2)–(5.6), and $\tau_N \le \tau^+$, therefore imply

$$\sum_{r=R}^{N-1} J_r \sum_{i=1}^{N-r} [\tau^+(r) - \tau_N(i, i+r)] = o(N^{2-p}).$$

Since $J_r \ge c_\delta N^{-p}$ on these separations, the corresponding unweighted deficit is $o(N^2)$. Separations below δN account for only $O(\delta N^2)$ pairs. Letting dyadic $\delta \downarrow 0$ proves $\mathbb{E}X_I X_H \to m^2$ for every pair of fixed macroscopic intervals, and thus the second assertion of (5.1).

There is also a ceiling on large averages. Tilt by field $h > 0$ just on I. Raising the remaining fields and comparing with the infinite plus system bounds each derivative of its log partition ratio by $M_J(h)$, so $\log \mathbb{E}e^{h|I|X_I} \le |I|[p_J(h) - p_J(0)]$. For small $h > 0$ the pressure difference is at most $h(m + \varepsilon/2)$. Chernoff and spin flip give $\mathbb{P}(|X_I| > m + \varepsilon) \le 2e^{-h\varepsilon|I|/2}$. Together with $\mathbb{E}X_I^2 \to m^2$ and boundedness, this forces $|X_I| \to m$ in probability and proves the first assertion. This completes the lemma at the stated scale. □

5.1 A quantitative finite ordering test

We give its coarse-chain input explicitly.

Lemma 5.2 (An effective coarse chain)**.** *Given rational $z > 1$ and $v < 1$, one can compute a nearest-neighbour strength D'_1 such that the chain with dimensionless $D'_j = z/[2(j+1)^2]$ for $j \ge 2$ has free mean-square spin average greater than v^2 along arbitrarily large explicit intervals.*

Proof. Choose $\max(3/4, v) < \gamma < 1$ with $s = z\gamma^2 > 1$, and rational $0 < \delta_0 \le 10^{-4}$ with $\gamma^2(1-\delta_0) > v^2$. Choose integer $n_* \ge 256$ so

$$\frac{4\,128^s (n_*-1)^{1-s}}{s-1} < \frac{1}{1000}, \qquad R_* = n_* + 10. \tag{5.7}$$

Let $p(K, \gamma')$ be the free probability that an interval of $2K$ sites has no cluster of size at least $2K\gamma'$, for $\gamma' \ge \gamma$. We establish, for $K \ge 2$ and integer $C > 2R_*$,

$$p(CK, \gamma' - R_*/C) \le p(K, \gamma')/1000 + 2C^2 p(K, \gamma')^2. \tag{5.8}$$

Take child intervals centered at iK, $-C+1 \le i \le C-1$. Successive good components intersect because their overlap has K sites and together they omit less than K there. Two failed children with disjoint interiors cost at most p^2, by conditioning and free comparison, and fewer than $2C^2$ such pairs occur. If no such pair occurs and i fails, the children outside $i-1, i, i+1$ are good. The adjacent two children need not be revealed. Alternating children tile each good run. For $n = C - |i| < n_*$, the longer run already has the required mass with the allowed R_*K loss. For $n \ge n_*$, joining the left and right runs loses at most $6K$ sites and suffices.

Conditional on the revealed internal edges of those runs and of the failed child, no edge between their chosen components has been revealed. If $W = 2\sum_{x\in C_-, y\in C_+} D'_{|x-y|}$, the probability of no bridge is at most $2e^{-W}$, by combining parallel edges with cluster weight two. Rank the selected sites from the gap. For $1 \le a, b \le A = \lfloor Kn/4 \rfloor$, their distance is at most $6K + (a+b)/\gamma'$. Hence, with $h = 6K + 5$,

$$W \ge z(\gamma')^2 \sum_{a,b=1}^{A} (6K + 3 + a + b)^{-2} \ge z(\gamma')^2 \log \frac{(h+A)^2}{h(h+2A)} \ge s \log(n/128).$$

The integral in the middle is the integral of $(h+x+y)^{-2}$ on $[0,A]^2$. The last bound uses $A \ge Kn/5$ and $h \le 9K$. The event that child i fails and the exterior runs are good has probability at most p and is measurable in the revealed interiors. Summing its missing-bridge cost over at most two positions for each n gives $4p\sum_{n\ge n_*}(128/n)^s \le p/1000$ by (5.7). This proves (5.8).

Set $K_0 = 2$, $\gamma_0 = (1+\gamma)/2$, choose integer $A_0 > 2R_*$ with $2R_*/A_0 < \gamma_0 - \gamma$, and put $C_j = A_0(j+1)^2$, $K_{j+1} = C_{j+1}K_j$. Reduce density by R_*/C_{j+1} at step j; its total loss is less than $2R_*/A_0$. With $X_j = C_j^2 p(K_j, \gamma_j)$, (5.8) implies $X_{j+1} \le (16/1000)X_j + 512X_j^2$. Choose D_1' with $6e^{-2D_1'} < \delta_0/A_0^2$. Three consecutive nearest bonds then give the initial good block, since their total failure probability is at most $6e^{-2D_1'}$. Induction gives $X_j \le \delta_0$. Independent FK colors yield mean square at least $\gamma^2(1-\delta_0) > v^2$.

This is the block bootstrap of Duminil-Copin–Garban–Tassion [12, Lemma 2] with the integral evaluated explicitly. In arXiv v2, the display before their (7) has an incorrect integral equality; its required lower bound holds with 32 in place of 8, since the integral is $\log[(h+A)^2/(h(h+2A))] \ge \log[A/(2h)]$. The constants above already incorporate that correction. □

Suppose a true lower tail certificate $J_r \ge b/r^2$ for $r \ge n_b$ is available. Choose rational parameters

$$0 < \theta < 1, \quad tb\theta^2 > a_0 > 1/2, \quad 0 < r_0 < 1, \quad z = 2r_0 a_0 > 1. \tag{5.9}$$

Let $D_j = a_0/(j+1)^2$ for $j \ge 2$ and choose $D_1 = D_1'/r_0$ from Lemma 5.2. Fix $q \ge 16D_1/a_0$, integer. Search dyadic L divisible by 2^q with $L/2^{q-1} \ge n_b$. A free internal block is good when one cluster occupies at least fraction θ of the whole block and of each left layer $(u/2, u]$, $u = L2^{-j}$, $0 \le j < q$, and their right reflections. Its probability p_L is a computable finite FK sum. Put

$$g = 4\sum_j D_j, \qquad \eta(p) = -g^{-1}\log[1-(1-e^{-g})p]. \tag{5.10}$$

The test accepts if $\eta(p_L)^2 > r_0$.

Lemma 5.3 (Soundness and completeness of the ordering test)**.** *Acceptance implies $m_J(t) \ge \theta v > 0$. If Lemma 5.1 applies at t and $2tb\,m_J(t)^2 > 1$, some choices of the parameters accept. Varying θ and v gives lower bounds exhausting $m_J(t)$ whenever the strict strength condition is available.*

Proof. In consecutive physical blocks, the interior marginal dominates the product of free interiors. For fixed sites the conditional connection probability after exterior edges are restored is increasing in those interiors. Average that function against the product law before choosing any representative components. In each good block choose one qualifying component deterministically, delete the other interblock edges and contract the chosen component. At block separation $d \ge 2$, its dimensionless coupling is at least $tb\theta^2/(d+1)^2$. For adjacent blocks the q pairs of facing layers give at least $q\,tb\theta^2/16 \ge D_1$; their distances exceed the cutoff by the choice of L. This dominates the coarse couplings $\xi_i\xi_j D_{|i-j|}$ with independent $\xi_i \sim \text{Bernoulli}(p_L)$.

For any finite coarse correlation $F = \langle S_i S_j\rangle$, allow site multipliers $s_k \in [0,1]$. In one multiplier, $F' = \mathrm{Cov}(S_iS_j, X) \ge 0$, with $X = S_k\sum_{l\ne k} D_{|k-l|}s_l S_l$. Also $F'' = \mathrm{Cov}(S_iS_j, X^2) - 2\langle X\rangle F' \ge -gF'$, since every coefficient of the spin monomials in X^2 is nonnegative. Since $\eta'' = g(\eta')^2$, $F(\eta(y_k))$ is separately convex. Coordinatewise Jensen therefore compares the diluted mean below by the pure correlation at couplings $\eta(p_L)^2 D$, hence by those at $r_0 D$. Summing correlations over physical pairs in distinct blocks gives mean square at least $\theta^2 v^2 - o(1)$ along the coarse lengths. The positive-field ceiling in Lemma 5.1's proof bounds any such limsup by $m_J(t)^2$. This proves soundness.

For completeness apply that lemma to the whole block and the finitely many layers. Color the free clusters by independent fair signs. Let a_i be their whole-block fractions and b_{ij} their fractions in layer j. Conditional on the graph,

$$\mathbb{E}\sum_j (X_j - X)^2 = \sum_{ij} (b_{ij} - a_i)^2,$$

and, writing $A_2 = \sum_i a_i^2$,

$$\mathbb{E}[(X^2 - m^2)^2 \mid \omega] = (A_2 - m^2)^2 + 2(A_2^2 - \sum_i a_i^4) \geq \tfrac{2}{3}(m^2 - a_{\max}^2)^2.$$

The final inequality uses $\sum a_i^4 \leq a_{\max}^2 A_2$ and completion of the square. For $\theta < \theta' < m$, the largest cluster therefore exceeds θ' with probability tending to one, and its fractions in every layer differ from its global fraction by less than $\theta' - \theta$ with probability tending to one. Thus $p_L \to 1$ along the lemma's dyadic lengths. Since $\eta(1) = 1$, acceptance follows. The choices $\theta \uparrow m$ and $v \uparrow 1$ prove the quantitative assertion. □

Theorem D (Critical jump and effective inverse-square data)**.** *Let $J \geq 0$ on the one-spin chain and $r^2 J_r \to B \in (0, \infty)$. Then*

$$0 < \beta_c < \infty, \qquad m(\beta_c) > 0, \qquad 2\beta_c B m(\beta_c)^2 = 1. \tag{5.11}$$

If J is computable in ℓ^1 and is supplied with an enumeration of true rational pairs (b, n) such that $J_r \geq b/r^2$ for $r \geq n$ and $\sup b = B$, then β_c is uniformly computable. The plus magnetization and every fixed plus marginal are uniformly computable on $\beta \neq \beta_c$. If B is also supplied as a computable real, they are computable on the promised closed side $\beta \geq \beta_c$ as well. In particular β_c is computable for $J_r = c/r^2$ with computable $c > 0$.

The equality in (5.11) is the open problem stated by Aizenman–Chayes–Chayes–Newman [10, p. 7]. It was first claimed in Bhattacharya's preprint [11, Theorem S1.1]; the introduction describes how the proof below relates to that preprint.

Proof. The necessary jump inequality is the refereed import from Aizenman–Chayes–Chayes–Newman [10, Theorem 3.1, (1.6), Section 1.1.2(c)]: at every ordered β, $2\beta B m(\beta)^2 \geq 1$. Their coefficient is $2J$ per unordered pair. Finiteness follows from Lemma 5.2. Fix a true certificate $J_r \geq b/r^2$ for $r \geq n$; one exists because $r^2 J_r \to B > 0$. Delete every bond not joining two sites of the sublattice $n\mathbb{Z}$; by GKS this lowers every correlation. At sublattice distance j the remaining couplings are at least $(b/n^2)j^{-2}$. Take $z = 2$ and $v = 1/2$ in Lemma 5.2. Once $\beta b/n^2 \geq 1$ and $\beta J_n \geq D_1'$, the sublattice chain dominates the lemma's chain, so along arbitrarily large intervals its free sublattice averages have mean square greater than $1/4$. Since all correlations are nonnegative, the free average over all sites of such an interval has mean square greater than $1/(4n^2)$. The positive-field ceiling at the end of the proof of Lemma 5.1 bounds the upper limit of these mean squares by $m(\beta)^2$. Hence $m(\beta) \geq 1/(2n)$ and $\beta_c < \infty$. The high-temperature row test gives positivity of β_c. The plus mean is increasing and upper semicontinuous, as the infimum of continuous finite plus-box means, hence right-continuous. The necessary inequality as $\beta \downarrow \beta_c$ proves $m(\beta_c) > 0$.

For every $t > \beta_c$, applying the necessary inequality at a smaller ordered point gives $2tBm(t)^2 > 1$. The supplied certificates approach B, so Lemmas 5.1 and 5.3 make every such rational t detectable. Enumerate strict upper bounds by accepting rational $u > t$ after a successful test at t, and combine with (2.2). These two enumerations compute β_c.

If the product in (5.11) exceeded one, the same test would exist at $t = \beta_c$, using some true tail lower bound and Lemma 5.1. Its finite probabilities and strength inequalities have strict

margins, which persist at a slightly smaller t. Soundness would contradict the definition of β_{c}. The necessary inequality proves equality.

Quantitative accepted bounds θv increase to $m(t)$ on the ordered open side. Start the lower list at zero; finite plus boxes, computed using the ℓ^1 tail to enclose exterior fields, give a decreasing upper list. They compute the mean also in the disordered phase. If B is computable, (5.11) computes $m(\beta_{\mathrm{c}})$, and adding it to the lower list handles the promised closed ordered side. Under the monotone coupling of a finite plus box and the infinite plus state, the probability of disagreement on a fixed finite set is at most half the sum of their one-site mean differences. Once the mean is computable, this bound computes every finite marginal. □

5.2 The half-line

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $V \in \{\mathbb{N}, \mathbb{Z}\}$ and a finite interval $\Lambda \subset V$, the free FK law on Λ has cluster weight 2 and parameters $1 - e^{-2\beta J_{|x-y|}}$ on the pairs of Λ. The wired law also keeps the couplings between Λ and $V \setminus \Lambda$, with $V \setminus \Lambda$ identified to one exterior vertex. Their limits along increasing intervals are $\phi_\beta^{V,\mathrm{f}}$ and $\phi_\beta^{V,\mathrm{w}}$. Let $\beta_{\mathrm{c}}^{V,\mathrm{f}}$ and $\beta_{\mathrm{c}}^{V,\mathrm{w}}$ be the suprema of the β at which these laws have almost surely no infinite cluster.

Lemma 5.4 (Anchored coarse chain)**.** *For every j, the free chain of Lemma 5.2 on the interval of $2K_j$ sites constructed in its proof joins the two endpoints of that interval with probability greater than $1 - \delta_0$. Hence its free endpoint correlation exceeds v^2.*

Proof. Keep the notation of the proof of Lemma 5.2. In a parent interval of $2CK$ sites, call the child centered at $(-C + 1)K$ its first child; it shares the parent's left end. All densities exceed $3/4$, so a good interval has a unique large cluster. Let $q(CK, \gamma'')$ be one minus the free probability that the first child is good and that the parent has a cluster of at least $2CK\gamma''$ sites containing the large cluster of the first child. We claim, with $p = p(K, \gamma')$,

$$q(CK, \gamma' - R_*/C) \le (n_* + 4)p + 2C^2p^2. \tag{5.12}$$

Exclude the events bounded in the proof of (5.8), two failed children with disjoint interiors and a missing bridge, of total probability at most $p/1000 + 2C^2p^2$. Exclude also the event that one of the $n_* + 3$ children nearest the left end fails, of probability at most $(n_* + 3)p$. If no child fails, consecutive large clusters intersect and all children lie in one cluster. Otherwise the failing children lie in $\{i, i + 1\}$, where $i \ge -C + n_* + 4$ is the least failing index. If $C - |i| < n_*$, then $i > 0$, and the left run, which is the longer one and begins with the first child, has the required mass. If $C - |i| \ge n_*$, the left and right runs are bridged and their union has the required mass. In both cases the large cluster of the parent contains that of the first child. This proves (5.12). The failure events counted by p and q are decreasing, so under the free law of a larger interval, which induces a boundary partition on the parent, their probabilities are at most the free values.

Follow the first children of the interval of $2K_j$ sites down to level 0, and likewise its last children. A good level-0 interval has a cluster of more than $4\gamma_0 > 3$ of its four sites, so it is connected and contains its endpoint. If both level-0 intervals are good and all the anchored events hold, both endpoints lie in the large cluster of the whole interval. The proof of Lemma 5.2 gives $p(K_i, \gamma_i) \le \delta_0/C_i^2$, and $C_{i+1} \le 4C_i$. By (5.12) the failure probability is therefore at most

$$\frac{2\delta_0}{A_0^2} + 2\sum_{i \ge 0} \frac{(n_* + 4)\delta_0 + 32\delta_0^2}{C_i^2} \le \frac{2\delta_0 + 4(n_* + 4)\delta_0 + 128\delta_0^2}{A_0^2} < \delta_0,$$

using $\sum_{i \ge 0} C_i^{-2} = \zeta(4)/A_0^2 < 2/A_0^2$ and $A_0 > 2R_* = 2n_* + 20$. In free FK the endpoint correlation equals the endpoint connection probability, and $1 - \delta_0 > \gamma^2(1 - \delta_0) > v^2$. □

Proposition 5.5 (Half-line and full-line thresholds)**.** *Let $J \geq 0$ on the one-spin chain with $r^2 J_r \to B \in (0, \infty)$. Then*

$$\beta_{\mathrm{c}}^{\mathbb{N},\mathrm{f}} = \beta_{\mathrm{c}}^{\mathbb{Z},\mathrm{f}} = \beta_{\mathrm{c}}^{\mathbb{N},\mathrm{w}} = \beta_{\mathrm{c}}^{\mathbb{Z},\mathrm{w}} = \beta_{\mathrm{c}}.$$

Consequently the Ising models on $\mathbb{N}$ and on $\mathbb{Z}$ have a unique Gibbs measure for $\beta < \beta_{\mathrm{c}}$ and more than one for $\beta > \beta_{\mathrm{c}}$.

For $J_r = r^{-2}$, $\phi_\beta^{\mathbb{N},\mathrm{f}}$ and $\phi_\beta^{\mathbb{Z},\mathrm{f}}$ are the FK-Ising Dyson measures at $\alpha = 2$ of Berger, Johansson and Öberg [6, Section 2], at their inverse temperature 2β. The proposition therefore gives $\beta_c^{\mathbb{N}} = \beta_c^{\mathbb{Z}}$ in their notation and answers their Problem 1 [6]; they proved this equality for $1 < \alpha < 2$ [6, Theorem 1]. The equality of the uniqueness thresholds proves, for every J of this class, the equality conjectured by Johansson, Öberg and Pollicott [7, Remark 1]. We found no earlier solution of either question. Nothing is claimed about boundary order at β_{c}: the proof uses the strict surplus $2\beta B m(\beta)^2 > 1$, which fails at β_{c} by (5.11), and whether a site of $\mathbb{N}$ has positive plus mean at β_{c} is left open.

Proof. By the ghost representation of Section 2, the plus mean of a site of V is the probability under $\phi_\beta^{V,\mathrm{w}}$ that it lies in an infinite cluster. On $\mathbb{Z}$, translation invariance then gives $\beta_{\mathrm{c}}^{\mathbb{Z},\mathrm{w}} = \beta_{\mathrm{c}}$. Every Gibbs measure on V lies between the plus and minus states, which coincide exactly when every plus mean vanishes. This gives the statement on Gibbs measures once the thresholds are identified.

Wired laws dominate free ones, so $\beta_{\mathrm{c}}^{\mathbb{Z},\mathrm{w}} \leq \beta_{\mathrm{c}}^{\mathbb{Z},\mathrm{f}}$ and $\beta_{\mathrm{c}}^{\mathbb{N},\mathrm{w}} \leq \beta_{\mathrm{c}}^{\mathbb{N},\mathrm{f}}$. Free laws increase with the interval, so on the pairs of $\mathbb{N}$ the law $\phi_\beta^{\mathbb{Z},\mathrm{f}}$ dominates $\phi_\beta^{\mathbb{N},\mathrm{f}}$, and $\beta_{\mathrm{c}}^{\mathbb{Z},\mathrm{f}} \leq \beta_{\mathrm{c}}^{\mathbb{N},\mathrm{f}}$. For $\beta_{\mathrm{c}}^{\mathbb{Z},\mathrm{w}} \leq \beta_{\mathrm{c}}^{\mathbb{N},\mathrm{w}}$, take the wired law of $\mathbb{Z}$ on $\{-n, \dots, n\}$ and set to zero every coupling with a negative endpoint, including the couplings to negative exterior sites. The negative sites then meet the other vertices only at the exterior vertex, so the weight factorizes there, and the law of the edges among $\{0, \dots, n\}$ and the exterior vertex is the wired law of $\mathbb{N}$ on $\{0, \dots, n\}$. Restoring the couplings raises the law, and $n \to \infty$ gives the claim. It remains to prove $\beta_{\mathrm{c}}^{\mathbb{N},\mathrm{f}} \leq \beta_{\mathrm{c}}$.

Fix $\beta > \beta_{\mathrm{c}}$ and $v \in (0, 1)$. As in the proof of Theorem D, $2\beta B m(\beta)^2 > 1$. Lemma 5.3 at $t = \beta$, with a true certificate $J_r \geq b/r^2$ for $r \geq n_b$ and b close to B, gives parameters (5.9) and a dyadic L whose test accepts; its coarse chain is the chain of Lemma 5.2 for these z and v. Let $M = 2K_j$ and split $\{1, \dots, ML\}$ into consecutive blocks $P_1, \dots, P_M$ of length L. The soundness argument of Lemma 5.3 applies to the pair of blocks P_1, P_M: fixed-site connection probabilities are averaged against the product of free block laws before clusters are chosen, a vacant block gives zero diluted correlation, and an occupied block has a chosen cluster of at least θL sites. Write $\langle S_1 S_M \rangle_{\xi\xi D}$ and $\langle S_1 S_M \rangle_{r_0 D}$ for the free coarse-chain correlations on $\{1, \dots, M\}$ with couplings $\xi_i \xi_k D_{|i-k|}$, where the ξ_i are independent Bernoulli(p_L) variables, and $r_0 D_{|i-k|}$. Then

$$\sum_{x \in P_1,\, y \in P_M} \phi^{\mathrm{f}}_{\{1,\dots,ML\},\beta}(x \leftrightarrow y) \geq \theta^2 L^2\, \mathbb{E}_\xi \langle S_1 S_M \rangle_{\xi\xi D} \geq \theta^2 L^2 \langle S_1 S_M \rangle_{r_0 D} > \theta^2 L^2 v^2.$$

The final step is Lemma 5.4, because the couplings $r_0 D$ are those of Lemma 5.2.

Hence some $x \in P_1$ and $y \in P_M$ satisfy $\phi^{\mathrm{f}}_{\{1,\dots,ML\},\beta}(x \leftrightarrow y) > \theta^2 v^2$. As j varies, $P_1 = \{1, \dots, L\}$ is fixed, so one x occurs for infinitely many j, with partners $y_j \geq (2K_j - 1)L + 1 \to \infty$. Free laws increase with the interval, so $\phi_\beta^{\mathbb{N},\mathrm{f}}(x \leftrightarrow y_j) > \theta^2 v^2$. The events that x is joined to a site at distance greater than R decrease, as $R \to \infty$, to the event that the cluster of x is infinite, since an infinite subset of $\mathbb{N}$ is unbounded. Thus $\phi_\beta^{\mathbb{N},\mathrm{f}}(x \leftrightarrow \infty) \geq \theta^2 v^2 > 0$. As $\beta > \beta_{\mathrm{c}}$ was arbitrary, $\beta_{\mathrm{c}}^{\mathbb{N},\mathrm{f}} \leq \beta_{\mathrm{c}}$, and all four thresholds equal β_{c}. □

6 What the tail data determine

6.1 The joint endpoint at fixed row and coefficient

Fixing both masses gives a stronger restriction than the row alone.

Theorem E (The sharp joint endpoint)**.** *Let* $2\sum_r J_r = T > 0$ *and* $r^2 J_r \to B \in (0,\infty)$*. Let* $z_* \in (0,1)$ *be the unique solution of*

$$2Bz_* \operatorname{arctanh} z_* = T, \qquad a_* = (2Bz_*^2)^{-1}. \tag{6.1}$$

Then $\beta_c > a_*$*. For computable* T, B*, the bound is sharp among strictly positive completely monotone exchanges with explicit lower tail certificates, and their critical points are exactly the computable reals greater than* a_**. Here the exchange is computable in* ℓ^1 *and the certificates are supplied as in Theorem D. For* $J_r = 1/r^2$*,*

$$0.5645 < \beta_c < 16. \tag{6.2}$$

Proof. At any ordered β, write $m = m(\beta)$. We first prove the strict mean-field comparison

$$\operatorname{arctanh} m < \beta T m. \tag{6.3}$$

Retain only one distance r with $k = \beta J_r > 0$. Its reference system splits into nearest-neighbour chains. The dimensionless field producing mean u is

$$h_k(u) = \operatorname{arsinh}\left(e^{-2k}\frac{u}{\sqrt{1-u^2}}\right).$$

This follows by differentiating the larger transfer eigenvalue $e^k[\cosh h + \sqrt{\sinh^2 h + e^{-4k}}]$. At fixed u, $h_0(u) = \operatorname{arctanh} u$ and $\partial_k h_k(u) = -2\tanh h_k(u) > -2u$ for $k > 0$.

For completeness the GHS comparison needed here is as follows. Interpolate a reference kernel K_0 to the full kernel, with added dimensionless row τ. In a periodic volume put $h(s) = \varepsilon + (1-s)\tau u$. If its common mean is M_s and susceptibility χ_s, GHS gives $\operatorname{Cov}(\sigma_z, \sigma_i\sigma_j) \le M_s[\operatorname{Cov}(\sigma_z,\sigma_i) + \operatorname{Cov}(\sigma_z,\sigma_j)]$, hence $M_s' \le \tau(M_s - u)\chi_s$. An integrating factor keeps $M_s < u$ if $M_0 < u$. Take the volume limit at positive ε, then $\varepsilon \downarrow 0$. Therefore if the reference mean at field τu is smaller than u, the full spontaneous mean is at most u.

In the one-distance reference $\tau = \beta T - 2k$. If $\tau m < h_k(m)$, this remains true for some $u < m$, a contradiction. Hence $\tau m \ge h_k(m) > \operatorname{arctanh} m - 2km$, proving (6.3). Finite energy gives $m < 1$. Let $m_{\mathrm{MF}}(x)$ be the positive root of $m = \tanh(xm)$ for $x > 1$. Equation (6.3) and the necessary jump bound at β_c imply $2\beta_c B m_{\mathrm{MF}}(\beta_c T)^2 > 1$. This increasing continuous expression crosses one exactly at (6.1), proving strict necessity.

We give the sharpness argument explicitly. Fix $\beta > a_*$. Reserve a small row for the tail $B\int_0^x te^{-rt}\,\mathrm{d}t$, with rational $x > 0$, and let T_c be the remaining row. Choose $\theta < m_{\mathrm{MF}}(\beta T_c)$ with $2\beta B\theta^2 > 1$. This tail supplies the explicit bound $r^2 J_r \ge B[1 - (1+rx)e^{-rx}]$. Choose the strict constants in (5.9), then its finite collection of layers.

Use geometric core $(T_c/2)(1-q)q^{r-1}$ with $q = 1 - 1/(Qs)$, where Q is a large integer and s the number of sites in one cell. For cells at separation $0 \le j \le D$, the core dominates a weaker coupling a_j/s, with a_j chosen just below $(T_c/(2Q))e^{-(j+1)/Q}$, once s is large. Choose Q large, then D/Q large, and these constants close enough to their bounds that $t_0 = a_0 + 2\sum_{j=1}^{D} a_j$ satisfies $m_* = m_{\mathrm{MF}}(\beta t_0) > \theta$. The limiting geometric sum approaches T_c.

In n free cells, binomial counting as $s \to \infty$ concentrates the cell means v_i at the maximizers of

$$\mathcal{F}_n(v) = \sum_i S(v_i) + \frac{\beta}{2}\sum_{|i-j|\le D} a_{|i-j|}v_i v_j,$$

where $S(v)$ is binary entropy. The bounds $(s+1)^{-1}e^{sS(v)} \le \binom{s}{s(1+v)/2} \le e^{sS(v)}$ prove concentration on any neighborhood of the compact maximizer set. If $F(z) = S(z) + \beta t_0 z^2/2$, then $\mathcal{F}_n(v) \le \sum_i F(v_i)$, whereas the constant m_* profile attains $nF(m_*) - C$, with C independent of n. The scalar maxima are exactly $\pm m_*$. Thus at a maximizer only a bounded number of cells can have $||v_i| - m_*| > \varepsilon$. Every maximizer has one common sign: taking absolute values improves unlike nonzero neighbours; a zero beside a positive neighbour is improved to first order, and connectivity propagates positivity. Entropy excludes endpoints ± 1.

Fix the finitely many prescribed layers. The limits are taken in this order. First choose dyadic n so large that the bounded number of exceptional cells is a negligible fraction of the smallest prescribed layer. Then, for that n, choose dyadic $s = s(n)$ so large that binomial concentration puts the cell means within the required distance of the maximizer set with probability close to one. Along such pairs $(n, s(n))$ with $n \to \infty$, the whole-block and all prescribed layer means obey (5.1) with m_* in place of m. The FK moment calculation of Lemma 5.3 gives a good block with probability arbitrarily close to one in the weaker system and hence in the physical geometric-core system. Its finite ordering test accepts at β. This proves the sharp infimum, with a single computable completely monotone exchange for each such construction. This sharpness argument does not use the record scales of Lemma 5.1: the finite cell calculation supplies the block input, and soundness of the ordering test uses only the positive-field ceiling proved at the end of that lemma's proof. Lemma 5.1 enters Theorem E only through Theorem D, in the computability of the critical points of certified exchanges.

There are also certified centers with arbitrarily large critical points at the same T, B: send the tail cutoff $x \downarrow 0$ and the geometric parameter $q \downarrow 0$. They approach the row-T nearest-neighbour chain in ℓ^1. Its finite-set witness at any prescribed finite temperature persists for sufficiently close centers. The affine segment between two strictly bracketing centers preserves row, coefficient and complete monotonicity, and has common effective lower tail certificates. The two searches of Theorem D make its critical point a computable continuous function. A computable target has a computable preimage: either some rational midpoint attains it, or strict bisection at every midpoint terminates. The preimage is thus found separately for each computable target.

For the lower number in (6.2), use $T = 2\zeta(2)$, $B = 1$ in (6.1). Rational series and tail bounds, specified in Appendix C, prove $a_* > 0.5645$. For the upper number, the pure chain at $\beta = 16$ dominates Lemma 5.2 with

$$z = 32, \quad \gamma = 4/5, \quad \gamma_0 = 9/10, \quad n_* = 512,$$
$$R_* = 522, \quad A_0 = 20000, \quad \delta_0 = 10^{-4}, \quad D_1' = 16.$$

The strict tests (5.7) and $6e^{-32} < 10^{-4}/20000^2$ hold. They persist slightly below 16, proving the strict upper bound. □

Corollary 6.1 (Certified bracket for the unit inverse-square chain)**.** *For $J_r = r^{-2}$,*

$$\frac{631}{1000} < \beta_c < \frac{11544}{7225} = 1.59778\ldots, \qquad m\Big(\frac{11544}{7225}\Big) \ge \frac{17}{20}.$$

This bracket is computer-assisted, verified by two independently written programs. Its proof is Theorem B.1 in Appendix B. It sharpens the bracket (6.2), which remains an independent proof that uses none of those programs:

- the lower number 0.5645 comes from the strict bound $\beta_c > a_*$ of Theorem E at $T = 2\zeta(2)$, $B = 1$; its only numerical input is one exact rational inequality, $a_* > 0.5645$, checked by the standard-library verification of Appendix C;
- the upper number 16 comes from the explicit coarse chain of Lemma 5.2.

6.2 Degrees in the approach to the asymptote

Theorem F (An ineffective inverse-square limit)**.** *Fix rational $B, T > 0$ and a c.e. set H. There is an exactly rational-valued computable exchange $J_r > 0$, with a computable ℓ^1 tail, such that*

$$2\sum_r J_r = T, \qquad r^{2+k}(-1)^k \Delta^k J_r \to (k+1)!B \quad (k \geq 0), \qquad \beta_{\mathrm{c}} \equiv_T H.$$

All alternating finite differences are strictly positive. The least Turing degree computing a modulus for $r^2 J_r \to B$, or enumerating true eventual lower certificates approaching B, is $\deg_T(H)$.

Proof. We give the rational moment construction directly. Choose rational $z \in (1/2, 1)$ with $2B(1-z) < T/2$, and a rational interval $I \subset (0, 1/2)$. Start with density $f = B$ on $(z, 1)$ and a positive rational constant P_0 on I, zero elsewhere, chosen so $2\int f = T$. Define

$$J_r = \int_0^1 u^{r-1}(1-u) f(u)\, \mathrm{d}u. \tag{6.4}$$

At every finite stage f is piecewise rational polynomial, equals B on a known interval adjacent to 1, and stays between $P_0/2$ and $3P_0/2$ on I. Its other alterations are disjoint intervals in $(z, 1)$ on which $f = B(1-\eta)$, $0 < \eta \leq 1/2$. We call this finite object a center.

Each center has a uniformly computable critical point by Theorem D, hence ultimately by Lemma 5.1. Indeed if $f = B$ on $(y, 1)$, then

$$r^2 J_r \geq B\frac{r}{r+1} - Bry^r,$$

which supplies lower certificates approaching B. The remaining bounded density below $y < 1$ also gives an explicit two-sided convergence rate. The ordinary row tail has the uniform bound

$$2\sum_{r>R} J_r = 2\int_0^1 u^R f(u)\, \mathrm{d}u \leq \frac{2B}{R+1} + 3P_0|I|(\sup I)^R. \tag{6.5}$$

All centers dominate the fixed kernel with density $(B/2)\mathbf{1}_{(z,1)}$. It dominates some computable positive multiple of r^{-2}, by its explicit asymptote and the minimum over the remaining finite initial segment. Thus the centers' critical points have one computable finite upper bound, using (6.2).

At stage s a cut of depth η on a rational interval $D \subset (z, 1)$ will preserve the row and the first s moments by a compensator on I. Use the rational polynomial basis $b_0 = 1$, $b_j = (1-u)u^{j-1}$ for $1 \leq j \leq s$. It spans all polynomials of degree at most s, by the nonzero successive leading coefficients. Solve for $P = \sum_{i=0}^s c_i b_i$ in

$$\int_I b_j P\, \mathrm{d}u = B\eta \int_D b_j\, \mathrm{d}u, \qquad 0 \leq j \leq s. \tag{6.6}$$

The rational Gram matrix is positive definite: a nonzero polynomial has strictly positive square integral on I. Its inverse is computable and bounds $\|P\|_\infty \leq C_s B|D|$ for a computable rational C_s. The map from the cut indicator to P is linear. Since the bilateral kernel norm change is at most $2\int |\delta f|$, both it and the compensator's sup norm can be made arbitrarily small by moving the eligible window close to 1. This also applies to a temporary interval ending at 1 and its later restoration.

Here is the marker action. At a center C and allotted rational depth η, the jump equality and right continuity give rationals

$$a_C < l < t < h < v$$

and a finite plus box with $2v(1-\eta)B[M_\Lambda^+(C,v)]^2 < 1$. They are found by strict search: as $v \downarrow a_C$, the corresponding infinite expression tends to $1-\eta$, and plus boxes decrease to that mean. Store these parameters and rational $0 < \xi < 1/2$ with $a_C/(1-\xi) < l$. Compute a row-norm radius $d > 0$ preserving the strict plus-box test. Such radii are effective: in an n-spin plus box a row perturbation δ changes configuration exponents by at most $vn\delta$, so likelihood ratios are bounded by $e^{2vn\delta}$. Finite-set disorder tests have the same effective continuity, with their outgoing sums bounded using tanh's Lipschitz constant.

If this marker's index enters H, choose an untouched interval $(x,1)$ small enough for all current budgets, temporarily put $f = B(1-\eta)$ there, and apply (6.6). The coefficient is now $(1-\eta)B$. The protected finite plus mean and the necessary ACCN inequality exclude order at v, so $h < v$ has a strict finite-set witness by Proposition 2.2. Find it by enumeration. Restore $f = B$ on $(y,1)$ for rational $y > x$ sufficiently close to 1, using the compensator for (x,y) instead. Linearity makes the restoration norm arbitrarily small, so the witness at h persists. Commit only the cut (x,y) and record a radius preserving this new witness under all future changes. The marker's action therefore forces the final critical point above $h > t$.

We specify the simultaneous budgets. Every strict witness ever committed remains in a permanent list with an effective radius $d_i > 0$. At stage s, require both the temporary and committed change from the entry center to have row norm below

$$2^{-s-6}\min(1, d_1, \dots, d_j),$$

where the list includes the plus tests of all currently retained markers, including any initialized that stage. Require its reservoir compensator to have sup norm below $P_0 2^{-s-6}$ and, for each retained marker i, below $\xi_i P_0 2^{-s-6}$. These finitely many strict requirements are possible by (6.6). Within the action, choose the restoration endpoint so its newly found witness survives, then add its radius to the permanent list. The geometric sums of later budgets are smaller than each stored radius. Consequently all permanent witnesses and all retained marker tests survive even during a later temporary change. The unconditional reservoir budgets keep its density within the asserted bounds. Erasing a marker removes only its future no-action budget, never a permanent disorder witness or a committed cut.

At global stage s, inspect the first s steps of an enumeration of H. Initialize missing markers in increasing order through index s. Give a newly initialized index e a depth $\eta_e \le 2^{-e-2}$ and below every retained smaller index's ξ_i. Act on the least entered index with an unacted current marker, if any; mark it acted and erase larger markers. There is at most one action per stage. After each stage move the future eligible window beyond every committed cut and into $(1-2^{-s},1)$. Each finite search terminates as just proved. Induction on the index shows finite injury: once all smaller markers are stable, an entered index acts once and a nonmember retains its unacted marker.

On the final untouched or altered tail, $f(u) \to B$ as $u \uparrow 1$. For any fixed e, only finitely many cuts belong to indices at most e, while the depths of all others are at most 2^{-e-2}. The density is bounded and nonnegative. Equation (6.4) gives

$$(-1)^k \Delta^k J_r = \int_0^1 u^{r-1}(1-u)^{k+1} f(u)\,\mathrm{d}u.$$

Compare on a neighborhood of 1 with the constant density B, and use exponential decay on its complement. The beta integral $\int_0^1 u^{r-1}(1-u)^{k+1}\,\mathrm{d}u = (k+1)!/[r(r+1)\cdots(r+k+1)]$ proves every stated asymptote. Positive density adjacent to 1 makes every difference strict. At each action of stage s, (6.6) preserves J_r for $r \le s$. To compute the exact rational J_r, simulate through stage r and integrate its finite rational polynomial density. No later action changes it. The preserved row is T, and (6.5) supplies the computable ordinary tail without stabilization information.

We verify the two oracle reductions. For a final unacted marker e, every later action has depth at most ξ_e on an interval untouched at initialization. On the reservoir the total possible decrease is less than $\xi_e P_0/8$, whereas its initial density was at least $P_0/2$. Thus the final density, and hence the kernel, dominates $(1-\xi_e)C_e$ pointwise. Its critical point is below $l_e < t_e$. If the final marker acts, its permanently protected witness gives $\beta_c > h_e > t_e$.

Given β_c as a real oracle, decide membership in increasing index order. Once membership below e is known, simulate until all those members have acted on their current markers and all smaller markers and marker e are present. There can be no further smaller injury. If e has already entered, answer yes. Otherwise compare β_c with t_e: if it never enters the no-action bound gives $\beta_c < t_e$; if it enters later the action gives $\beta_c > t_e$. The comparison always terminates, proving $H \le_T \beta_c$.

Conversely an H oracle finds a stage $s \ge n+3$ stable through index n. All future cuts have depth at most $\varepsilon_n = 2^{-n-2}$, and the unconditional compensator budgets give relative reservoir change less than 2^{-s}. For its finite center C,

$$(1-\varepsilon_n)C_r \le J_r \le (1+2^{-s})C_r, \qquad \frac{a_C}{1+2^{-s}} \le \beta_c \le \frac{a_C}{1-\varepsilon_n}.$$

The uniform upper bound on a_C makes these effective Cauchy brackets, so $\beta_c \le_T H$. The same pointwise bounds, together with the center's explicit rate to B, give an H-computable modulus for $r^2 J_r \to B$. Either such a modulus or a certificate enumeration approaching B computes β_c by Theorem D, hence computes H. A modulus supplies lower certificates since B is rational. This proves both least-degree assertions. □

6.3 Powers below two

Theorem G (Comparable powers)**.** *Suppose J is computable in ℓ^1 and $cr^{-p} \le J_r \le Cr^{-p}$ for all sufficiently large r, where $1 < p < 2$ and $0 < c \le C$. Then β_c is finite and computable, $m(\beta_c) = 0$, and m is computable on $(0,\infty)$. The algorithms are uniform from the ordinary ℓ^1 data and one rational triple (c', p', R_0) with $c' > 0$, $p' < 2$ and $J_r \ge c' r^{-p'}$ for $r \ge R_0$, on the comparable-power promise. At fixed computable $T > 0$ and $p \in (1,2)$, the possible onsets with row T are exactly the computable reals greater than $1/T$.*

Proof. Finiteness follows by the inverse-square sublattice comparison. Lemma 5.1 applies with scale N^{2-p}. Since $r^2 J_r$ eventually exceeds every desired b, at any ordered t one can choose $2tb\theta^2 > 1$ with $\theta < m(t)$. Lemma 5.3 then supplies an ordering certificate. If $m(\beta_c) > 0$, the same argument at $t = \beta_c$ would give a strict finite test persisting below β_c, a contradiction. Thus $m(\beta_c) = 0$. The supplied lower triple computes cutoffs for every b using $c' r^{2-p'} \ge b$. These upper certificates and (2.2) compute β_c. Quantitative lower bounds starting at zero, together with finite plus-box upper bounds, compute m even across the continuous transition.

For the exact spectrum use reflection-positive centers mixing normalized r^{-p} with geometric hopping of row T. Write $D(k) = \sum_{r \ge 1} 2J_r(1-\cos kr)$ for the band of the kernel, with normalized integrals over $[-\pi,\pi]$. Their reciprocal band integral is computable, since the power contribution gives $D(k) \ge c|k|^{p-1}$ near zero by summing over $1/|k| \le r \le 2/|k|$. The finite-torus infrared inequality [14, (4.31), proof of Lemma 4.7] implies $\beta_c \le \int 1/D$. Its normalization here is $1/[\beta D(k)]$: divide the kernel by T and use inverse temperature βT. Positive mixtures of e^{-tr} are through-bond reflection positive, since their crossing quadratic form is a positive integral of squares. This checks the hypothesis for these centers.

Here is the thermodynamic step for that use of the inequality. For a fixed finite-support zero-sum vector, torus sums converge by the integrable $|k|^{1-p}$ majorant. Below β_c, uniqueness and finite susceptibility follow from Proposition 2.2. Subtract from a single-site vector the uniform vector on a long interval. Its spin mean square vanishes as that interval grows, by the

finite susceptibility, and dominated convergence removes its Fourier transform. The infrared bound then gives $1 \le \beta^{-1} \int 1/D$ for every $\beta < \beta_c$, as asserted.

Spread the geometric law and then let the power weight decrease to zero. Dominated convergence gives $\int 1/D \to 1/T$, so finite centers have onsets approaching $1/T$. At the other end let both the power weight and the geometric range tend to zero, approaching a nearest-neighbour chain. Any fixed finite-temperature disorder witness persists, so the finite onsets are unbounded. A segment between strictly bracketing centers retains the comparable-power bounds and their effective lower triple. Its critical point is computable by the preceding searches and continuous, either from their open witnesses or from multiplicative comparison. The pointwise computable-preimage bisection used in Theorem E realizes each computable target. The singleton test proves strict necessity above $1/T$. □

7 Finite-range periodic pair models

The finite-range result combines a lower search, made complete by subcritical sharpness on quasi-transitive graphs, with an upper search constructed in this section.

Proposition 7.1 (Subcritical sharpness on weighted quasi-transitive graphs)**.** *Let $G = (V, E)$ be a connected, infinite, locally finite graph with weights $J : E \to (0, \infty)$, and let a group of weight-preserving graph automorphisms act on V with finitely many orbits. For finite $\Lambda \subset V$ let $\phi^{\mathrm{w}}_{\Lambda,\beta}$ be the FK measure of cluster weight 2 and parameters $p_e = 1 - e^{-2\beta J_e}$ on the edges with both endpoints in Λ, in which all vertices of Λ incident to an edge leaving Λ are identified. Let $\phi^{\mathrm{w}}_{\beta}$ be its infinite-volume limit and $\beta_c(G) = \inf\{\beta : \phi^{\mathrm{w}}_{\beta}(x \leftrightarrow \infty) > 0\}$, which does not depend on x. Write $\Lambda_n(x)$ and $S_n(x)$ for the graph-distance ball and sphere of radius n about x. For every $\beta < \beta_c(G)$ there is $c_\beta > 0$ such that, for all $x \in V$ and $n \ge 1$,*

$$\phi^{\mathrm{w}}_{\Lambda_n(x),\beta}[x \text{ is joined to } S_n(x) \text{ by an open path}] \le e^{-c_\beta n}.$$

Duminil-Copin, Raoufi and Tassion prove this for vertex-transitive weighted graphs and state that their theorem extends to quasi-transitive weighted graphs [4, Theorem 1.2 and the remark after it, p. 80]; Duminil-Copin's lecture notes state the same extension [5, Section 2.3]. Neither gives a proof of the extension, and Appendix A writes one. Their couplings are $2J$ in our normalization. The proposition is used only for completeness of the lower search below.

Theorem J (Periodic rational pair models)**.** *The input is $d \ge 2$, a nonempty finite type set F, a finite list of positive rational bond orbits (i, j, v, J) on $\mathbb{Z}^d \times F$, and positive rational weights w_i summing to one. Exclude $(i, i, 0)$, identify (i, j, v) with $(j, i, -v)$, and sum parallel orbits. There is one Ising spin per vertex and energy $-\sum_e J_e \sigma_x \sigma_y$ per unordered pair. Put $M = \sum_i w_i \sigma_{(0,i)}$ and $\beta_c = \inf\{\beta : \mu^+_\beta(M) > 0\}$.*

An algorithm decides by a rank test whether $\beta_c = \infty$, and reports that value exactly in this case. Otherwise, on input k, it returns rationals $\ell \le \beta_c \le u$ with $u - \ell < 2^{-k}$. For $\beta < \beta_c$ there is exactly one Gibbs measure; for $\beta > \beta_c$, the distinct plus and minus states are invariant under cell translations, exchanged by spin flip, and have opposite nonzero means of M. The algorithm is claimed to terminate, with no running-time estimate.

The ordered-side certificate is constructed directly. The method adapts Severo's architecture [20]: our surface arm estimate corresponds to his Lemma 2.1, the good-block lemma to Proposition 2.2, upper-certificate soundness to Proposition 2.4, and the domination lemma to Lemma 2.5. The interface step adapts his Appendix A and the earlier arguments of Lebowitz–Pfister and Bodineau [18, 19]. An interior field sheet handles periodic decorations, and a closed-edge path cost handles rank two.

7.1 Connected components and regular temperatures

First work on a connected component with cell group $\mathbb{Z}^r$, $r \geq 2$. After blocking, every edge moves at most one cell in maximum norm. Let $B_n(x)$ contain all types in its radius-n cell cube. Its boundary layer means every vertex in its outermost cell shell, which contains every vertex incident to an exterior edge. All sheets below likewise contain every type. Call $t > 0$ regular if the pressure per cell is differentiable there. Write μ_t^{f} for the infinite-volume free spin law and $\phi_t^{\mathrm{f}}, \phi_t^{\mathrm{w}}$ for the free and wired FK laws.

Lemma 7.2 (Regular temperatures)**.** *All but countably many $t > 0$ are regular. At a regular t,*

$$\mu_t^{\mathrm{f}} = (\mu_t^{+} + \mu_t^{-})/2, \qquad \phi_t^{\mathrm{f}} = \phi_t^{\mathrm{w}}, \tag{7.1}$$

this common bond law has at most one infinite cluster, and it is mixing. If moreover $t > \beta_{\mathrm{c}}$, the infinite cluster exists almost surely, and the probability that $B_k(x)$ meets it tends to one as $k \to \infty$, uniformly in x.

Proof. Tiling and the finite-range surface bound give one convex pressure for plus and free boundaries, so the exceptional set is countable. At a regular t, their energy derivatives agree. Spatial averages of each bond moment converge: discard a fixed-width boundary strip, squeeze by the appropriate finite-box and infinite moments, then let that width grow. Plus pair moments dominate free pair moments, and every edge weight is positive. Equality of the finite energy sums therefore gives equality for each bond type.

All even moments agree as well. Every finite even vertex set on a connected graph is a symmetric difference of positive-edge endpoint pairs. Write u and v for plus and free moments. If $u_A = v_A$ for an even set A and B is a positive edge, then (2.3) for the sets $A \triangle B$ and B gives $0 = u_A - v_A \geq u_B |u_{A \triangle B} - v_{A \triangle B}|$, since $u_B = v_B$; and $u_B > 0$. Induction over the decomposition proves the claim. Spin-flip symmetry then gives the first identity in (7.1). This recovers Lebowitz's phase-mixture argument [9, Theorems 2–3, pp. 470–471] for a connected periodic component. The bond identity follows because each edge marginal is $p_e(1 + \langle \sigma_x \sigma_y \rangle)/2$ and an ordered coupling with equal marginals agrees on all edges.

The common bond law has at most one infinite cluster. The free two-point function is the physical connection probability, whereas the wired two-point function is the probability of physical connection or two infinite arms. These statements follow by the increasing free and decreasing wired box couplings. Equality rules out two distinct infinite clusters, first for each pair of vertices and then by countability. The free law is mixing: for an increasing cylinder event A, its conditional probability given any exterior of a larger box is bounded below by its finite free probability. The mean excess tends to zero, uniformly bounding covariance with exterior cylinder events. Approximation and linearity give mixing and ergodicity. If $t > \beta_{\mathrm{c}}$, the infinite cluster consequently exists almost surely, and growing boxes meet it with probability tending to one, uniformly in their finitely many types. □

7.2 Interior sheets and a surface-order arm estimate

Put a dimensionless field $s > 0$ on the full sheet $P = \{z_r = 0\}$. For free or plus boxes B_k about a vertex v denote its site means by $m_k^{\mathrm{f},s}(v)$ and $m_k^{+,s}(v)$, and write m_v^{+} for the zero-field plus mean.

Lemma 7.3 (Interior field sheet)**.** *At every regular t, every $s > 0$ and every vertex v,*

$$m_k^{+,s}(v) - m_k^{\mathrm{f},s}(v) \longrightarrow 0 \qquad (k \to \infty). \tag{7.2}$$

Proof. Below the onset this follows from zero-field uniqueness and GHS monotonicity of the difference in s. Above it, choose j distant sheet sites of one type whose plus mean is $m > 0$,

all far from v. The plus state clusters: the conditional mean at a site given the exterior of a radius-k box is bounded by its finite plus mean and has expectation m_v^+. Its L^1 deviation from m_v^+ is at most twice the excess of that finite mean, which tends to zero. Apply this successively to separated spins. Tilt the free state by fields s at the selected sites. Equation (7.1) and clustering give the limiting mean

$$m_v^+ \frac{(\cosh s + m \sinh s)^j - (\cosh s - m \sinh s)^j}{(\cosh s + m \sinh s)^j + (\cosh s - m \sinh s)^j}.$$

It is bounded by the free full-sheet mean. Letting $j \to \infty$ shows $m^{\mathrm{f},s}(v) \geq m_v^+$. The plus-sheet mean tends to m_v^+ as $s \downarrow 0$, by squeezing with every fixed plus box. At finite volume GHS makes $m_k^{+,s} - m_k^{\mathrm{f},s}$ nonincreasing in s: its derivative is a sum of field derivatives of a magnetization, each decreasing when the nonnegative boundary fields are raised. Hard pins follow by limits. The nonnegative limiting gap tends to zero at $s = 0+$, hence is zero for every $s > 0$, proving (7.2). □

Lemma 7.4 (Surface-order arm bound)**.** *Let $t > \beta_{\mathrm{c}}$ be regular. There are $a, b > 0$ such that for all large L and all integers $1 \leq l \leq aL/2$,*

$$\phi^{\mathrm{f}}_{B_{5L},t}[B_l(0) \text{ has no physical arm to cell distance } \lfloor aL \rfloor] \leq e^{-bl^{r-1}}. \tag{7.3}$$

The same bound holds for the FK law at t on any domain containing B_{5L}, with any boundary partition.

Proof. Take $s_0 \in (\beta_{\mathrm{c}}, t)$, an aspect ratio $\kappa \in (0, 1/4)$ fixed below, and the rectangle $\mathcal{Q} = [-L, L]^{r-1} \times [-\bar{L}, \bar{L}]$, $\bar{L} = \lfloor \kappa L \rfloor$. Pin its whole boundary first plus, then with opposite signs below and above height zero. Include physical edges between pinned spins. Write the partition sums $Z_+, Z_\pm$. Every plus mean at $\beta \in [s_0, t]$ is at least the positive minimum $m_*(s_0)$ over types. Equation (2.3) gives each edge-energy difference at least $J_{xy} u_x u_y |v_x/u_x - v_y/u_y|$ for the plus and mixed means u, v.

There are $c_0 L^{r-1}$ edge-disjoint bottom-to-top paths, with c_0 independent of κ: concatenate a fixed path from one vertex to its one-cell vertical translate, trim it between faces, erase loops and translate it into disjoint spaced columns. Along each path the endpoint ratios are $-1, +1$. Integrating the resulting derivative of $\log(Z_+/Z_\pm)$ from s_0 to t gives $Z_\pm/Z_+ \leq e^{-c_1 L^{r-1}}$. Freeing one lateral pinned spin changes a partition sum by a factor between 1 and $1 + e^{2tD_J}$, where D_J bounds its row. There are $O((\bar{L}+1)L^{r-2})$ such spins. Choose κ sufficiently small. With only the end faces pinned plus, the probability of no physical end-to-end crossing is therefore at most $e^{-c_1 L^{r-1}/2}$.

Embed $\mathcal{Q}$ in the free cube B_{2L}. Let $\mathcal{D}$ be no crossing of its middle rectangle between heights $\pm\lfloor \bar{L}/2 \rfloor$, omitting the end-plate edges from this event. With the two original plates hard pinned, its probability has that same exponential bound by volume comparison. Replace hard pins by field s on the plate set P_L. For $p_s = 1 - e^{-2s}$, direct differentiation gives

$$\frac{\mathrm{d}}{\mathrm{d}s} \log \mathbb{P}_s(\mathcal{D}) = \frac{2}{p_s} \sum_{v \in P_L} [\mathbb{P}_s(e_{vg} = 1 \mid \mathcal{D}) - \mathbb{P}_s(e_{vg} = 1)].$$

For plate sites at least $\sqrt{L}$ from the rim, condition outside a radius-$\lfloor \sqrt{L} \rfloor$ cube. It sees a full field sheet, and $\mathcal{D}$ is measurable outside it. The term is bounded by the plus/free sheet gap of Lemma 7.3. The rim fraction vanishes. For every site the single-edge rule bounds the absolute term by $1 - \tanh s$, which is integrable on $(0, \infty)$. Dominated convergence proves that removing the fields costs $o(L^{r-1})$ in log probability. Thus the middle crossing fails with probability at most $e^{-c_2 L^{r-1}}$ under the free law.

Cover its middle section by at most $C(L/l)^{r-1}$ radius-l cubes. If all have no arm to height $\lfloor \bar{L}/2 \rfloor$, no middle crossing occurs. These failures are decreasing, so FKG and the crossing estimate show that at least one cube's arm failure is at most $e^{-(c_2/C)l^{r-1}}$. Enlarge to the radius-$5L$ cube about that center and translate. Choosing $a < \kappa/2$ proves (7.3). The event in (7.3) is decreasing, so domain and partition comparison give the last assertion. □

7.3 Paths with few closed edges and good blocks

The cost of a path in a sampled FK configuration is its number of closed edges.

Lemma 7.5 (Paths with few closed edges)**.** *Let $t > \beta_c$ be regular. For every $C \geq 1$ and $\eta > 0$ there are $c, C_1 > 0$ and l_0 such that the following holds for all $l \geq l_0$. Let x, y be vertices whose cells are at distance at most Cl, and consider the FK law at t on any domain containing $B_{(C+10)l}(x)$, with any boundary partition. The probability that no path from x to y inside $B_{(C+10)l}(x)$ has cost at most ηl is at most $C_1 e^{-cl^2}$.*

Proof. Reference walks. Fix a reference type. Using connectivity, choose finite paths that change type within a cell and that join reference vertices of adjacent cells, and translate them. This gives a constant A such that each such move takes at most A edges inside the A-neighbourhood of the move. Vertices whose cells are at distance u are then joined by a deterministic walk of length at most $A(ru + 2)$ near their coordinate rectangle. Fix $0 < \zeta < \min(1, \eta/r)$, so that $r\zeta l + 2A \leq \eta l$ for large l. The route from x to y consists of two type changes and at most r coordinate segments between reference vertices. By a union bound and loop erasure it therefore suffices to prove a segment estimate: a coordinate segment of length at most Cl is crossed by a walk of cost less than ζl within distance $2l$ of the segment, except with probability at most $e^{-c'l^2}$, under any boundary partition. All these walks lie in $B_{(C+10)l}(x)$ for large l.

A fixed local estimate. Put $C' = C + 1$ and choose $0 < \alpha < \min(1/2, \zeta/(16AC'))$ and $0 < \rho < \alpha/4$. By Lemma 7.2 choose k such that every radius-k cube meets the unique infinite cluster with probability greater than $1 - \rho/4$. Then choose K with $2A(rk + 2)/K < \zeta/(16C')$. For a coordinate segment of K steps, both radius-k cubes about its endpoints meet the infinite cluster with probability greater than $1 - \rho/2$, and uniqueness then joins them by a finite open path. Free finite-volume laws increase to the infinite free law, and connection inside a box increases with the box. Hence there is M such that, under the free law of the radius-M box about the segment, the two endpoint cubes are joined inside that box with probability greater than $1 - \rho$; enlarge M to contain the deterministic access walks. This connection event is increasing, so its probability stays above $1 - \rho$ under conditioning on all edges outside the box, with any partition. On this event the segment endpoints are joined inside the box at cost at most $2A(rk + 2)$. Otherwise the deterministic walk of length at most AK is used.

One row. Let a coordinate segment have length $T \leq Cl$. If $T \leq \zeta l/(4A)$, its deterministic walk costs less than $\zeta l/2$. Otherwise cut it into $N = \lfloor T/K \rfloor$ consecutive K-step pieces and a remainder costing at most AK. Split the pieces into a fixed number q_0 of residue classes so that test boxes in one class are disjoint with gaps of two cells. Within one class of m pieces, condition successively on all edges outside each test box. Each failure then has conditional probability at most ρ, so the number F of failures satisfies $\mathbb{E}e^{\lambda F} \leq (1 - \rho + \rho e^{\lambda})^m$. Since $\rho < \alpha$, some $\lambda > 0$ has $\log(1 - \rho + \rho e^{\lambda}) < \lambda\alpha$, and then $\mathbb{P}(F \geq \alpha m) \leq e^{-c''m}$. Every nonempty class has $m = N/q_0 + O(1)$ pieces, and N is of order l. Hence, outside an event of probability at most $e^{-c_0 l}$, at most αN pieces fail, and the row costs at most

$$N \cdot 2A(rk + 2) + \alpha N \cdot AK + AK < \frac{\zeta l}{16} + \frac{\zeta l}{16} + AK < \frac{\zeta l}{2}$$

for large l, using $N \leq Cl/K$ and the choices of K and α. This bound holds conditionally on all edges outside the union of the test boxes, which we call the tube of the row.

Parallel tubes. Since $r \geq 2$, there is a coordinate direction transverse to the segment. Choose $0 < b_0 < \min(1, \zeta/(8A))$. Translates of the tube by transverse displacements at most $b_0 l$, separated by gaps of two cells, give at least $c_1 l$ disjoint tubes for large l, because the tube width is fixed. Conditioning on all edges outside one tube at a time, each tube fails with conditional probability at most $e^{-c_0 l}$. All of them therefore fail with probability at most $e^{-c_0 c_1 l^2}$. A successful tube is reached from both ends of the original segment by deterministic walks of total cost at most $2Ab_0 l + O(1) < \zeta l/4 + O(1)$. This proves the segment estimate with cost less than $3\zeta l/4 + O(1) < \zeta l$; all walks lie within $2l$ of the segment for large l. □

Independently sprinkle each physical edge with probability $\epsilon \in (0,1)$, writing γ for the sprinkled edges and ω for FK at t.

Lemma 7.6 (Good blocks)**.** *Let $t > \beta_c$ be regular and $\epsilon \in (0,1)$. Define $G_n(x)$ by two finite conditions:*

1. *In $B_{8n}(x)$ an ω-path joins $B_n(x)$ to the radius-$8n$ sphere.*
2. *All ω-clusters of the induced $B_{4n}(x)$ meeting $B_{2n}(x)$ and its radius-$4n$ sphere belong to one $\omega \cup \gamma$ component in $B_{4n}(x)$.*

There is a fixed integer $D > 9$ for which

$$\inf_{\xi} \mathbb{P}^{\xi}_{B_{Dn},t,\epsilon}(G_n(0)) \longrightarrow 1, \tag{7.4}$$

the infimum being over all boundary partitions ξ of $B_{Dn}(0)$.

Proof. Two likely properties of ω. Let a, b be as in Lemma 7.4, and put $L = \lceil 20n/a \rceil$ and $l = \lceil A_0 \log n \rceil$ with $bA_0 > r + 3$. Choose $D > 9$ so large that, for every cell $y \in B_{8n}(0)$, the cubes $B_{5L}(y)$ and $B_{16l}(y)$ lie in $B_{Dn}(0)$. Let E_1 be the event that for every cell $y \in B_{8n}(0)$ some vertex of $B_l(y)$ has an ω-path to cell distance at least $19n$ from y. By Lemma 7.4, translation, $l^{r-1} \geq l$ and a union bound over $O(n^r)$ cells, $\mathbb{P}(E_1^c) \leq O(n^r) n^{-bA_0} \to 0$, uniformly in ξ. Choose $\eta > 0$ with $\eta A_0 \log(1/\epsilon) \leq 1/4$, so that $\epsilon^{\eta l} \geq n^{-1/3}$ for large n. Let E_2 be the event that every pair of vertices at cell distance at most $5l$, the first in $B_{4n}(0)$, is joined by a path of cost at most ηl inside the radius-$16l$ cube about the first. Lemma 7.5 with $C = 5$ and a union bound over $O(n^r l^r)$ pairs give $\mathbb{P}(E_2^c) \leq O(n^r l^r) e^{-cl^2} \to 0$, uniformly in ξ. On E_1, the case $y = 0$ gives condition 1: stop the path at its first visit to the radius-$8n$ sphere, and note $l \leq n$.

Layers. Fix $\omega \in E_1 \cap E_2$; from now on only γ is random. Call original clusters the ω-clusters of the graph induced on $B_{4n}(0)$ that meet its radius-$4n$ sphere. Put $W = \lceil 100l \rceil$, $R_i = 3n - iW$ and $T = \lfloor n/W \rfloor$, so $R_T \geq 2n$ and $T \asymp n/\log n$. Layer $i < T$ consists of the physical edges whose two endpoints have cell radius strictly between R_{i+1} and R_i; its sprinkles are exposed at step i. Before step i, two original clusters are equivalent if they are joined inside $B_{4n}(0)$ by ω together with the sprinkles of the earlier layers. A class is active at step i if it contains an original cluster meeting $B_{R_i}(0)$; let N_i be the number of active classes, so $N_0 \leq |B_{4n}|$.

Participants and links. Let $R_i' = \lfloor (R_i + R_{i+1})/2 \rfloor$. A class participates at step i if it contains an original cluster with a vertex within cell distance l of a cell of the radius-R_i' sphere. An original cluster meeting $B_{R_{i+1}}$ also reaches the radius-$4n$ sphere, and an edge changes cell radius by at most one, so it has a vertex on the radius-R_i' sphere. Thus every class active at step $i+1$ comes from a participant. For each cell y of the radius-R_i' sphere, E_1 supplies a vertex within l of y with an ω-path leaving B_{4n}; stopping that path at its first visit to the radius-$4n$ sphere shows that the vertex lies in an original cluster. Select one such vertex for each sphere cell. For $r \geq 2$ the cell sphere is connected by coordinate steps, since each face is a grid and adjacent faces meet. Selected vertices of adjacent sphere cells are within cell distance $2l + 1$, and each participant has a vertex within $2l$ of a selected vertex. These pairs

form a connected graph on the k participants. Fix a spanning tree by a deterministic rule. For each of its $k-1$ edges, E_2 provides a path of cost at most ηl inside the radius-$16l$ cube about one endpoint, again chosen by a fixed rule. All these paths lie strictly inside layer i, since their vertices are within $17l+1 < W/2$ of the radius-R_i' sphere.

Expected number of classes. A tree edge succeeds if γ opens every closed edge of its path. Given ω and the earlier layers, this has probability at least $\epsilon^{\eta l} \geq p := n^{-1/3}$; the successes may be dependent. A set of S_i successful tree edges is a forest, so it merges the k participant classes into at most $k - S_i$ classes. Hence $N_{i+1} \leq k - S_i$. Since $\mathbb{E}[S_i \mid \omega, \text{earlier layers}] \geq p(k-1)_+$ and $k \leq N_i$,

$$\mathbb{E}[(N_{i+1}-1)_+ \mid \omega, \text{earlier layers}] \leq (1-p)(k-1)_+ \leq (1-p)(N_i-1)_+.$$

Iterating, $\mathbb{P}(N_T \geq 2) \leq \mathbb{E}(N_T - 1)_+ \leq |B_{4n}|(1-n^{-1/3})^T \to 0$. Every original cluster meeting B_{2n} meets B_{R_T}, so it is active at the end. With probability tending to one all of them therefore lie in a single $\omega \cup \gamma$ component of $B_{4n}(0)$, which is condition 2. Together with the bounds on E_1 and E_2 this proves (7.4). □

Lemma 7.7 (Sprinkling domination)**.** *Let $0 < t < u$ and*

$$0 < \epsilon < \tfrac{1}{2} \min_e [p_e(u) - p_e(t)]. \tag{7.5}$$

On every finite graph with any boundary partition, FK at t joined with independent Bernoulli(ϵ) sprinkles is stochastically dominated by FK at u with the same partition.

Proof. For an edge marginal a conditional on a prefix, its derivative in its own parameter is $a(1-a)/[p(1-p)] \geq 1/2$, since $p/(2-p) \leq a \leq p$. Other parameter increases and ordered prefixes increase this marginal. Sequentially sample the lower edge and an independent Bernoulli(ϵ) mark; their union probability is $a + (1-a)\epsilon \leq a + \epsilon$, within the higher conditional marginal. Complete it with a further uniform draw. Induction proves domination with independent sprinkles. This argument applies to arbitrary partitions and unequal edge weights. □

7.4 The two finite certificates

At a candidate upper bound $u > 0$, enumerate $0 < t < u$, $\epsilon > 0$, integers $D > 9, n \geq 1$, and accept if (7.5) and

$$\min_{\xi} \mathbb{P}^{\xi}_{B_{Dn},t,\epsilon}(G_n(0)) > 1 - 2^{-40(2D+3)^2} \tag{7.6}$$

hold. The minimum is over every partition of the full outermost cell shell.

Soundness uses an elementary percolation bound.

Lemma 7.8 (A planar counting bound)**.** *Let $q \geq 1$ and $\delta = 2^{-40q^2}$. Suppose each site of $\mathbb{Z}^2$ is bad with conditional probability at most δ given any statuses of the other sites in its residue class modulo q. Then with probability greater than $1/2$ the origin is good and lies in an infinite nearest-neighbour good cluster.*

Proof. Any specified j sites contain at least j/q^2 sites of one residue class, so they are all bad with probability at most δ^{j/q^2}, by iterating the conditional bound. Suppose the origin is good but its cluster is finite. Orient the unit squares centred on the cluster counterclockwise and cancel common edges. The remaining edges split into edge-disjoint simple cycles, and since their total winding number about the origin is one, some edge-simple dual cycle of length $m \geq 4$ surrounds the origin. Each of its edges has a bad neighbour outside the cluster, and at least $m/4$ of these neighbours are distinct. Such a cycle has a vertex within distance $m+1$ of

the origin, so there are at most $2(2m+5)^2 4^m$ oriented candidates. The failure probability is therefore at most

$$\delta + 2\sum_{m\ge 4}(2m+5)^2[4\delta^{1/(4q^2)}]^m < \frac{1}{2},$$

using $4\delta^{1/(4q^2)} = 1/256$ and $(2m+5)^2 \le 16^m$ for $m \ge 4$. □

For soundness put coarse sites at $n(i,j,0,\dots,0)$, call a coarse site x good if $G_n(x)$ holds, and put $q = 2D+3$. Large cubes of distinct sites in one residue class modulo q are disjoint. Conditional on all other statuses of that class, the bad probability is at most $\delta = 2^{-40q^2}$ by (7.6): condition first on the exterior edges and sprinkles of the chosen cube, which induces some FK partition and independent interior marks. Lemma 7.8 gives an infinite good cluster with positive probability. Neighbouring good sites glue their condition-1 paths by condition 2 in the shared radius-$4n$ box. It follows that $\omega \cup \gamma$ has an unbounded physical cluster. Lemma 7.7 gives order at u, so $\beta_c \le u$.

For completeness take a regular $t_0 \in (\beta_c, u)$, then ϵ strictly satisfying (7.5). Lemma 7.6 fixes D and then gives n passing (7.6). Its finite minimum is continuous, so a nearby parameter on any dense computable temperature grid also passes. This proves upper-certificate completeness. It also proves finiteness for a rank-$r \ge 2$ component: choose a fixed large finite box in which the all-open configuration satisfies G_n by the reference paths, then take t large. Uniformly over partitions the all-open probability tends to one. Small β is disordered by bounded-degree path counting and $p_{\max}(\beta) \to 0$.

For a candidate lower bound $s \ge 0$ let $\mathcal{A}_n$ mean that some vertex of $B_n(0)$ is joined by an open path inside B_{3n} to the radius-$3n$ sphere. Accept if

$$\phi^{\mathrm{w}}_{B_{4n},s}(\mathcal{A}_n) < (2\cdot 3^r)^{-10^r} =: \delta_-. \tag{7.7}$$

Tile by side-n cells. Any visited tile of an infinite physical cluster has its translated $\mathcal{A}_n$. Conditioning outside the radius-$4n$ cube bounds its probability by the wired value. Along a self-avoiding maximum-norm-neighbour path of j tiles, select a residue class modulo 10 with at least $j/10^r$ indices. Exterior conditioning bounds the joint probability by $\delta_-^{j/10^r}$. There are at most $(3^r)^j$ such paths. The product tends to zero, so acceptance excludes wired infinite arms and implies $s \le \beta_c$.

Completeness of this lower test uses Proposition 7.1. The component is a connected, infinite, locally finite graph on which the cell translations act with finitely many orbits. By the ghost representation of Section 2, the plus mean of a vertex is the probability under the wired law that it lies in an infinite cluster, so the onset β_c of the component is its wired FK onset $\beta_c(G)$. Let $s < \beta_c$. Since one edge moves at most one cell, an open path from a vertex $v \in B_n(0)$ to cell distance $3n$ reaches the graph sphere $S_n(v)$, and the graph ball $\Lambda_n(v)$ lies in $B_{2n} \subset B_{4n}$. Wired domain comparison, Proposition 7.1 and a union bound over the $O(n^r)$ vertices of $B_n(0)$ give $\phi^{\mathrm{w}}_{B_{4n},s}(\mathcal{A}_n) \le O(n^r)e^{-c_s n} \to 0$.

Every test is finite. If Q is a common denominator of $J_e = k_e/Q$, use rational $y = e^{-2\beta/Q} \in (0,1)$, so $p_e = 1 - y^{k_e}$. With rational y, ϵ, both finite probabilities and the minimum over partitions are rational; clearing denominators verifies their strict tests exactly. The grid is dense and the preceding continuity arguments apply. Dovetail accepted lower and upper bounds and convert their y endpoints to outward rational β enclosures. Their strict-open-side completeness yields a closed bracket of every requested width, including at rational critical points.

7.5 Rank reduction and the Gibbs conclusions

In each connected type quotient choose a spanning tree with displacement a_i from a root to type i. The subgroup $H \le \mathbb{Z}^d$ generated by $a_i + v - a_j$ over the listed bonds is exactly

the displacement group of root-to-root walks: the generators have those tree-bond-tree realizations, and conversely displacements telescope along any walk. A lifted component is $\{(a_i + h, i) : h \in H\}$; all components of that type are translates. Integer row and column operations compute a basis and rank r of H. In that basis, and after finite blocking, the component has the presentation used above.

Rank zero is finite. In rank one there are at most A bonds crossing any layer cut, so its conditional all-closed probability is at least $(1 - p_{\max})^A > 0$ at every finite β. Iterating over spaced disjoint cuts makes an infinite physical arm impossible even in the wired limit. Thus these ranks never order. If all component types have rank at most one, report infinity; otherwise compute the finite values of the remaining finitely many types and take their minimum. Positive readout weights make that exactly the onset of M.

Every Gibbs measure is sandwiched between μ^- and μ^+. Below the minimum onset their one-site means all agree at zero, so an ordered coupling has zero disagreement probability at each vertex and their full laws coincide. Above it one component orders and $\mu^+(M) > 0$. Translation invariance, spin flip and strict positivity of the weights give the stated opposite periodic states. This completes the proof of Theorem J. □

8 A finite first moment

The one-spin long-range corollary uses transitive sharpness on its lower side and the upper construction of Theorem J.

Corollary 8.1 (Finite-first-moment kernels)**.** *Put one Ising spin at every site of $\mathbb{Z}^d$, with nonnegative inversion-symmetric pair kernel J, computable entries and a supplied ℓ^1 tail modulus. Assume $\sum_z |z|_1 J(z) < \infty$. For $J_R = J\mathbf{1}_{|z|_1 \le R}$,*

$$\beta_c(J_R) \downarrow \beta_c(J).$$

For $d \ge 2$, β_c is uniformly computable on the promise that it is finite; without that promise $1/(1 + \beta_c)$ is uniformly computable, with value zero at infinity. Finiteness holds exactly when the support generates a subgroup of rank at least two. In $d = 1$ the onset is infinite. A modulus for the first moment is unnecessary.

Proof. We first give the finite path inequality. In any finite pair ferromagnet with nonnegative fields suppose all unfixed means are at least $q > 0$, permitting exterior spins fixed plus. For a route $z_0 = x, \dots, z_k = y$, write $f_j = \langle \sigma_x \sigma_{z_j} \rangle$, $g_j = \langle \sigma_{z_{j-1}} \sigma_{z_j} \rangle$, and a dot for a derivative in an additional field at v. Differentiating (2.3) between the systems before and after that field increment gives

$$\dot{g}_j \ge |f_{j-1}\dot{f}_j - f_j \dot{f}_{j-1}|.$$

Since $f_j \ge q^2$, $f_0 = 1$, and $\dot{f}_0 = 0$, divide by $f_{j-1} f_j$ and sum the differences of $\dot{f}_j / f_j$. Interchange a field and a pair derivative in the finite partition sum. For dimensionless coefficients K_e this yields

$$0 \le \partial_{K_{xy}} m_v \le q^{-4} \sum_{j=1}^{k} \partial_{K_{z_{j-1} z_j}} m_v. \tag{8.1}$$

Repeated sites and fixed vertices are allowed; a pair with one fixed endpoint is a field, and one with two is a constant.

Let H be the subgroup generated by the positive displacements. Choose a finite symmetric positive subset S generating H: take an integer basis for H, express each basis element using finitely many support elements, and collect those elements. If its basis matrix is B_H, a real

left inverse bounds the sum of the absolute basis coordinates of $z \in H$ by a constant times $|z|_1$. Choosing fixed S-routes for each signed basis element therefore gives a template for every z of length at most $A|z|_1$. Translate that template for every bond. Each occurrence of one step determines at most one translate using a fixed unoriented backbone bond, so its total routed tail load is at most AT_R, where $T_R = \sum_{|z|_1 > R} |z|_1 J(z) \to 0$. Counting both signs is an upper bound. This proves the backbone construction for arbitrary, possibly lower-rank support.

Fix $m_J(t) = m > 0$ and $u > t$. Put $q = m/2$ and $\lambda = (u - t) \min_{z \in S} J(z) > 0$. In a finite plus box interpolate by deleting tJ beyond R and adding λ to every backbone coefficient. Include its summable induced exterior fields. While all site means are at least q, (8.1) and nonnegative derivatives give

$$\frac{\mathrm{d}m_v}{\mathrm{d}s} \geq [\lambda - tq^{-4} AT_R] \sum_{e \text{ in backbone}} \partial_{K_e} m_v.$$

Choose R beyond the backbone and with the bracket positive. The initial means are at least m, and continuous finite-volume interpolation cannot have a first hit of q because all derivatives before it are nonnegative. They remain at least m. The final coefficients are bounded above by uJ_R, so comparison and the infinite-volume limit give $m_{J_R}(u) \geq m_J(t)$. This proves truncation convergence of critical points; monotonicity gives the other direction. If $\beta_c = \infty$, monotonicity already gives equality for every truncation.

For effective lower bounds use Proposition 2.2; its exterior sums need only the supplied ordinary tail. For upper bounds enumerate finite rational kernels L with $0 < L(z) < J(z)$ on their finite positive support. These strict comparisons are semidecidable even when zero entries of J are undecidable. Enumerate their finite upper certificates (7.6). Each accepted upper bound for L is one for J. If $u > \beta_c(J)$, choose $\beta_c < t < u_0 < u' < u$. Truncation gives some J_R ordered at u_0. A rational finite $L < J_R$ can be chosen so close that $u'L \geq u_0 J_R$ on its positive support; thus it is ordered at u' and eventually gives an upper bound below u. Both searches are complete on their respective open sides. They compute a finite β_c on that promise. Starting with lower bound zero and upper bound infinity and transforming by $x \mapsto 1/(1 + x)$ computes the compactified value also when $\beta_c = \infty$, since the lower bounds then diverge.

If the support has rank at least two, a finite positive rational minorant of rank at least two has finite onset by the upper construction in Section 7. The full kernel does too. If the rank is at most one, every truncation has infinite onset by its cut argument; truncation convergence proves the same for J. In dimension one this covers every support. □

9 Canonical observables and volume sequences

The saturation integral specifies the continuum excited density. A periodic grid can sample its singularity differently. We first give an exact finite identity, then describe this dependence.

Proposition 9.1 (Return-time identity)**.** *Let h be any finite symmetric hopping Laplacian with nonnegative hopping, and $A = e^{-\beta h}$. Let D_N have the canonical occupation law of a chosen constant zero mode at particle number N. Independently run the discrete-time stochastic A-chain from site x, and let S_k be its kth strictly positive return time, with $S_0 = 0$. Then*

$$\mathbb{P}_N(n_x = k) = \mathbb{P}(S_k \leq D_N < S_{k+1}). \tag{9.1}$$

Translation invariance is unnecessary for this identity.

Proof. Let $Z_N = \operatorname{Tr} \Gamma_N(A)$ and $\mathcal{Z}(z) = \sum_N Z_N z^N = \det(I - zA)^{-1}$. Removing j particles from the chosen zero mode gives $\mathbb{P}(D_N \geq j) = Z_{N-j}/Z_N$, with $Z_k = 0$ for $k < 0$. The local

multiplier t^{n_x} is $\Gamma_N(I-(1-t)|x\rangle\langle x|)$. The symmetric-power trace identity and the rank-one determinant formula give

$$\sum_N z^N Z_N \mathbb{E}_N t^{n_x} = \frac{\mathcal{Z}(z)}{1+(1-t)[G(z)-1]}, \qquad G(z) = \langle x, (I-zA)^{-1}x\rangle.$$

It follows first for positive factors by similarity with $A^{1/2}$ and diagonalization, then at zero factors by continuity. The matrix A is stochastic since $-h$ is a jump generator, and symmetric by hypothesis. If $F(z)$ is the first-positive-return generating function, renewal gives $G = 1+FG$. The ratio is therefore $(1-F)/(1-tF)$. The coefficient of t^k is $F^k - F^{k+1}$; the coefficient of z^N in $\mathcal{Z}F^k$, divided by Z_N, is $\mathbb{P}(S_k \le D_N)$. Subtract to prove (9.1). □

For example take two sites with heat matrix $A = \frac{1}{4}\left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$ and $N = 2$. The canonical vacancy is exactly $9/28$. Appendix C reproduces it in rational arithmetic by both sides of (9.1).

Theorem H (All ring subsequences)**.** *Let $d = 1$ and let the hopping satisfy Section 2, with finite saturation a. Fix $\beta > a$ and put $n = n_{\mathrm{c}}(\beta)$, $C = \rho - n > 0$. On periodized rings of size L, in the canonical state at $N = \lfloor \rho L \rfloor$, every subsequential vacancy limit belongs to*

$$\left[\frac{e^{-C/(1+n)}}{1+n}, \frac{1}{1+\rho}\right]. \tag{9.2}$$

The lower endpoint occurs along a subsequence exactly when $N_0/L \to C$ and $\max_{j\neq 0} N_j/L \to 0$ in probability. The upper endpoint occurs exactly when $\max_j N_j/L \to 0$ in probability.

For every left-c.e. $a > T^{-1}\log(1+1/\rho)$ there is one strictly positive computable hopping, of row T with an effective ℓ^1 tail and saturation a, for which the vacancy cluster set is the entire interval (9.2), simultaneously at every $\beta > a$. For that hopping and computable $\beta > a$, the limits along computable increasing ring subsequences are exactly the limit-computable reals in that interval.

Proof of the bounds and endpoint characterizations. The finite partition coefficients Z_N are log-concave. To verify this inductively when adjoining a geometric sequence of weight t, let $U_n = tU_{n-1} + u_n$ be the convolution. Log-concavity of u implies $u_n U_n \ge u_{n+1}U_{n-1}$ by summing $u_n u_{n-i} \ge u_{n+1}u_{n-i-1}$. Substitution in the recurrence proves log-concavity of U.

Thus $Q(l) = Z_{N-l}/Z_N$ has decreasing successive ratios and $Q(i+j) \le Q(i)Q(j)$. By (9.1) and independent return increments, the site tails $p_k = \mathbb{P}(n_x \ge k) = \mathbb{E}Q(S_k)$ satisfy $p_{i+j} \le p_i p_j$. Since $p_1 = 1 - v_L$,

$$\frac{N}{V} = \sum_{k\ge 1} p_k \le \frac{1-v_L}{v_L}, \qquad v_L \le (1+N/V)^{-1}, \quad V = L.$$

The mean equality uses translation invariance, and so does the finite lower bound $v_L \ge (1-1/V)^N$, which holds on these periodized rings: in each mode number vector project all creation vectors onto the complement of site x. Each plane wave retains squared norm $1-1/V$, and successive creation operators satisfy $\|(a^\dagger(b))^l\psi\|^2 \ge l!\|b\|^{2l}\|\psi\|^2$.

We need the diffuse thermal mass on arbitrary rings. For fixed $\delta > 0$, the occupation outside $|k| < \delta$, divided by V, converges in probability to $n_\delta = \int_{|k|\ge\delta} f(\beta E(k))\,\mathrm{d}k/(2\pi)$, and its maximum fraction tends to zero. Here are the large-deviation details. Tilt the independent grand-canonical modes by $z_V = e^{-1/V}$ and condition their total to equal N. The shell probability is $\exp[-o(V)]$. Indeed the quadratic neighbour lower bound controls the logarithmic singularity uniformly in both grids and integrals, so $V^{-1}\log\mathcal{Z}(z_V) \to p(1) = \int -\log(1-e^{-\beta E})$. Conversely, for fixed $z < 1$, the independent excited total concentrates at density $n(z) < \rho$

with bounded geometric moments. Restrict to that typical total and fill the zero mode; undoing the tilt gives $\liminf V^{-1}\log Z_N \geq p(z)$ (the extra nonnegative term $-n(z)\log z$ can be discarded). Send $z \uparrow 1$. The coefficient upper bound gives the reverse inequality, proving the shell estimate. Outside the fixed arc the parameters are uniformly bounded below one; exponential Markov bounds give probability e^{-cV} for a fixed density deviation, or for a mode exceeding εV. Dividing by the shell preserves convergence. Finally $n_\delta \uparrow n$.

Sort all mode fractions decreasingly. On a further subsequence they converge in distribution in the compact product space to $(a_i)_{i\geq 1}$, with $\sum_i a_i \leq \rho$. The preceding thermal bound gives diffuse remainder $d = \rho - \sum_i a_i \geq n$ almost surely. Conditional on these fractions, take independent uniform phases ζ_i and a circular Gaussian Y_d of variance d. The limiting site law is mixed Poisson with intensity

$$\left| Y_d + \sum_i \sqrt{a_i}\zeta_i \right|^2 . \tag{9.3}$$

For a direct verification, the one-site Weyl characteristic in a mode number vector is $e^{-t/2}\prod_j L_{N_j}(t/V)$, $t \geq 0$. Replacing each Laguerre factor by $J_0(2\sqrt{tN_j/V})$ costs $O_{\rho,t}(N_j/V^2)$, using $0 \leq N_j^l - (N_j)_l \leq \binom{l}{2} N_j^{l-1}$. Summing gives $O_{\rho,t}(1/V)$; other factors are bounded by $e^{t/(2V)}$ by Weyl unitarity and Bessel factors by one. After the largest k fractions, the maximum is at most $\rho/(k+1)$. The remaining product differs from its Gaussian limit by $O_{\rho,t}(1/k)$, since $J_0(2\sqrt{ta}) = 1 - ta + O_t(a^2)$. Pass to the distributional limit, then $k \to \infty$. This proves (9.3); tightness follows from the site mean. The phase series converges in mean square.

Separate a Gaussian of variance n and call the remaining amplitude X. Conditional on the fractions, $\mathbb{E}|X|^2 = C$ and $\mathrm{Var}(|X|^2) = C^2 - \sum_i a_i^2$. Integrating the variance-n Gaussian gives vacancy $(1+n)^{-1}\mathbb{E}e^{-|X|^2/(1+n)}$. Jensen yields the lower endpoint. Strict convexity gives equality exactly when $|X|^2 = C$ almost surely; then $\sum_i a_i^2 = C^2$ with $\sum_i a_i \leq C$, forcing one fraction C, the rest zero and $d = n$.

That unique macroscopic fraction must be the zero mode. Otherwise on an event of positive limiting probability an excited mode exceeds $7CV/8$ and every other mode is below $CV/8$. Move $\lfloor CV/3 \rfloor$ particles from it to the zero mode. The weight cannot decrease; the map is injective, because the image's unique macroscopic excited mode identifies the source. Its image has two macroscopic fractions, an event of vanishing probability. This contradiction proves the lower characterization and its converse through (9.3).

If the finite upper bound is approached, then $\sum_{k\geq 1}[(1-v_L)^k - p_k] = (1-v_L)/v_L - N/V \to 0$. The site law tends to the geometric law of mean ρ with uniformly geometric tails. In a number vector its second factorial moment is $[2N^2 - N - \sum_j N_j^2]/V^2$. Hence $\mathbb{E}\sum_j N_j^2/V^2 \to 0$, implying $\max_j N_j/V \to 0$ in probability. Conversely vanishing maximum makes every $a_i = 0$ in (9.3), giving that same geometric law. This proves the bounds and equivalences. □

Proof of the simultaneous realization. We use a larger family of computable centers than in Theorem B. Let $\nu(r) = r^{-3/2}/\zeta(3/2)$ be a probability on positive distances, and let Φ be its row-T band. Summing r between $1/|k|$ and $2/|k|$ gives $\Phi(k) \geq c\sqrt{|k|}$ near zero. A center is a strictly positive hopping with known such a lower bound on an effective neighborhood of zero and a positive nearest-neighbour coefficient. Its I and saturation point are computable, by this integrable bound near zero and the neighbour bound on the complement.

There are centers below every rational target above the floor. Mix a small fraction of ν with geometric distance law $g_q(r) = (1-q)q^{r-1}$. The latter's normalized band is

$$\frac{(1+q)(1-\cos k)}{(1-q)^2 + 2q(1-\cos k)} \longrightarrow 1 \quad (k \neq 0,\ q \uparrow 1).$$

First keep a positive power fraction to justify domination, then let that fraction decrease to zero. At the selected inverse temperature b the density tends to $f(bT)$, strictly below ρ. Strict search finds the initial center.

In every ball about any center J, critical values of centers vary continuously along a path to infinity. For $0 < u < 1$, exponentially damp its positive-distance probability law and renormalize: $p_u(r) = u^r p(r)/\sum_n u^n p(n)$. For rational u sufficiently close to one this is as close to p in row norm as required. Along $(1-t)p + tp_u$, $t < 1$, the known residual $\sqrt{|k|}$ lower bound gives computable integrals and local continuity in t. At $t = 1$ the second moment is finite, so the saturation integral diverges. Lower semicontinuity forces the finite saturation values to tend to infinity as $t \uparrow 1$.

The required effective lower semicontinuity holds on the fixed-row space: for rational b, $b < a$ exactly when some continuous capped integral of $f(bE)$ exceeds ρ. A strict capped-integral witness has a computable row-norm neighborhood, since the band depends uniformly on that norm. Given $l_s \uparrow a$ strictly from below, tune centers successively into (l_s, l_{s+1}) inside nested balls, then protect the witness at l_s in a closed smaller ball of radius below 2^{-s}. Preserve positivity of the first s coefficients there. The centers converge effectively in ℓ^1 to a strictly positive hopping. The protected lower witnesses give saturation at least a, while lower semicontinuity bounds it above by the liminf of the center values, namely a. This describes a fully effective calibration in any sufficiently small ball, using the path just constructed.

We now add the grid requirements. Fix a computable schedule of positive rational Δ_s visiting every open subinterval of $(0, \infty)$ infinitely often. At a tuned center of stage s, write A_R for its band through distance $R \geq 1$ and τ for the fraction of row beyond R. Since the center is strictly positive, $\tau > 0$. Consider

$$E_{R,L,\xi}(k) = (1-\xi)[A_R(k) + \tau\Phi(Lk)] + \xi\Phi(k), \qquad 0 < \xi < 1. \tag{9.4}$$

It has row T and changes the hopping norm by at most $2\tau T + 2\xi T$. It is a center and is strictly positive by the last term. On its own L-grid it samples exactly $B_{R,\xi} = (1-\xi)A_R + \xi\Phi$, since $\Phi(2\pi j) = 0$.

We prove the continuum averaging used to preserve the strict crossing bracket. At fixed R,

$$\int \frac{\mathrm{d}k}{A_R(k) + \tau\Phi(Lk)} \longrightarrow \iint \frac{\mathrm{d}k\,\mathrm{d}t}{A_R(k) + \tau\Phi(t)} \quad (L \to \infty), \tag{9.5}$$

with normalized measures. Cap the reciprocal at M. It is a continuous two-variable function, extending by M at its only singular point. The averages of any continuous $F(k, Lk)$ converge to its product average: for Fourier monomials $e^{i(mk+nt)}$ the diagonal integral vanishes unless $m + nL = 0$, and for a fixed nonconstant monomial this fails for all large L. Uniform trigonometric approximation proves the assertion. The cap error is uniform in L: it is at most the integral of $[\tau\Phi(Lk)]^{-1}$ where this exceeds M, whose distribution is independent of integer L. Its tail integral tends to zero by the integrable square-root bound. The same estimate applies to product measure, proving (9.5).

As $R \to \infty$, the right side tends to the old I. Indeed

$$\int \frac{\mathrm{d}t}{A_R(k) + \tau\Phi(t)} \leq \frac{C}{A_R(k) + \tau T}, \qquad C = \int \max(1, T/\Phi(t))\,\mathrm{d}t < \infty.$$

The old band obeys $E(k) \leq A_R(k) + 2\tau T$, so this is bounded by $2C/E(k)$, integrable at the center. Pointwise almost everywhere it tends to $1/E(k)$. Dominated convergence proves the successive limit.

Choose a small rational $\xi_{\max} > 0$. For all $0 < \xi \leq \xi_{\max}$ the reciprocal integral of (9.4) is at most $(1 - \xi_{\max})^{-1}$ times the left side of (9.5). Its band is uniformly close to the old band when $\xi_{\max}, \tau$ are small. The regular part in (4.2) is uniformly continuous under this

perturbation. Thus choosing $\xi_{\max}$ small, then R large and L large, preserves the strict upper test at l_{s+1} uniformly over $0 < \xi \le \xi_{\max}$. The protected ball preserves the lower test at l_s.

Also choose $\xi_{\max}$ small enough that

$$\int [(1-\xi_{\max})E + \xi_{\max}\Phi]^{-1} < I_s + \Delta_s/4,$$

where I_s is the old center's reciprocal integral. This is possible by domination by $2/E$ as $\xi_{\max} \downarrow 0$. For fixed $\xi_{\max}$, increasing R makes $\int 1/B_{R,\xi_{\max}}$ tend to this value, dominated by $1/(\xi_{\max}\Phi)$. In contrast, for fixed R, $\int 1/B_{R,\xi} \to \infty$ as $\xi \downarrow 0$, because $A_R(k) = O(k^2)$. On the positive ξ interval the integral is computable and continuous. Rational $\xi \le \xi_{\max}$ can therefore make it arbitrarily close to $I_s + \Delta_s$. At that fixed R, ξ, the square-root bound gives convergence of its reciprocal grid sums and maximum reciprocal divided by L tending to zero.

Consequently one can find an arbitrarily large special integer grid L_s, and a center (9.4), inside the prescribed ball, with crossing in (l_s, l_{s+1}) and

$$\left| \frac{1}{L_s} \sum_{j \neq 0} \frac{1}{E_{R,L_s,\xi}(2\pi j/L_s)} - (I_s + \Delta_s) \right| < 2^{-s-1}, \qquad \max_{j \neq 0} \frac{1}{L_s E_{R,L_s,\xi}(2\pi j/L_s)} < 2^{-s-1}. \quad (9.6)$$

All choices are effective without a rate for the successive limits: enumerate rational ξ, integers R, L and center calibration parameters, and test the strict norm, crossing and finite-grid inequalities. Each candidate is a center with computable integrals; the preceding existence argument proves that the search terminates. Shrink the new closed ball so these finite tests persist with tolerance 2^{-s} against the same fixed computable number I_s. Include all earlier ball constraints. Take successive L_s strictly increasing. This completes the effective construction of one hopping.

The old and new center bands tend uniformly to the final band, and their crossings tend to a. Equation (4.3) therefore gives $I_s \to I$ even without an effective rate. On any selected stages with $\Delta_s \to \Delta < \infty$, the final hopping's grid tests and the continuous regular part imply, at every fixed $\beta > a$,

$$\frac{1}{L_s} \sum_{j \neq 0} f(\beta E_j) \to n + \Delta/\beta, \qquad \max_{j \neq 0} f(\beta E_j)/L_s \to 0.$$

The first term uses the protected reciprocal sum; the regular part has ordinary Riemann convergence. The canonical argument for slabs below applies to any finite limiting excited density and vanishing normalized maximum. Thus the zero-mode density is $(\rho - n - \Delta/\beta)_+$ and the vacancy is (4.7) at $n_* = \min(\rho, n + \Delta/\beta)$. The schedule realizes every $\Delta \ge 0$, including zero as a limit of positive values. This fills (9.2); its bounds already exclude all other answers. The same reciprocal tests work simultaneously for every $\beta > a$.

For computable $\beta > a$, n and both endpoints are limit-computable by (4.2). A desired limit-computable vacancy in the interval has a limit-computable $n_* \in [n, \rho]$, using the computable continuous inverse of (4.7), and hence a nonnegative limit-computable $\Delta = \beta(n_* - n)$. Let computable rationals $d_j \to \Delta$ be given. Select successively later scheduled stages with $|\Delta_s - \max(0, d_j)| < 2^{-j}$. The corresponding increasing grid subsequence is computable and has the desired limit. Conversely finite-ring expectations are uniformly computable from the hopping and β by finite diagonalization or finite-sector partition sums. Every convergent computable real sequence has a limit computable relative to $0'$. This proves the exact effective subsequence classification. □

Theorem I (Slab onsets for one fixed gas)**.** *Fix rational nearest-neighbour hopping $J > 0$ on $\mathbb{Z}^3$ and rational density $\rho > 0$. In periodic boxes (L, L, H), assume $L \ge H \to \infty$ and*

$\log L/H \to \ell \in [0,\infty)$. *At* $N = \lfloor \rho L^2 H \rfloor$, *put*

$$n_* = \min\left(\rho, n_{\mathrm{c}}(\beta) + \frac{\ell}{2\pi\beta J}\right).$$

The zero-mode density tends in probability and mean to $\rho - n_*$, *and the canonical vacancy tends to* $(1+n_*)^{-1}e^{-(\rho-n_*)/(1+n_*)}$. *Among computable nested exhaustions of this form, the finite inverse temperatures of macroscopic zero-mode onset are exactly the limit-computable reals* $a' \geq a_0$, *where* $n_{\mathrm{c}}(a_0) = \rho$ *is the computable cubic onset of this fixed gas.*

Proof. The band is $E(k) = 2J\sum_{i=1}^{3}(1-\cos k_i)$. In the plane $k_3 = 0$, $f(\beta E(k',0)) = (\beta J|k'|^2)^{-1} + O_{\beta,J}(1)$ uniformly off the origin. Comparing unit squares about nonzero integer points with radial integration gives $\sum_{0<|j|_\infty \leq L/2} |j|^{-2} = 2\pi \log L + O(1)$: the local errors are $O(|j|^{-3})$, summable in dimension two, and changing circular to square cutoffs costs a bounded amount. The plane contributes, per volume,

$$\frac{\log L}{2\pi\beta JH} + O(1/H).$$

For $k_3 \neq 0$ the grid sum tends to the ordinary three-dimensional $n_{\mathrm{c}}(\beta)$. To check uniform integrability, points with three active coordinates are bounded by the adjacent inward rectangular cell integral of $C|k|^{-2}$, giving $O(\delta)$ near zero. The two-active-coordinate planes containing the short axis cost $O((1+\log H)/L) = o(1)$ by summing first along the longer coordinate. Its isolated line costs $O(H/L^2) = o(1)$. These cover all points outside the zero plane. The largest nonzero mean is $O(L^2) = o(L^2H)$. Thus the excited mean density tends to $d_\ell = n_{\mathrm{c}}(\beta) + \ell/(2\pi\beta J)$ and its normalized variance vanishes.

For $\rho > d_\ell$, conditioning has probability tending to one, proving the ground fraction. At every density, the excited maximum remains submacroscopic after conditioning on the decreasing event that their total is at most N: the product law associates an increasing event negatively with that decreasing event. Its unconditional probability tends to zero by the second-moment bound. For $\rho \leq d_\ell$, the zero occupation increases stochastically with total particle number. Indeed $\mathbb{P}_N(N_0 \geq k) = Z_{N-k}/Z_N$ increases with N by log-concavity of Z. Compare with density $d_\ell + \varepsilon$, whose zero fraction tends to ε, and let $\varepsilon \downarrow 0$. The factorial-moment calculation for (4.8) now gives the stated vacancy on both sides, with n_* as its diffuse density.

The onset a_ℓ is the unique root of $n_{\mathrm{c}}(a_\ell) + \ell/(2\pi a_\ell J) = \rho$; it is at least a_0. A computable exhaustion makes ℓ limit-computable, and strict root bracketing computes a_ℓ from it. Conversely, for a finite limit-computable target $a' \geq a_0$, set

$$\ell = 2\pi J a'[\rho - n_{\mathrm{c}}(a')].$$

This is nonnegative and limit-computable, since n_{c} is a computable continuous function in three dimensions. Choose computable rational $e_s \to \ell$, clip it at zero, choose $H_s \geq L_{s-1}$ increasing fast enough that $\log H_s/H_s \to 0$, and let $L_s = H_s 2^{m_s}$ with a nonnegative integer m_s within two of $H_s e_s/\log 2$. Fixed absolute-precision approximation finds such an integer without deciding equality. Then the boxes are nested and $\log L_s/H_s \to \ell$. They realize exactly a'.

The distinction between ordinary saturation and a second zero-mode onset in exponentially anisotropic boxes has classical precedents, including Beau–Zagrebnov [23]. The coefficient and canonical law here use the periodic lattice normalization just computed. □

10 Discussion

For the classes proved here, a computable finite-volume description and a computable limiting threshold are different properties. On the Ising line, sharpness always enumerates the lower cut.

The realizations in Theorem A prevent the opposite approximation from having a computable rate. Finite range in periodic models supplies an ordered-side certificate; with subcritical sharpness on quasi-transitive graphs it gives Theorem J's terminating algorithm. For one-spin kernels a finite first moment supplies the extension in Corollary 8.1. These statements concern ferromagnetic pair interactions.

The inverse-square case separates a tail coefficient from an effective approach to it. The exact critical jump fixes a relation between the coefficient and the spontaneous mean. Lower certificates approaching the coefficient make the critical point computable. At the same rational row and coefficient, Theorem F permits every c.e. degree when the approach has no supplied rate. Comparable powers strictly below two instead have a continuous computable magnetization under the stated lower-power data. On the half-line the critical point is the full-line one (Proposition 5.5); whether a half-line site has positive plus mean at the critical point itself is not decided here. The remaining value classification at fixed (T, B) without effective tail certificates, and tails with slowly varying corrections such as $r^{-2} \log r$, are natural questions. The long-range higher-dimensional problem with infinite first moment also remains open.

For bosons, the exceptional one- and two-dimensional hoppings place information in a singular inverse-band integral. The hoppings constructed for Theorem B undergo canonical single-mode condensation at their saturation points, as does every gas of the class in dimension at least three, and the specified monotone one-dimensional rings have the limiting local law of Theorem C. General ring subsequences can give the range in Theorem H. The fixed nearest-neighbour slab example places the information in the exhaustion itself: its continuum saturation point is computable while its zero-mode onset along that exhaustion can be any limit-computable value above it.

These mechanisms distinguish sources of missing convergence information. The thresholds belong to infinite-volume models with specially chosen tails or volume sequences. It remains useful to ask which analogous classifications hold for other interactions, including higher-state random-cluster models in dimension at least three, where an ordered-side certificate at every supercritical parameter is a separate issue.

How this work was done

The assistance named on the first page was used in developing and checking the mathematical arguments. The programs described in Appendix C check the displayed finite constants and identities in exact rational arithmetic; the infinite-volume statements rest on the proofs in the text. The authors take responsibility for the proofs, their scope and the use of the cited results.

A Proof of Proposition 7.1

The proof follows Duminil-Copin–Raoufi–Tassion's proof for transitive graphs [4, Section 3]. Two of their results hold on arbitrary finite sets and are used as stated.

1. *OSSS inequality for monotonic measures* [4, Theorem 1.1]. Let μ be a monotonic measure on $\{0,1\}^E$, E finite, let $f : \{0,1\}^E \to [0,1]$ be increasing, and let T be a decision tree. Then $\mathrm{Var}_\mu(f) \le \sum_{e \in E} \delta_e(f,T)\, \mathrm{Cov}_\mu(f, \omega_e)$, where the revealment $\delta_e(f,T)$ is the probability that T queries e no later than the first time at which the queried values determine f. Finite FK measures of cluster weight 2 with parameters in $(0,1)$, free or wired, are monotonic.
2. *Analytic lemma* [4, Lemma 3.1]. Let $f_n : [0, \beta_0] \to [0, M]$ be nondecreasing differentiable functions converging pointwise to f, with $f_0 > 0$ and $f_n' \ge n f_n / \sum_{k<n} f_k$ for $n \ge 1$. Then

there is $\beta_1 \in [0, \beta_0]$ such that, for each $\beta < \beta_1$, $f_n(\beta) \le e^{-c_\beta n}$ for some $c_\beta > 0$ and all large n, and $f(\beta) \ge \beta - \beta_1$ for $\beta > \beta_1$.

We also use the FK facts of Section 2: comparison of boundary conditions, monotonicity in the parameters, and conditional opening probabilities between $p_e/(2-p_e)$ and p_e. Distances are those of G, connections use open edges, and the event that u is joined to $S_l(u)$ depends only on edges with both endpoints in $\Lambda_l(u)$, by stopping a path at its first visit to $S_l(u)$. Since the vertex orbits are finitely many and G is locally finite, $W = \max_x \sum_{e \ni x} J_e$ is finite.

Lemma A.1 (Exploration)**.** *Let $o \in V$ and $n \ge 1$. Let μ be a monotonic measure on the edges of a finite subgraph of G containing every edge with both endpoints in $\Lambda_n(o)$, and let f be the indicator that o is joined to $S_n(o)$ by an open path with all edges in $\Lambda_n(o)$. If μ satisfies the FKG inequality, then*

$$\sum_e \mathrm{Cov}_\mu(f, \omega_e) \ge \frac{n\,\mu(f)(1-\mu(f))}{4\Theta}, \qquad \Theta = \max_{u \in \Lambda_n(o)} \sum_{l=0}^{n-1} \mu(u \leftrightarrow S_l(u)),$$

where the $l = 0$ term equals 1.

Proof. For $0 \le k \le n-1$ define a decision tree T_k. Fix an ordering of the edges with both endpoints in $\Lambda_n(o)$ and start with the set $\mathcal{D} = S_k(o)$. While some unqueried edge of $\Lambda_n(o)$ has at least one endpoint in $\mathcal{D}$, query the first such edge; if it is open, add both endpoints to $\mathcal{D}$. Then query the remaining edges in a fixed order. At the end of the first stage, $\mathcal{D}$ is the union of the open clusters of $\Lambda_n(o)$ that meet $S_k(o)$, and every edge of $\Lambda_n(o)$ with an endpoint in $\mathcal{D}$ has been queried, including the edges with both endpoints in $\mathcal{D}$. The decision trees printed in [4, p. 90] and [5, proof of Lemma 2.11] need this standard modification: their first stage queries only edges with exactly one endpoint in the discovered set. Along an edge the distance to o changes by at most one, so every path from o to $S_n(o)$ meets $S_k(o)$. Hence $f = 0$ if $o \notin \mathcal{D}$, and otherwise f is read from the known open subgraph on $\mathcal{D}$: the first stage determines f. Every vertex of $\mathcal{D}$ is joined to $S_k(o)$ by queried open edges, so an edge uv queried before f is determined has an endpoint joined to $S_k(o)$, and

$$\delta_{uv}(f, T_k) \le \mu(u \leftrightarrow S_k(o)) + \mu(v \leftrightarrow S_k(o)).$$

Let $d = d(o,u) \le n$. A path from u to $S_k(o)$ reaches distance at least $|k-d|$ from u, so it joins u to $S_{|k-d|}(u)$. As k runs over $0, \dots, n-1$, each value $l \ge 1$ of $|k-d|$ occurs at most twice. If $d \le n-1$ all values lie in $\{0, \dots, n-1\}$. If $d = n$ they are $1, \dots, n$, and since $\mu(u \leftrightarrow S_l(u))$ is nonincreasing in l their sum is at most the sum over $0, \dots, n-1$. Thus $\sum_k \mu(u \leftrightarrow S_k(o)) \le 2\Theta$ and $\sum_k \delta_e(f, T_k) \le 4\Theta$ for every edge e. The covariances are nonnegative by FKG. Summing the OSSS inequality over k gives $n\mu(f)(1-\mu(f)) \le 4\Theta \sum_e \mathrm{Cov}_\mu(f, \omega_e)$. □

Proof of Proposition 7.1. The critical point does not depend on the vertex. If a path π joins x to y, FKG and the conditional opening probabilities give $\phi^{\mathrm{w}}_\beta(x \leftrightarrow \infty) \ge \prod_{e \in \pi} [p_e/(2-p_e)]\, \phi^{\mathrm{w}}_\beta(y \leftrightarrow \infty)$.

Orbit sums. Choose representatives $o_1, \dots, o_s$ of the vertex orbits. For $n \ge 1$ put

$$\theta^a_n(\beta) = \phi^{\mathrm{w}}_{\Lambda_{2n}(o_a),\beta}[o_a \leftrightarrow S_n(o_a)], \qquad \theta^a_0 = 1, \qquad H_n = \sum_{a=1}^{s} \theta^a_n, \qquad A_n = \sum_{k<n} H_k.$$

Each θ^a_n is a ratio of finite sums, hence differentiable, and nondecreasing in β.

Comparison across orbits. Fix $\beta_0 > 0$ and $0 < \beta \le \beta_0$, and apply Lemma A.1 to $\mu = \phi^{\mathrm{w}}_{\Lambda_{2n}(o_a),\beta}$ and $o = o_a$, so that $\mu(f) = \theta^a_n$. Let $u \in \Lambda_n(o_a)$. Since $l \mapsto \mu(u \leftrightarrow S_l(u))$ is nonincreasing, pairing l with $\lfloor l/2 \rfloor$ gives $\sum_{l<n} \mu(u \leftrightarrow S_l(u)) \le 2 \sum_{l \le (n-1)/2} \mu(u \leftrightarrow S_l(u))$. For $l \le (n-1)/2$ the ball $\Lambda_{2l}(u)$ lies in $\Lambda_{2n-1}(o_a)$. The event $u \leftrightarrow S_l(u)$ is increasing

and depends only on edges of $\Lambda_l(u)$, so comparison of boundary conditions bounds its μ-probability by its probability under $\phi^{\mathrm{w}}_{\Lambda_{2l}(u),\beta}$. A weight-preserving automorphism carrying u to the representative o_b of its orbit preserves distances, so this probability is $\theta^b_l \le H_l$. Hence $\Theta \le 2A_n$, and Lemma A.1 gives

$$\sum_e \mathrm{Cov}(f,\omega_e) \ge \frac{n\,\theta^a_n(1-\theta^a_n)}{8A_n}.$$

Differential inequality. The finite FK weight is proportional to $2^{k(\omega)}\prod_e (e^{2\beta J_e}-1)^{\omega_e}$, where k counts clusters after the boundary identification. Differentiation gives

$$(\theta^a_n)' = \sum_e \frac{2J_e}{1-e^{-2\beta J_e}}\,\mathrm{Cov}(f,\omega_e) \ge \frac{1}{\beta_0}\sum_e \mathrm{Cov}(f,\omega_e),$$

since $1-e^{-y}\le y$. For $n\ge 1$, closing every edge at o_a prevents o_a from reaching $S_n(o_a)$; each such edge is closed with conditional probability at least $1-p_e = e^{-2\beta J_e}$, so $1-\theta^a_n \ge \kappa := e^{-2\beta_0 W}$. Combining and summing over a, with the same A_n in every term,

$$H_n' \ge c\,\frac{nH_n}{A_n}, \qquad c=\frac{\kappa}{8\beta_0}, \qquad n\ge 1,$$

on $(0,\beta_0]$. It also holds at $\beta=0$, where $H_n=0$ and $H_n'\ge 0$ for $n\ge 1$.

Limits. Comparison of boundary conditions shows that θ^a_n is nonincreasing in n and at least $\theta^a = \phi^{\mathrm{w}}_\beta(o_a \leftrightarrow \infty)$. For fixed $k\le n$, $\theta^a_n \le \phi^{\mathrm{w}}_{\Lambda_{2n}(o_a),\beta}[o_a \leftrightarrow S_k(o_a)]$, which tends to $\phi^{\mathrm{w}}_\beta[o_a \leftrightarrow S_k(o_a)]$ as $n\to\infty$; this decreases to θ^a as $k\to\infty$. Hence $H_n \to H = \sum_a \theta^a$.

Conclusion. Let $\beta<\beta_{\mathrm{c}}(G)$ and choose $\beta_0\in(\beta,\beta_{\mathrm{c}}(G))$, or any $\beta_0>\beta$ if $\beta_{\mathrm{c}}(G)=\infty$. Apply the analytic lemma to $f_n = H_n/c$ on $[0,\beta_0]$, with $f_0 = s/c$ and $M = s/c$. Since $H=0$ on $[0,\beta_0]$, its second conclusion forces $\beta_1=\beta_0$. Thus $H_n(\beta)\le e^{-c'n}$ for some $c'>0$ and all large n, and $\theta^a_n(\beta)\le H_n(\beta)$ for each a. For x in the orbit of o_a and $N\ge 2$ put $n=\lfloor N/2\rfloor$. Then $\Lambda_{2n}(x)\subset\Lambda_N(x)$, joining x to $S_N(x)$ requires joining it to $S_n(x)$, and comparison of boundary conditions and the automorphism give

$$\phi^{\mathrm{w}}_{\Lambda_N(x),\beta}[x\leftrightarrow S_N(x)] \le \phi^{\mathrm{w}}_{\Lambda_{2n}(x),\beta}[x\leftrightarrow S_n(x)] = \theta^a_n(\beta).$$

For the finitely many remaining radii the probability is at most $1-\kappa<1$, by the closed-edge bound. Decreasing the constant proves the proposition, uniformly in x. □

If $\beta_0>\beta_{\mathrm{c}}(G)$, then $\beta_1=\beta_{\mathrm{c}}(G)$: exponential decay below β_1 forces $\beta_1\le\beta_{\mathrm{c}}(G)$, and $H=0$ below $\beta_{\mathrm{c}}(G)$ forces $\beta_1\ge\beta_{\mathrm{c}}(G)$. The analytic lemma then yields $\sum_a \phi^{\mathrm{w}}_\beta(o_a\leftrightarrow\infty)\ge c(\beta-\beta_{\mathrm{c}}(G))$ for $\beta\in(\beta_{\mathrm{c}}(G),\beta_0]$, the quasi-transitive form of the mean-field lower bound in [4, Theorem 1.2]. Theorem J does not use it.

B A certified bracket for the pure inverse-square chain

This appendix proves the following bracket for $J_r = r^{-2}$, in the conventions of Section 2: energy $-\sum_{\{x,y\}} J(|x-y|)\sigma_x\sigma_y$ with each unordered pair counted once, dimensionless field $h\sum_x\sigma_x$, and $\beta_{\mathrm{c}} = \inf\{\beta>0 : m(\beta)>0\}$.

Theorem B.1 (Certified bracket; computer-assisted, verified by two independently written programs)**.** *For $J_r = r^{-2}$,*

$$\frac{631}{1000} < \beta_{\mathrm{c}} < \frac{11544}{7225} = 1.59778\ldots, \qquad m\Big(\frac{11544}{7225}\Big) \ge \frac{17}{20}. \tag{B.1}$$

The lower number certifies the absence of order, and the upper number certifies order. Both reduce to finitely many strict inequalities between explicit rationals. Two programs, written independently and using different arithmetic models, check each of them (Section B.4). Every other step is proved below. Throughout, $\beta_- = 631/1000$ and $\beta_+ = 11544/7225$.

B.1 Imported results

Two external theorems enter, stated as imports with exact locators. The remaining inputs are recalled or proved in the text.

Proposition B.2 (Refereed jump import)**.** *For FK percolation on $\mathbb{Z}$ with cluster weight $q \geq 1$, edge probabilities $p_{ij} = 1 - \exp(-\beta'|i-j|^{-2})$ for $|i-j| \geq 2$ and $p_{i,i+1} = 1 - e^{-\lambda}$, and percolation density $\theta(q, \beta', \lambda)$ of the wired infinite-volume measure: if $\theta > 0$ then $\beta'\theta^2 \geq 1$.*

This is Duminil-Copin–Garban–Tassion [12, arXiv v2, Theorem 4(ii)], which they attribute to Aizenman–Chayes–Chayes–Newman [10, Theorem 3.1, (1.6), Section 1.1.2(c)]. By the FK representation recalled in Section 2, the Ising chain at β is the case $q = 2$, with edge parameter $1 - e^{-2\beta J_e}$, so $\beta' = \lambda = 2\beta$. The wired density is $m(\beta)$. Hence

$$m(\beta) > 0 \implies 2\beta\, m(\beta)^2 \geq 1, \tag{B.2}$$

which is the case $B = 1$ of the import used in Theorem D. The convention-free form is $\lim_r r^2 p_{0r} \cdot \theta^2 \geq 1$; here $r^2 p_{0r} \to 2\beta$.

Proposition B.3 (Refereed GHS import)**.** *For a finite Ising system with nonnegative pair couplings and nonnegative fields, $\mathrm{Cov}(\sigma_z, \sigma_i\sigma_j) \leq \langle\sigma_i\rangle \mathrm{Cov}(\sigma_z, \sigma_j) + \langle\sigma_j\rangle \mathrm{Cov}(\sigma_z, \sigma_i)$. Also $\mathrm{Cov}(\sigma_z, \sigma_i) \geq 0$ and $\langle\sigma_i\rangle \geq 0$. These are Lebowitz [8, (1.5)–(1.8)]; the displayed inequality is the expanded third-cumulant inequality.*

We also use the finite Edwards–Sokal and FK comparison facts recalled at the end of Section 2, and two results proved in the text:

- Lemma 2.3;
- the positive-field ceiling proved at the end of the proof of Lemma 5.1: for the free measure on an interval I and the average X_I of its spins, $\limsup_{|I|\to\infty} \mathbb{E}X_I^2 \leq m(\beta)^2$.

B.2 The lower bound: absence of order at β_-

For a translation-invariant kernel $G \geq 0$, write $p_G(h)$ for its pressure at the fixed β as a function of the field.

Lemma B.4 (GHS comparison)**.** *Let $0 \leq G \leq J$ be summable and translation invariant. Let $\tau \geq 2\beta \sum_{r\geq 1}(J_r - G_r)$ and $0 < u < 1$. If*

$$[p_G(\tau u + s_0) - p_G(\tau u)]/s_0 < u$$

for some $s_0 > 0$, then $m(\beta) \leq u$. If moreover $2\beta u^2 < 1$, then $m(\beta) = 0$.

Proof. This is the comparison in the proof of Theorem E; here are the details we need.

Choice of ε. By convexity, for $0 < \varepsilon < s_0$ at which p_G is differentiable at $\tau u + \varepsilon$,

$$p_G'(\tau u + \varepsilon) \leq [p_G(\tau u + s_0) - p_G(\tau u + \varepsilon)]/(s_0 - \varepsilon).$$

The right side tends to the assumed secant as $\varepsilon \downarrow 0$. So for all small ε outside a countable set, $p_G'(\tau u + \varepsilon) < u$ and p_J is differentiable at ε.

Torus setup. On the torus $\mathbb{Z}/L\mathbb{Z}$ use the periodized couplings $G^{(L)}(x) = \sum_{n\in\mathbb{Z}} G_{|x+nL|}$ and similarly $J^{(L)}$. The self-images at $x \equiv 0$ are constants. Comparing with free boxes changes $\log Z$ by at most $\beta \sum_{x\in[0,L)} \sum_{z\notin[0,L)} J(|z-x|) = o(L)$, as in the proof of Lemma 2.3. So the torus pressures converge to p_G and p_J. By translation invariance the torus mean is the derivative of the torus pressure, hence converges at differentiability points.

Interpolation. Interpolate $G_s = G^{(L)} + s(J^{(L)} - G^{(L)})$ and $h_s = \varepsilon + (1-s)\tau u$, and let M_s be the common mean and $\chi_s = \sum_i \operatorname{Cov}(\sigma_z, \sigma_i) \geq 0$. Then

$$M_s' = \sum_{\{i,j\}} \beta (J^{(L)} - G^{(L)})_{ij} \operatorname{Cov}(\sigma_z, \sigma_i \sigma_j) - \tau u \chi_s.$$

Apply Proposition B.3. Every row of $J^{(L)} - G^{(L)}$ off the diagonal sums to at most $2\sum_r (J_r - G_r)$. Hence

$$M_s' \leq \tau (M_s - u) \chi_s,$$

At $s = 0$ the torus system is G at field $\tau u + \varepsilon$, so $M_0 \to p_G'(\tau u + \varepsilon) < u$, and $M_0 < u$ for large L. With $\Phi(s) = \tau \int_0^s \chi_t \, \mathrm{d}t$, the function $(M_s - u)e^{-\Phi(s)}$ is nonincreasing, so $M_1 - u \leq (M_0 - u)e^{\Phi(1)} < 0$. At $s = 1$ the torus system is J at field ε, so Lemma 2.3 gives $\mu^{+,\varepsilon}(\sigma_0) = p_J'(\varepsilon) \leq u$. Letting $\varepsilon \downarrow 0$ gives $m(\beta) \leq u$. Finally, if $2\beta u^2 < 1$, then (B.2) excludes $m(\beta) > 0$. □

Lemma B.5 (A two-exponential minorant)**.** *Let $x_{1,2} = 3 \mp \sqrt{3}$, $w_{1,2} = (1 \pm 1/\sqrt{3})/2$ and $\kappa > 0$. For every $r > 0$,*

$$\sum_{i=1}^{2} \frac{w_i}{\kappa^2} e^{x_i} e^{-r x_i/\kappa} \leq \frac{1}{r^2}. \tag{B.3}$$

Consequently, take $R \geq 1$ and numbers $0 < q_i \leq e^{-x_i/\kappa}$ and $0 < a_i \leq w_i e^{x_i} q_i^{R+1}/\kappa^2$. Then the kernel

$$G_r = r^{-2} \ (r \leq R), \qquad G_r = a_1 q_1^{r-R-1} + a_2 q_2^{r-R-1} \ (r > R)$$

satisfies $0 \leq G \leq J$ at every distance.

Proof. The two-node rule. The weights are positive and integrate the moments of orders 0 to 3 of $xe^{-x}\,\mathrm{d}x$ on $[0, \infty)$, namely $1, 2, 6, 24$. The mass is $w_1 + w_2 = 1$ and the mean is $3 - \sqrt{3}(w_1 - w_2) = 2$. The identity $x_i^2 = 6x_i - 6$ gives the second and third moments.

Interpolation error. Let $f(x) = e^{(1-r/\kappa)x}$ and let P be its cubic Hermite interpolant, matching f and f' at both nodes. For x off the nodes, apply Rolle's theorem repeatedly to $f - P - c\,(x - x_1)^2(x - x_2)^2$. This gives

$$f(x) - P(x) = f''''(\xi)(x - x_1)^2(x - x_2)^2/24 \geq 0,$$

because $f'''' = (1 - r/\kappa)^4 f \geq 0$.

Conclusion. Integrating against xe^{-x} and using exactness on cubics gives

$$\sum_i w_i f(x_i) \leq \int_0^\infty x e^{-rx/\kappa} \, \mathrm{d}x = \kappa^2/r^2,$$

which is (B.3). For $r > R$ the hypotheses give $a_i q_i^{r-R-1} \leq w_i e^{x_i} q_i^r/\kappa^2 \leq w_i e^{x_i} e^{-r x_i/\kappa}/\kappa^2$. Summing over i proves $G_r \leq r^{-2}$. □

Transfer representation of the reference pressure. Fix G as in Lemma B.5 and put

$$b_i = \frac{\beta a_i}{1 - q_i}, \qquad B = b_1 + b_2, \qquad A_i = \frac{4 b_i}{1 - q_i}, \qquad H_R = \sum_{r \leq R} r^{-2}.$$

A state is a pair (η, y). Here $\eta \in \{0,1\}^R$ records the last R spins as $\sigma_r = 2\eta_r - 1$, with $r = 1$ the newest. The pair $y = (y_1, y_2) \in [0,1]^2$ is defined by $y_i = (1 + x_i)/2$, where $x_i = (1 - q_i)\sum_{k \geq 0} q_i^k \sigma_{R+1+k}$ is the weighted older history.

One step. Appending a spin $e = \pm 1$ drops the oldest bit $d = \eta_R$ into the history and gives $y_i' = q_i y_i + (1 - q_i)d$. The tail contributes $\beta e \sum_{r > R} G_r \sigma_r = \sum_i b_i e(2y_i - 1)$ to the exponent.

Conjugating by the bounded gauge $\prod_i \exp(2b_i y_i/(1-q_i))$ turns the factor $\exp(b_i e(2y_i-1))$ into $\exp(-b_i e - 2b_i d)\, e^{s_{i,e} y_i}$, with $s_{i,+} = 4b_i$ and $s_{i,-} = 0$; this is direct algebra. Subtract the field-independent constant βH_R per step.

The operator. Define the positive operator

$$(T_h f)(\eta, y) = \sum_{e=\pm 1} W_h(\eta, e)\, e^{s_{1,e} y_1 + s_{2,e} y_2}\, f\big(\eta',\, q_1 y_1 + (1-q_1)d,\, q_2 y_2 + (1-q_2)d\big),$$

$$W_h(\eta, e) = \exp\Big(eh + \beta e \sum_{r \le R} r^{-2}\sigma_r - \beta H_R - eB - 2Bd\Big),$$

where η' is η shifted by one with the new bit $(1+e)/2$ in front.

Relation to the pressure. Start from the all-plus history. The interaction of n new spins with a fixed history is at most $\beta \sum_r r G_r < \infty$, and the gauge ratio is bounded. Hence $(T_h^n 1)^{1/n} \to \rho_T(h)$ with $\log \rho_T(h) = p_G(h) - \beta H_R$. The constant cancels in pressure differences.

Lemma B.6 (Finite matrix)**.** *Let $\mathcal{K}$ be the cone of power series $f(\eta, \cdot)$ with nonnegative coefficients satisfying $\partial_i f \le A_i f$ coefficientwise. Let Π delete the monomials $y_1^j y_2^k$ with $j > P_1$ or $k > P_2$, and suppose $A_i < P_i + 2$. Put*

$$\delta = \sum_{i=1}^{2} \frac{A_i^{P_i+1}}{(P_i+1)!\,(1 - A_i/(P_i+2))} < 1, \qquad S_h = \Pi T_h.$$

Then $\rho(S_h) \le \rho_T(h) \le \rho(S_h)/(1-\delta)$, where $\rho(S_h)$ is the Perron root of the finite nonnegative matrix S_h. In the basis $y_1^j y_2^k\, A_1^j A_2^k/(j!k!)$ the entries of S_h are $W_h(\eta, e) C^{(1)}_{j'j} C^{(2)}_{k'k}$, with

$$C_{ij} = \frac{i!}{j!} A^{j-i}\, [y^i]\, e^{sy} (qy + (1-q)d)^j \ge 0. \tag{B.4}$$

Proof. T_h preserves $\mathcal{K}$. The substitution $y_i \mapsto q_i y_i + c$ with $c \ge 0$ maps the coefficientwise bound A_i to $q_i A_i$. Multiplication by $e^{s_{i,e} y_i}$ adds at most $4b_i = A_i(1-q_i)$. Positive sums preserve the cone.

Truncation error. Deletion preserves $\mathcal{K}$. For $f \in \mathcal{K}$ the coefficients satisfy $a_{jk} \le a_{0k} A_1^j / j!$, so on $[0,1]^2$ the deleted y_1-tail is at most the $i = 1$ term of δ times $\sum_k a_{0k} y_2^k \le f$. The second coordinate is symmetric. Hence

$$(1-\delta) T_h f \le S_h f \le T_h f \quad \text{for } f \in \mathcal{K},$$

and by induction $(1-\delta)^n T_h^n 1 \le S_h^n 1 \le T_h^n 1$.

Primitivity. For $(d, e) = (1, +)$ the matrix (B.4) is the product of the lower-triangular matrix of multiplication by e^{sy}, with entries $\binom{i}{k}(1-q)^{i-k}$, and the upper-triangular matrix of the substitution, with entries $q^i (A(1-q))^{j-i}/(j-i)!$. The leftmost column of the former and the top row of the latter are positive, so the product is strictly positive. All the matrices have positive diagonals. So S_h^{2R+1} is strictly positive, $(S_h^n 1)^{1/n} \to \rho(S_h)$ at every point of $[0,1]^2$, and the two bounds follow. □

The test. For any positive vector v,

$$\min_x (S_h v)_x / v_x \le \rho(S_h) \le \max_x (S_h v)_x / v_x.$$

If $L_0 \le \rho(S_H)$ and $U_1 \ge \rho(S_{H+s_0})$, then by Lemma B.6,

$$\frac{p_G(H+s_0) - p_G(H)}{s_0} \le \frac{1}{s_0} \log \frac{U_1}{(1-\delta) L_0}. \tag{B.5}$$

Proposition B.7 (Lower certificate)**.** *At $\beta = \beta_-$ the hypotheses of Lemma B.4 hold with $u = 222541/250000$. Hence $m(\beta_-) = 0$ and $\beta_c > \beta_-$.*

Proof. Data. Take $R = 16$, $\kappa = 40$, $(P_1, P_2) = (20, 10)$, $s_0 = 1/20000$ and

$$\begin{aligned} q_1 &= 122809788365170259142635083642 1 \cdot 2^{-100}, \\ q_2 &= 112621673787270679199735141883 3 \cdot 2^{-100}, \\ a_1 &= 129541568564927173982053116 5 \cdot 2^{-100}, \\ a_2 &= 127199303292555728018994968 9 \cdot 2^{-99}. \end{aligned}$$

Minorant. Both programs verify the hypotheses of Lemma B.5 with outward enclosures of $\sqrt{3}$, e^{x_i} and $e^{-x_i/\kappa}$. Implementation A also checks $G_r \le r^{-2}$ exactly from $r = 17$ to the largest individual peak $r = 63$, beyond which every term is nonincreasing; Implementation B checks it to $r = 400$.

Field. Put $\bar{\zeta} = \sum_{r \le 8192} \lceil 2^{90} r^{-2} \rceil 2^{-90} + 1/8192 \ge \zeta(2)$, by the integral test. Set $\bar{M} = \bar{\zeta} - H_{16} - \sum_i a_i/(1 - q_i)$, $\tau = 2\beta\bar{M}$ and $H = \tau u$. Then $\tau \ge 2\beta \sum_r (J_r - G_r)$.

Truncation. The exact truncation constant satisfies $\delta < 1.85 \cdot 10^{-11}$.

Collatz bounds. With trial vectors that carry no proof burden, the outward Collatz bounds give

$$\begin{aligned} L_0 &= 4810294620658127 \cdot 2^{-52} = 1.0680999686170\ldots, \\ U_1 &= 2405254308990383 \cdot 2^{-51} = 1.0681474855679\ldots. \end{aligned}$$

Comparison. The exact comparison

$$\frac{U_1}{(1-\delta)L_0} = 1.00004448738137\ldots < \sum_{k=0}^{24} \frac{(us_0)^k}{k!} = 1.00004450919050\ldots$$

holds, and the right side is at most e^{us_0}. So the secant (B.5) is below u; it is $0.88973\ldots$. The margin is $2.18 \cdot 10^{-8}$. Moreover

$$1 - 2\beta u^2 = 42594289/31250000000000 > 0.$$

Strictness. All these inequalities are strict and continuous in β at fixed a_i, q_i and trial vectors. They therefore persist at some $\beta > \beta_-$, and Lemma B.4 gives $\beta_c > \beta_-$. □

B.3 The upper bound: order at β_+

We work in the FK representation with cluster weight 2 and $p_{xy} = 1 - e^{-2\beta|x-y|^{-2}}$. A block of $2K$ consecutive sites is *good at density* γ' if its internal FK graph has a cluster with at least $2K\gamma'$ sites. Let $p(K, \gamma')$ be the probability that a block is not good, under its free measure. Free interiors are dominated by every induced boundary partition (Section 2), so the same bound holds conditionally on anything outside the block.

Lemma B.8 (Overlap recursion)**.** *Let $3/4 < \gamma \le \gamma' \le \gamma_0 < 1$ and $s = 2\beta\gamma^2 > 1$. Let $K \ge K_{\min} = 4096$, and let $D \ge n_{\min} = 100$, $C \ge 2$ and R_* be integers with*

$$R_* \ge \max\left(2\gamma_0,\ \gamma_0(D+1)/2\right).$$

Put

$$F = \frac{2[\gamma(6-4\gamma) + 5/K_{\min}]}{\gamma(1 - 4/n_{\min}) - 1/(K_{\min} n_{\min})}, \qquad \varepsilon > \frac{4F^s (D-1)^{1-s}}{s-1}. \tag{B.6}$$

Then

$$p(CK, \gamma' - R_*/C) \le \varepsilon\, p(K, \gamma') + 2C^2 p(K, \gamma')^2. \tag{B.7}$$

Proof. The parent is $[-CK, CK) \cap \mathbb{Z}$. Its children are $B_i = [K(i-1), K(i+1))$ for $|i| \le C-1$, so neighbouring children overlap in K sites. Write $p = p(K, \gamma')$. The parent is good at density $\gamma' - R_*/C$ as soon as it has a cluster of at least $2CK\gamma' - 2KR_*$ sites.

Joining. The large cluster of a good child misses at most $2K(1-\gamma')$ of its sites. So it meets the K-site overlap with a neighbour in at least $K - 2K(1-\gamma')$ sites. Two such sets in the overlap intersect because $2[K - 2K(1-\gamma')] > K$, that is, $\gamma' > 3/4$. Hence along any run of consecutive good children, the large clusters lie in one cluster of the parent. Its size is at least $2K\gamma'$ times the number of pairwise disjoint children in the run.

Cases. If all children are good, the C disjoint children $-C+1, -C+3, \dots, C-1$ give a cluster of at least $2CK\gamma'$ sites. If two disjoint children ($|i-j| \ge 2$) are bad, condition on all edges except the internal edges of one of them. The conditional law of those edges dominates the free law, and badness is decreasing, so the probability is at most p^2. There are fewer than $2C^2$ such pairs.

Otherwise there are bad children, and they all lie in some $\{i, i+1\}$. Then the event E_i holds: B_i is bad and every B_j with $|j-i| \ge 2$ is good. Its probability is at most p. By reflection take $i \ge 0$ and put $n = C - i$. The left run $\{-C+1, \dots, i-2\}$ has $n_L = C + i - 2 \ge n - 2$ children and the right run $\{i+2, \dots, C-1\}$ has $n_R = n - 2$. Let U_L, U_R be the unions of their large clusters. They are determined by edges of length less than $2K$. Every edge joining $U_L \subset [-CK, K(i-1))$ to $U_R \subset [K(i+1), CK)$ is longer than $2K$ and is internal to no child.

Short distance to the end, $n \le D-1$. The alternate children $-C+1, -C+3, \dots$ of the left run number $\lceil n_L/2 \rceil \ge n_L/2$. They give

$$|U_L| \ge K\gamma' n_L = 2CK\gamma' - K\gamma'(n+2) \ge 2CK\gamma' - 2KR_*,$$

because $R_* \ge \gamma_0(D+1)/2 \ge \gamma'(n+2)/2$. So the parent is good, and E_i contributes nothing.

Long distance, $n \ge D$. If some edge joining U_L to U_R is open, the joined cluster has at least $K\gamma'(n_L + n_R) = 2CK\gamma' - 4K\gamma'$ sites. This suffices since $R_* \ge 2\gamma_0$. Hence the parent can fail on E_i only if every such edge is closed.

No bridge. Fix possible values $U_L = \mathcal{A}$, $U_R = \mathcal{B}$, and put

$$W = \sum_{x \in \mathcal{A},\, y \in \mathcal{B}} 2\beta |x-y|^{-2}.$$

The event $E_i \cap \{U_L = \mathcal{A}, U_R = \mathcal{B}\}$ depends only on internal edges of children. Condition on all edges not joining $\mathcal{A}$ to $\mathcal{B}$; both sets are internally connected. If they are already joined, all joining edges are closed with probability e^{-W}. If not, opening any nonempty set of them merges two clusters, and the probability is $2e^{-W}/(1+e^{-W}) \le 2e^{-W}$.

Lower bound on W. Let $g = \lceil 2K\gamma' \rceil$ and $A = \lfloor K\gamma'(n-4) \rfloor$. On the right use the disjoint children $i+2, i+4, \dots$. There are $\lceil n_R/2 \rceil$ of them, holding at least $K\gamma'(n-2) \ge A$ sites of U_R; list these sites as $y_1 < y_2 < \cdots$.

If y_b lies in the $(t'+1)$-st of these children, the earlier ones contain at least $t'g$ smaller listed sites, so $t' + 1 \le \lceil b/g \rceil$. Inside its child, y_b has rank $b' \le b - t'g$ among at least g sites, hence offset at most $2K - g + b' - 1$. Therefore

$$y_b - K(i+1) \le \lceil b/g \rceil (2K - g) + b - 1 \le 2Kb/g + 2K - g - 1.$$

The same holds on the left for the sites $x_1 > x_2 > \cdots$ of the children $i-2, i-4, \dots$, measured from $K(i-1) - 1$. Adding the gap of $2K+1$ and using $g \ge 2K\gamma'$ gives, for $1 \le a, b \le A$,

$$\gamma'(y_b - x_a) \le a + b + h_0, \qquad h_0 = K(6\gamma' - 4\gamma'^2) - \gamma'.$$

Since W contains the terms with $1 \le a, b \le A$, it is at least $2\beta\gamma'^2 \sum_{a,b=1}^{A} (a+b+h_0)^{-2}$. Put $h = K(6\gamma' - 4\gamma'^2) + 5 \ge h_0 + 2$. Since $(a+b+h_0)^{-2}$ dominates $(x+y+h_0+2)^{-2}$ on

$[a-1,a]\times[b-1,b]$,

$$W \geq 2\beta\gamma'^2 \int_0^A\int_0^A \frac{\mathrm{d}x\,\mathrm{d}y}{(x+y+h)^2} = 2\beta\gamma'^2 \log\frac{(h+A)^2}{h(h+2A)} \geq 2\beta\gamma'^2 \log\frac{A}{2h}.$$

The last step is $2(h+A)^2 - A(h+2A) = 2h^2+3hA \geq 0$.

Now $A \geq K\gamma'(n-4)-1$ and $h \leq K[\gamma'(6-4\gamma')+5/K]$. Also $\gamma'(6-4\gamma')$ decreases on $[3/4,1]$, $\gamma' \geq \gamma$, $n \geq n_{\min}$ and $K \geq K_{\min}$. So $A/(2h) \geq n/F$. When $n \geq F$ this gives $W \geq s\log(n/F)$ for every admissible pair $(\mathcal{A},\mathcal{B})$. Summing the conditional bound over these pairs gives

$$\mathbb{P}(E_i,\ \text{parent bad}) \leq 2(F/n)^s\, p.$$

The bound is trivial when $n < F$.

Summation. There are at most two values of i for each n. Using $\sum_{n\geq D} n^{-s} \leq (D-1)^{1-s}/(s-1)$, the single-candidate failures cost at most $4F^s(D-1)^{1-s}p/(s-1) < \varepsilon p$. This proves (B.7). □

Lemma B.9 (Iteration)**.** *In the setting of Lemma B.8, let $\varrho > 1$ be rational and A_0 an integer. Put $w = \varrho + 1/A_0$ and $x_* = (w^{-2}-\varepsilon)/2$, and assume*

$$x_* > 0, \qquad R_*\varrho/(A_0(\varrho-1)) < \gamma_0 - \gamma.$$

If $p(K_0,\gamma_0) \leq x_*/A_0^2$ for some $K_0 \geq K_{\min}$, then $m(\beta) \geq \gamma$.

Proof. Scales. Let $C_j = \lceil A_0\varrho^j\rceil$, $K_{j+1} = C_jK_j$ and $\gamma_{j+1} = \gamma_j - R_*/C_j$, starting from γ_0. Then $\gamma_0 \geq \gamma_j \geq \gamma_0 - R_*\varrho/(A_0(\varrho-1)) > \gamma$. So Lemma B.8 applies at every level, and $C_{j+1}/C_j \leq w$.

Invariant. With $X_j = C_j^2 p(K_j,\gamma_j)$,

$$X_{j+1} \leq w^2X_j(\varepsilon+2X_j) \leq X_j \quad \text{whenever } X_j \leq x_*.$$

Since $X_0 \leq x_*$, we get $p(K_j,\gamma_j) \leq x_*/C_j^2 \to 0$.

Conclusion. Colouring free clusters by independent fair signs, the free spin average X of the block of $2K_j$ sites has $\mathbb{E}X^2 \geq \gamma^2(1-p(K_j,\gamma_j))$. The positive-field ceiling (Section B.1) gives $m(\beta)^2 \geq \gamma^2$. □

The seed. It remains to bound $p(K_0,\gamma_0)$ for one block of $N = 2K_0$ sites.

Split of the coupling. For an integer d_0, write $J = \phi + l$ with $\phi_r = \min(r^{-2},(d_0+1)^{-2})$, so $l_r = 0$ for $r > d_0$. For a word $\sigma \in \{\pm1\}^N$ let U_ϕ, U_l, U_J be the sums of $\phi_{y-x}, l_{y-x}, J_{y-x}$ over unlike pairs $x < y$.

Residual matrix. Let M be the nonnegative matrix on the states (a,b), $1 \leq a \leq d_0$, $0 \leq b \leq d_0$, with transitions

$$\begin{aligned} &(a,b) \to (\min(d_0,a+1),b) && \text{with weight } e^{-2\beta\sum_{a<r\leq\min(d_0,a+b)} l_r},\\ &(a,b) \to (1,a) && \text{with weight } e^{-2\beta\sum_{r\leq a} l_r}. \end{aligned}$$

Tile weights. For $1 \leq j \leq 16$ let

$$c_j = 2\Big[\sum_{r\leq j}\frac{1}{r} + j\sum_{r>j}\frac{1}{r^2}\Big].$$

Lemma B.10 (Seed bound)**.** *Let N be even, $2\gamma_0^2+\gamma_0 \geq 2$ and $\nu = \lceil(1-\gamma_0)N/2\rceil$, with $d_0 < \nu \leq (N-1)/2$. Suppose:*

- *$v > 0$ satisfies $v \geq M^{n_0}\mathbf{1}$ and $Mv \leq \lambda_+ v$, for some $n_0 < N-1$;*

- $0 < w_j \le e^{-2\beta c_j}$ *and* $\lambda_- \le 1 + \sum_{j\le 16} w_j H^{-j}$, *where* $H = 1 + \sum_{j\le 16} w_j$.

Then

$$\log p(N/2, \gamma_0) \le \log 2 - 2\beta C_\phi + \log v_{(1,0)} + (N - 1 - n_0)\log\lambda_+ - (N - 16)\log\lambda_-, \quad \text{(B.8)}$$

$$C_\phi = \log\frac{\nu+1}{d_0+1} + \frac{d_0}{2(d_0+1)} + \frac{\nu}{\nu+1} - \frac{\nu}{N-\nu+1}.$$

Proof. Colour factor. Colour the free FK clusters by independent fair signs and let X be the spin average. Suppose no cluster has mass fraction γ_0, and let $u_0 = (1-\gamma_0)/2$. Then $\mathbb{P}(|X| \le \gamma_0 \mid \omega) \ge 1/2$, by three exhaustive cases:

- two clusters of mass at least u_0 receive opposite colours with probability $1/2$;
- if the largest mass c lies in $[1/2, \gamma_0)$, the remaining signed mass opposes it with probability at least $1/2$, giving $|X| \le c$;
- otherwise $\mathbb{E}[X^2 \mid \omega] \le c^2 + u_0(1-c) \le 1/4 + u_0/2 \le \gamma_0^2/2$, using $2\gamma_0^2 + \gamma_0 \ge 2$, and Markov's inequality applies.

Since $|X| \le \gamma_0$ exactly when the minority count is at least ν, Edwards–Sokal gives $p \le 2\,\mathbb{P}(\min(\#+, \#-) \ge \nu)$. Here $\mathbb{P}$ is the free Ising law, with weights $e^{-2\beta U_J}$ after dividing by the aligned weight.

Smooth part. The internal ϕ-energy of j sites is at most that of an interval of j sites, because ordered gaps exceed rank gaps and ϕ is nonincreasing. Write $I(j) = \sum_{r<j}(j-r)\phi_r$ and $E = I(N)$. Applying the interval bound to both signs shows that U_ϕ at minority size j is at least $f(j) = E - I(j) - I(N-j)$, the value of an end segment. Moreover $f(j) = f(N-j)$, and $f(j+1) - f(j) = \sum_{r=j+1}^{N-j-1}\phi_r \ge 0$ for $j < N/2$. Counting r crossing pairs for $r \le \nu$ and ν pairs for $\nu < r \le N-\nu$, and comparing harmonic and square sums with integrals, gives $f(\nu) \ge C_\phi$. Hence the minority event has weight at most $e^{-2\beta C_\phi}\sum_\sigma e^{-2\beta U_l(\sigma)}$.

Residual part. Discard every unlike pair not joining adjacent sign runs; this increases $e^{-2\beta U_l}$. The state (a, b) holds the current and previous run lengths, capped at d_0; the caps are exact since $l_r = 0$ for $r > d_0$. Same sign and flip cost exactly the new adjacent-run pairs. So

$$\sum_\sigma e^{-2\beta U_l} \le 2(M^{N-1}\mathbf{1})_{(1,0)} \le 2\lambda_+^{N-1-n_0} v_{(1,0)}.$$

Denominator. Words ending in $+$ parse uniquely into tiles $+$ and $-^j+$. Every minus run of length j costs at most c_j, its cost in an infinite plus background. The tiled weights satisfy

$$W_n = W_{n-1} + \sum_{j\le 16} w_j W_{n-j-1}, \qquad W_0 = 1,$$

and $W_n \ge 1$ for $n \le 16$. Since $\lambda_- \le H$, induction gives $W_n \ge \lambda_-^{n-16}$. Spin reversal doubles this, so $\sum_\sigma e^{-2\beta U_J} \ge 2\lambda_-^{N-16}$.

Combining the three bounds proves (B.8). □

Proposition B.11 (Upper certificate)**.** *At* $\beta = \beta_+$, *Lemmas B.8–B.10 apply with*

$$\gamma = \tfrac{17}{20}, \quad \gamma_0 = \tfrac{91}{100}, \quad s = \tfrac{1443}{625}, \quad F = \tfrac{9057160}{1671163}, \quad D = 121, \quad R_* = 56, \quad \varrho = \tfrac{7}{5}, \quad A_0 = 3268,$$
$$\varepsilon = 2589056032828385 \cdot 2^{-53}, \quad N = 497156, \quad \nu = 22373, \quad d_0 = 88, \quad n_0 = 1056.$$

Hence $m(\beta_+) \ge 17/20$ *and* $\beta_c < \beta_+$.

Proof. Exact rational checks. The following hold:

- $R_* \ge \max(1.82, 55.51)$;
- $R_*\varrho/(A_0(\varrho-1)) = 49/817 < 3/50 = \gamma_0 - \gamma$;
- $w = 22881/16340$ and $x_* > 0$;

- $K_0 = 248578 \geq K_{\min}$, $2\gamma_0^2 + \gamma_0 \geq 2$ and $88 < \nu \leq (N-1)/2$.

The bridge constant. Condition (B.6) holds, but tightly: $\log \varepsilon$ exceeds the logarithm of the right side by only $2.5 \cdot 10^{-16}$. Implementation A compares ε^{625} exactly with the rational $(4/(s-1))^{625} F^{1443} (D-1)^{-818}$. Implementation B compares outward logarithms. Any change of constants must certify (B.6) afresh.

The seed. The outward Collatz bound gives

$$\lambda_+ = 4503753150775793 \cdot 2^{-52} = 1.0000340890438\ldots,$$
$$v_{(1,0)} = 592292193680279 \cdot 2^{-49} = 1.0521\ldots.$$

Sixteen outward tile weights give

$$\lambda_- = 126768566819840539559231720595 9 \cdot 2^{-100} = 1.0000276637506\ldots,$$

and $C_\phi > 6.97423$. Implementation A evaluates (B.8) with the weaker bounds $\log v \leq v - 1$ and $\log \lambda_+ \leq \lambda_+ - 1$, and obtains

$$\log p(K_0, \gamma_0) \leq -18.382414 < -18.379671 \leq \log(x_*/A_0^2).$$

The exact margin is greater than 0.00274. Lemma B.9 gives $m(\beta_+) \geq \gamma = 17/20$.

Strictness. The true seed probability is a finite sum of continuous functions of β. Every inequality above is strict and continuous in β at fixed constants. So they persist at some $\beta < \beta_+$, which gives $\beta_c < \beta_+$. □

Theorem B.1 follows from Propositions B.7 and B.11.

B.4 Arithmetic model and the two implementations

Implementation A (the certificate programs).

- Floating point and BLAS only propose trial vectors and constants.
- Every acceptance step uses binary64 elementary operations, each widened by one unit in the last place in the safe direction. This is valid since round-to-nearest errs by at most half a unit.
- Rationals are converted to binary64 with one unit of widening.
- Exponentials are enclosed by positive Taylor polynomials of degree 16 on arguments at most $1/2$, with remainder below 2^{-60}, followed by repeated squaring.
- Logarithms use $\log x = 2\sum_j z^{2j+1}/(2j+1)$, $z = (x-1)/(x+1)$, with a geometric remainder.
- Every final comparison is between exact rationals.

Implementation B (written independently from the inequalities above).

- All parameters and exponents are exact rationals.
- The upper side uses decimal arithmetic with directed rounding throughout, and the decimal library's correctly rounded exp and log, widened by two units. It uses its own burn length $n_0 = 1500$, a telescoping bound $\sum_{r>M} r^{-2} < (M + 1/2)^{-1}$, and exact logarithms of $v_{(1,0)}$ and $\lambda_\pm$.
- The lower side builds the matrices (B.4) twice (closed formula and explicit polynomial products) and compares them exactly.
- It evaluates the scalar weights with correctly rounded decimal exponentials.
- It computes the matrix action in binary64 round to nearest, without BLAS or directed rounding. For nonnegative data, each term of each sum passes through at most 36 roundings, counting the conversion of its two matrix entries. Its weight carries relative error at most $\omega + 2^{-53}(1+\omega)$, with $\omega < 10^{-48}$. So the computed values lie within a factor $(1+2^{-53})^{36}(1+\omega+2^{-53}(1+\omega))$ of the exact ones, and the final division adds one more rounding. These bounds are applied in exact rationals.

Margins. The two implementations reach the following margins.

	Implementation A	Implementation B
lower: e^{us_0} bound minus $U_1/((1-\delta)L_0)$	$2.18091 \cdot 10^{-8}$	$2.18092 \cdot 10^{-8}$
upper: $\log(x_*/A_0^2)$ minus seed bound	0.00274329	0.00434514
upper, with the exact end-segment cost $f(\nu)$	—	0.02577136
bridge constant: $\log \varepsilon$ minus required	exact power test	$2.5 \cdot 10^{-16}$

B.5 Reproduction

The directory `programs-bracket` contains the programs, a `README.txt` with exact commands, the expected outputs and their SHA-256 sums. The programs need Python 3.10 or later. Numpy is needed for `lower_two_tail_certificate.py`, `upper_refine_certificate.py` and `second_lower_engine.py`. From that directory run, in this order:

```
python3 lower_two_tail_certificate.py 16 20 10 0.631 0.890164 1/20000
python3 upper_refine_certificate.py 88 11544/7225 17/20 91/100
python3 check_tail_minorants.py
python3 check_upper_scalars.py
python3 check_seed_combinatorics.py
python3 second_upper_engine.py
python3 second_lower_engine.py
```

The lower certificate program implements Proposition B.7. It takes about eight minutes, peaks near 0.4 GB of memory, and writes two trial vectors of 121 MB. The upper certificate program implements Proposition B.11 in about one second.

The three checkers re-derive, in exact rational arithmetic:

- the all-distance minorant and the lower acceptance constants;
- the bootstrap, tile-weight and logarithm inequalities of the upper side;
- finite instances of the colour, minority-cost and tiling steps of Lemma B.10.

The last two commands are Implementation B. They take about ten and twenty seconds respectively. Each program prints `"success": true` and writes a JSON record.

C Reproducibility

The finite verifications use only the standard library of Python 3, version 3.2 or later. From the source directory run

```
python3 programs/verify.py
```

It finishes in well under a second.

The program uses rational arithmetic throughout. For the endpoint, take $z = 94113/100000$. A positive partial series for $\operatorname{arctanh} z$ and the integral-test upper bound $\zeta(2) \le \sum_{r=1}^{10000} r^{-2} + 1/10000$ prove $z \operatorname{arctanh} z > \zeta(2)$. Thus $z_* < z$ and $(2z^2)^{-1} > 0.5645$. For the coarse chain it checks every numerical choice in the proof of (6.2). In particular, with $s = 512/25 > 20$,

$$\frac{4\,128^s 511^{1-s}}{s-1} < \frac{4 \cdot 511}{s-1}\left(\frac{128}{511}\right)^{20} < \frac{1}{1000}.$$

A positive partial series for e^{32} proves the strict seed bound; rational inequalities check the density loss and the induction. The contour bound in the periodic upper certificate is also checked by its geometric majorant.

For the two-site heat matrix, one computation extracts the local generating polynomial from its determinant. An independent recurrence computes first-return probabilities and averages over the exact zero-mode occupation law. They agree on every site probability for $N = 1, \ldots, 8$, including vacancy $9/28$ at $N = 2$. The grand-canonical density-one vacancy is $1/2$.

Expected output ends with

```
PASS: all finite arithmetic verifications
```

These calculations verify the finite displayed constants and identities. The infinite-volume classifications, including the constructions in Theorems F and H, are proved by the arguments in the text. The analytic inverse-square bracket (6.2) uses the rational endpoint estimate and the explicit coarse chain. The certified bracket of Corollary 6.1 uses the separate programs of Appendix B; Section B.5 gives their commands, requirements and running times.

The source builds with three pdfLaTeX passes, without a bibliography processor or shell escape:

```
pdflatex -no-shell-escape -interaction=nonstopmode \
  -halt-on-error main.tex
```

Repeat that command three times. MiKTeX users may add `-disable-installer`. All figures are generated within the source.

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