The Frame Bus: Exact Amplification and Catalysis of Complexness References in Real Quantum Networks

Seth Douglas
ORCID: 0009-0007-4708-3252 apsiape@gmail.com

Draft v1 — August 2026

Abstract

We construct, for every $n \geq 2$, an n-pair real-network resource Σ_n — the *frame bus* — whose transpose-charge code is the even-parity pair-block code: it has no weight-2 codewords, so by the per-pair dichotomy of [1] it has zero unassisted pairwise reference rate under fine-partition real separable operations, for every holder pair and any number of copies. We then prove an exact amplification theorem: Σ_n together with a *single* maximal pairwise complexness reference Ω, held by the two holders of one bus pair (placement is necessary), yields $\Omega^{\otimes n}$ — one exact maximal reference on every pair, in product — deterministically, from one copy, by local real operations and one round of classical broadcast. The input reference is returned in kind, so the conversion is a strict catalytic cycle with catalytic yield $n-1$ references per copy of Σ_n. The simultaneous-activation cost profile of Σ_n is therefore completely flat: one auxiliary reference exposes at once all n pairwise references, each inaccessible to unassisted fine-partition protocols. A declared two-holder coarsening makes Σ_n useful in the network sense for every n, so the family is bound complexness relative to the fine partition, generalizing the known $n = 2$ case. We locate the mechanism: the extracted settlement syndrome is a single classical bit selecting between two globally replicated correction programs, and it is the *commutativity* of the frame-character algebra — the same fact that prevents frame characters alone from generating nonclassical behavior — that makes the extraction nondemolition and lets one broadcast bit correct all n pairs. The imported span monotone pins the servable pairs to the n bus pairs and caps the number of distinct pairs receiving exact references at n — a bound the amplifier attains exactly; what no known invariant bounds is multiplicity and asymptotic rate, and we pose the construction of a quantitative real-sector monotone as the paper's principal open problem. The $n = 2$ member of the family is exactly the τ_4 state of [1], whose corrected accessibility accounting (v1.0.1 of that paper) is the seed of the present results.

1 Introduction

The resource theory of complexness references [1] classifies the multipartite structure of the Weilenmann–Gisin–Sekatski "reference frame of complexness" [2] in real quantum networks: real network parties who lack a shared convention for the complex unit can establish one by consuming pairwise reference states $\Omega = \frac{1}{4}(I - J \otimes J)$, and which pairs can obtain Ω from a given resource is governed — exactly for frame-code states, and as a one-sided obstruction in general — by an $\mathbb{F}_2$-valued transpose-charge code. That paper's activation theorem (Thm. 6.6 there) showed that *individually*, every used high-weight frame-code charge admits a declared-auxiliary descent to a pair with *positive* rate — exactly and deterministically in the τ_4 instance;

its accessibility corollary (Cor. 7.3 there, as corrected in v1.0.1) showed the accounting of that descent is net-positive and catalytic rather than unit-neutral. (All numbered imports refer to `complexness-networks` v1.0.1.)

This paper asks the simultaneous question, and answers it with a construction. We exhibit a family Σ_n of n-pair resources — the *frame bus* — that is maximally locked (zero unassisted pairwise rate everywhere, Theorem 3) and maximally unlockable (one auxiliary reference releases every pair at once, exactly and deterministically, Theorem 6); a declared two-holder coarsening makes it useful for every n, hence *bound complexness* relative to the fine partition (Proposition 4). The family interpolates the known $n = 2$ case ($\Sigma_2 = \tau_4$) and shows that the one-bit syndrome mechanism scales without loss:

$$\Sigma_n \otimes \Omega \;\longrightarrow\; \Omega^{\otimes n} \qquad \text{(exact, deterministic, one copy, real LOCC).}$$

Because the output includes an exact reference on the anchor pair, the seed is returned in kind and the conversion is a strict catalytic cycle (Theorem 10).

Four features distinguish these results from the unlockable bound-entanglement phenomena they structurally resemble [3, 4, 5, 6] — noting that exact deterministic unlocking is *already* present in that class (Smolin's unlocking is a Bell measurement, a broadcast, and a Pauli correction), so exactness alone is not the distinction. What is new here: in the amplifier and catalysis theorems the trigger is an *auxiliary reference*, not a joining of holders — no coarsening of the partition and no joint measurement by two parties occurs there (Proposition 4, by contrast, uses a declared coarsening, as any boundness statement must); the seed is *returned in kind*, making the conversion a strict catalytic cycle rather than a consumption; the activation profile is *flat* — one seed releases all n pairs simultaneously, where the ancestor protocols as described unlock a single target (we do not claim a simultaneous variant is impossible in their class, only that it is not stated); and the sector is complex-separable, so it is a resource theory of complexness [1, 21, 22], not of entanglement. The mechanism itself belongs to the known unlocking class, and we position it as such (§9).

Why it scales. The frame-character algebra of [1] is Abelian. That fact cuts both ways, and the frame bus sits exactly on the cut. Abelianness is why frame characters alone cannot generate nonclassical network behavior (the noncommuting payload is load-bearing for usefulness); it is simultaneously why the single hidden sign of Σ_n admits a *nondemolition* extraction at all, and why one group element corrects all n pairs — so the settlement layer is classical from birth, on the copyable, commuting side of the no-broadcasting dichotomy [8]. Two local measurements anchored on the auxiliary reference have individually random outcomes whose *product* reveals the common sign; one broadcast then triggers the same correction on all n pairs. The noncommuting payload is never copied; only the Abelian syndrome is. We make this precise in §7.

Honesty about converses. Every *positive* rate here is an achievability result. The converses inherited from the span monotone (Theorem 3, Lemma 7) do bound one quantity exactly: the weight-2 elements of the seeded input span are precisely the n pair words, so no rSEP protocol places an exact reference on any holder pair other than the n bus pairs, and at most n distinct pairs are ever served — the amplifier attains this bound (Remark 9). What nothing bounds from above is *multiplicity* on a repeated pair and asymptotic rate; the standard entanglement

monotones are vacuous on this sector because every state involved is complex-separable.[1] All rates in this paper are therefore lower bounds in the multiplicity/rate sense, and Open Problem 13 — a quantitative, non-magnitude-blind real-sector monotone — is in our view the most important question the construction raises.

2 Preliminaries

We use the conventions of [1], summarized briefly (real-Hilbert-space simulation of quantum systems goes back to [15]; the experimental real-vs-complex program to [16, 2]). A *rebit* is a two-dimensional real Hilbert space; J denotes the antisymmetric unit $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, with $J^{\mathsf{T}} = -J$ and $J^2 = -I$. Real states are symmetric positive semidefinite trace-one matrices; the free operations are real separable operations with charge-free ancillas (in particular real product ancillas, which carry charge zero) relative to a declared holder partition (rSEP), matching the hypothesis of [1]'s span monotone, and real local operations with classical communication (real LOCC) where noted. The maximal pairwise complexness reference on a rebit pair is

$$\Omega = \tfrac{1}{4}\,(I - J \otimes J),$$

the normalized projector onto the -1 eigenspace of $J \otimes J$. For a multi-rebit state ρ, its *transpose-charge support* is the set of J-string labels occurring in its expansion, and the *charge code* $C(\rho) \leq \mathbb{F}_2^m$ is the $\mathbb{F}_2$-span of that support, indexed by the m *holders*: a holder carrying several rebits has charge equal to the mod-2 sum of theirs ([1], Def. 5.1; the support description above is the frame-code case of that definition, which is stated via transpose sectors). All code-level statements in this paper are relative to the fixed $2n$-holder partition; auxiliaries add rebits to existing holders, never holders, except where Lemma 7 explicitly considers fresh-holder placements in order to rule them out. The span is an rSEP monotone, and the per-pair dichotomy states that holders a, b can obtain Ω one-shot from a frame-code state σ_C iff $e_a + e_b \in C$, while if C has no weight-2 codeword the asymptotic pairwise rate is zero [1].

Throughout, $n \geq 2$ pairs $P_i = (a_i, b_i)$ of rebits are held by $2n$ distinct parties (the *fine partition*); we write

$$K_i = J_{a_i} J_{b_i} \quad \text{(i.e. } J \otimes J \text{ on pair } i\text{)}, \qquad \rho_{\pm}^{(i)} = \tfrac{1}{4}\,(I \pm K_i),$$

so $\rho_{-}^{(i)} = \Omega_{P_i}$. Each K_i is real symmetric with $K_i^2 = I$; $\rho_{\pm}^{(i)}$ is half the projector onto the ± 1 eigenspace of K_i, so K_i is sharp (deterministic) on it.

3 The frame bus

Definition 1 (Frame bus)**.** *For $n \geq 2$,*

$$\Sigma_n = \tfrac{1}{2}\Big(\rho_{+}^{(1)} \otimes \cdots \otimes \rho_{+}^{(n)} + \rho_{-}^{(1)} \otimes \cdots \otimes \rho_{-}^{(n)}\Big).$$

Lemma 2 (Three presentations)**.** $\Sigma_n = 4^{-n} \sum_{|S| \text{ even}} \prod_{i \in S} K_i = 2^{-2n} \prod_{j=2}^{n} \big(I + K_1 K_j\big)$*, and Σ_n is a frame-code state σ_{C_n} in the sense of [1] with all signs $+1$, where*

$$C_n = \Big\{ \textstyle\sum_i x_i\,(e_{a_i} + e_{b_i}) : \; x \in \mathbb{F}_2^n, \; \sum_i x_i = 0 \Big\}$$

[1] For all n: $\rho_{\pm} = \frac{1}{4}(I \mp Y \otimes Y)$ is an equal mixture of two y-basis product states, so Σ_n is a mixture of tensor products of separable states.

is the even-parity pair-block code. In particular $\dim_{\mathbb{F}_2} C_n = n-1$*, every nonzero codeword has (rebit) weight* ≥ 4*, and* $\Sigma_2 = \tau_4 = \frac{1}{16}(I + J^{\otimes 4})$ *exactly.*

Proof. Expand $\rho_{\pm}^{\otimes n} = 4^{-n}\prod_i (I \pm K_i) = 4^{-n}\sum_S (\pm 1)^{|S|}\prod_{i\in S} K_i$ and average: odd-$|S|$ terms cancel. The product form follows since $2^{-2n}\prod_{j\geq 2}(I + K_1K_j)$ expands to exactly the even-subset sum. The generating set is $\{c_j = p_1 + p_j : 2 \leq j \leq n\}$ with $p_i = e_{a_i} + e_{b_i}$; all induced signs are $+1$ since $|c_j \wedge c_k| = 2$ is even. Each nonzero codeword occupies at least two pair blocks (parity) and is $\{0,1\}^2$-constant on each block, giving weight ≥ 4. Setting $n = 2$ (rebit order a_1, b_1, a_2, b_2): $\frac{1}{2}(\rho_+ \otimes \rho_+ + \rho_- \otimes \rho_-) = \frac{1}{16}(I + K_1K_2)$ and $K_1K_2 = J^{\otimes 4}$. □

Theorem 3 (Zero unassisted pairwise rate)**.** *Under the fine* $2n$*-holder partition,* Σ_n *has zero exact and zero asymptotic* Ω*-rate between* every *pair of holders, under* rSEP *with any number of copies. Moreover every two-holder marginal of* Σ_n *is exactly* $I/4$*.*

Proof. C_n has no weight-2 codeword (Lemma 2), so the per-pair dichotomy of [1] applies verbatim to the frame-code state Σ_n. For the marginal statement: tracing all but two rebits kills every term of the even-subset expansion except I, since every nonzero product $\prod_{i\in S} K_i$ with $|S| \geq 2$ has traceless factors outside any two fixed rebits. □

Proposition 4 (Usefulness through coarsening; Σ_n is bound for every n)**.** *For every* $n \geq 2$*,* Σ_n *is useful in the sense of [1] (§2), through one declared two-holder coarsening. Hence, with Theorem 3,* Σ_n *is* bound complexness relative to the fine $2n$-holder partition *in the sense of [1, Def. 7.1], for every* n*.*

Proof. The construction of [1, Thm. 7.2] applies verbatim with Σ_n in place of τ_4; usefulness is existential over declared embeddings, and we name the embedding. Code level: coarsening the holders $\{b_1, a_2\}$ into one location B sends the codeword $p_1 + p_2 \in C_n$ to the weight-2 word $e_{a_1} + e_{b_2}$ of the induced code. Operationally, route the shares b_1, a_2 to B and the shares a_1, b_2 to the two Renou sources of the four-source network of [1, Thm. 7.2]; the remaining $2n-4$ shares are routed anywhere and never touched — conditioned on the hidden variable their states are products and absorb into the receiving parties' POVMs, exactly the nonfreeness argument of [1, Thm. 7.2(i)], which is insensitive to the number of extra sources. In the preparation phase B measures the involution $L = J_{b_1}J_{a_2}$ on the two shares it holds *ab initio* (legal: resource descendants) and broadcasts the outcome h. In the even-subset expansion $\Sigma_n = 4^{-n}\sum_{|S| \text{ even}} K_S$, the only nonidentity term supported inside $\{a_1, b_1, a_2, b_2\}$ is K_1K_2 (coefficient $+1$), so the conditioned marginal on (a_1, b_2) is exactly $\frac{1}{4}(I + h\, J_{a_1}J_{b_2})$; for $h = +1$ the holder of a_1 applies $V = \mathrm{diag}(1, -1)$. Every branch yields exactly Ω on (a_1, b_2) before settings are drawn, and the round then runs the same exact WGS/McKague realification [2, 15] on the established reference. □

Remark 5 (On the word "bound"). Boundness here is partition-relative, exactly as for the Smolin state: the locking half (Theorem 3) is a property of the fine partition, the usefulness half (Proposition 4) lives at a declared coarser one. Neither half touches the other's partition, so there is no tension — this is the same geometry as [1, Thm. 7.2], scaled to all n.

4 The exact reference amplifier

Theorem 6 (Exact reference amplifier)**.** *Fix any anchor pair* P_j *and let the two holders of* P_j *share one auxiliary maximal reference* $\Omega_{\tilde{a}_j\tilde{b}_j}$*, with* $\tilde{a}_j$ *held by the holder of* a_j *and* $\tilde{b}_j$ *by the holder*

of b_j. Then there is a real-LOCC protocol using one copy of Σ_n, two local measurements, and one round of classical broadcast, whose output on the n pairs $P_1, \ldots, P_n$ (auxiliary traced out) is exactly

$$\Omega^{\otimes n},$$

deterministically (on every measurement branch); all statements are relative to the fixed $2n$-holder partition (the auxiliary rebits are held by the two existing anchor holders). The net gain is $n-1$ references, with the input reference returned in kind on the anchor pair.

Proof. Write $K_{\mathrm{aux}} = J_{\tilde{a}_j} J_{\tilde{b}_j}$. The two holders of P_j measure, respectively,

$$M_a = J_{a_j} J_{\tilde{a}_j}, \qquad M_b = J_{b_j} J_{\tilde{b}_j}.$$

Each is a real symmetric involution supported on that holder's two rebits, hence a lawful local real projective measurement (identical in form to the parity measurements of [1, Thm. 6.5]). They commute with each other and with K_j ($M_a K_j = J_{a_j} J_{\tilde{a}_j} J_{a_j} J_{b_j} = K_j M_a$ since distinct-site factors commute and the repeated J_{a_j} contributes $J^2 = -I$ on both orders), and

$$M_a M_b = K_j \, K_{\mathrm{aux}}.$$

On the branch of Σ_n with hidden sign s, K_j is sharp with value s; on the auxiliary, K_{aux} is sharp with value -1. Hence although the individual outcomes x_a, x_b are uniformly random, their product is deterministic:

$$h := x_a x_b = -s.$$

(The -1 is the auxiliary's sharp $K_{\mathrm{aux}} = -1$. In a frame-code descent each declared auxiliary inside the measured support enters through exactly such a sharp -1 — this is the ancilla moment in the coefficient computation of [1, Thm. 6.5] — so a descent with r auxiliaries has its correction convention flipped by $(-1)^r$; only $r = 1$ is used in this paper.) Because M_a, M_b act as identity outside P_j and the auxiliary, the remaining $n-1$ pairs are untouched, and conditioning on the outcome leaves them in $\rho_{-h}^{\otimes(n-1)}$; because $[M_{a,b}\,, K_j] = 0$, the measurement is nondemolition on K_j and the anchor pair itself retains its character sharp with value $s = -h$. Broadcast h. If $-h = +1$, every holder of one rebit per pair applies the local reflection $V = \mathrm{diag}(1, -1)$, which is real orthogonal with $VJV^{\mathsf{T}} = -J$, mapping $\rho_+ \mapsto \rho_- = \Omega$ on that pair. After correction, every branch of the conditioned global state, with the auxiliary traced out, equals $\Omega^{\otimes n}$ exactly: the only elements of the stabilizer-type expansion supported off the auxiliary are the $\langle K_1, \ldots, K_n \rangle$ products, and each generator K_i is sharp with value -1 after correction (an even product is correspondingly sharp at $+1$). (All claims in this paragraph are finite-dimensional computations — operator identities, outcome probabilities, and conditional states; they have been verified by exact computation at $n = 2, 3, 4$, over all branches and all anchor choices; see the Reproducibility section.) □

Lemma 7 (Seed placement is load-bearing)**.** *Let the auxiliary Ω be placed with its two rebits held (i) by the two holders of one pair P_j (as in Theorem 6); (ii) across two different pairs (e.g. holders of a_1 and a_2, or a_1 and b_2); or (iii) by two fresh holders. In case (i) the input charge code is $C_n + \langle e_{a_j} + e_{b_j} \rangle = \mathrm{span}\{e_{a_i} + e_{b_i} : i \le n\}$, which contains every pair word. In cases (ii) and (iii) — and likewise for the two remaining placements, both rebits at a single holder (contributed charge zero) or one rebit at an existing holder and one fresh — the input span contains* no *pair word $e_{a_i} + e_{b_i}$, and by the span monotone of [1] no* rSEP *protocol with*

charge-free ancillas produces an exact reference on any of the bus pairs $P_1, \ldots, P_n$. A mis-placed seed is not inert: in case (ii) the enlarged span acquires one new *cross-pair word (e.g. a seed on the holders of b_1, a_2 makes $e_{a_1} + e_{b_2}$ reachable — precisely the descent of [1, Thm. 6.5]); what fails in every non-(i) placement is the bus.*

Proof. $\mathbb{F}_2$ span arithmetic. In case (i), the odd-parity direction $e_{a_j} + e_{b_j}$ together with the even-parity code C_n spans all pair words. In case (ii) the auxiliary contributes a cross word (e.g. $e_{a_1} + e_{a_2}$); every element of the enlarged span is then either even on pair blocks or contains the cross word's off-block support, and no pair word is reachable. Case (iii) adds a word supported off the $2n$ rebits entirely; both-rebits-at-one-holder adds the zero word (the two charges cancel mod 2); one existing plus one fresh adds a word with off-bus support. (All five placements computed exhaustively at $n = 2, 3, 4$; see Reproducibility.) □

Remark 8 (Exactness needs an exact seed). Running the protocol with the graded partial reference $\eta_t = \frac{1}{4}(I - t\, J \otimes J)$, $0 \le t < 1$, in place of Ω leaves the outcome probabilities unchanged but yields $\frac{1+t}{2}\,\Omega^{\otimes n} + \frac{1-t}{2}\,\bar{\Omega}^{\otimes n}$ (where $\bar{\Omega} := \rho_+ = \frac{1}{4}(I + J \otimes J)$; the two branches flip globally), at trace distance $(1 - t)/2$ from the target; no robustness statement is claimed here, and the approximate theory is open.

Remark 9 (Monotone accounting). The input and output charge codes in Theorem 6 coincide: both equal $\mathrm{span}\{e_{a_i} + e_{b_i}\}$. The span monotone of [1] is thus *saturated*, and it is not silent: the weight-2 elements of the seeded input span are exactly the n pair words (any sum of two or more pair blocks has weight ≥ 4), so the monotone pins the servable pairs to the n bus pairs, caps the number of *distinct* pairs receiving exact references at n, and caps the number of $\mathbb{F}_2$-independent references at n — all three bounds attained by the amplifier. What it does not bound is multiplicity on a repeated pair and asymptotic rate; it forces no sacrifice (none occurs). See Open Problem 13.

5 Strict catalysis

Theorem 10 (Strict catalytic cycle)**.** *For every $n \ge 2$: the conversion of Theorem 6 returns an exact Ω on the* original auxiliary carriers*, decoupled from all outputs, so that*

$$\Sigma_n \otimes \Omega_{\mathrm{cat}} \;\longrightarrow\; \Omega^{\otimes(n-1)} \otimes \Omega_{\mathrm{cat}}$$

is a strict catalytic transformation with yield $n - 1$ maximal references per copy of Σ_n, at zero net reference cost (one pair of carriers — the anchor bus pair — is discarded).

Proof. Trace out the two anchor-pair carriers a_j, b_j of the corrected conditional state. In its group expansion, every element with odd J-support on a_j or b_j — equivalently, every element outside $\langle \{K_i\}_{i \ne j}, K_{\mathrm{aux}} \rangle$ — has a traceless factor there and dies; what survives is generated by $\{K_i\}_{i \ne j}$ and K_{aux} — disjoint pair blocks, each generator sharp with value -1 after correction. The state on the $n - 1$ non-anchor pairs and the original auxiliary carriers is therefore exactly $\Omega^{\otimes(n-1)} \otimes \Omega_{\mathrm{cat}}$ in product, independent of the broadcast record h (both branches yield the same corrected product). The catalyst is returned on its original carriers with no relabelling: its residual correlation runs only through the anchor pair, which is the discarded carrier pair. (Verified exactly at $n = 2, 3, 4$, all anchors, all branches; see Reproducibility.) □

Remark 11. Iterating over m copies of Σ_n (each copy's anchor pair held by the same two parties as the catalyst) gives $m(n-1)$ references; defining the catalytic *aggregate* rate as the exact references produced per copy with the catalyst returned — summed over distinct pairs, a different quantity from the per-pair rate $D^{a:b}$ of Theorem 3 — the aggregate rate of Σ_n is thus at least $n-1$, against an unassisted rate of exactly zero on every pair (Theorem 3). We emphasize *at least*: no converse is known (§8).

6 The activation profile is flat

Count auxiliary references as the activation cost. Define, for $1 \le k \le n$: $a_k(\Sigma_n)$ is the minimum, over placements of declared auxiliary Ω's, of the number of auxiliaries such that some one-copy real-LOCC protocol outputs, deterministically and exactly, a product of Ω's on k distinct holder pairs whose pair words are $\mathbb{F}_2$-independent.

Theorem 12 (Flat simultaneous-activation profile)**.** $a_1(\Sigma_n) = a_2(\Sigma_n) = \cdots = a_n(\Sigma_n) = 1$.

Proof. $a_k \ge 1$ for every $k \ge 1$: with zero auxiliaries the exact rate is zero on every pair (Theorem 3), so no $k \ge 1$ is achievable. Theorem 6 achieves all n at cost one (extras may be discarded). □

This is an extreme instance of shared activation infrastructure: by analogy with the strictly increasing generalized Hamming weight hierarchy of linear codes [14], one would generically expect the cost of exposing k independent resources to grow with k; the frame bus collapses the hierarchy completely. The mechanism is the subject of the next section.

7 Why one bit suffices: syndrome compression and broadcastability

The frame bus is a two-branch mixture: its entire unresolved structure is the hidden sign $s = \pm 1$. Conditioned on s, every pair requires the *same* correction (identity if $s = -1$, one local reflection if $s = +1$). In the language of correction signatures: the map from hidden branches to the vector of required per-pair corrections has an image of size two, so the minimal sufficient settlement syndrome is one bit, *independently of* n. The information-theoretic form: the per-pair correction demands $C_1, \ldots, C_n$ are perfectly correlated, with total correlation [13] $\sum_i H(C_i) - H(C_1 \ldots C_n) = n - 1$ bits. One bit of syndrome carries n correction instructions because those instructions are redundant, not because information is compressed away.

Two structural facts complete the picture.

Extraction. Abelianness enters here, twice: the syndrome observable $M_a M_b$ lies in the commutant of every pair character, so extraction is nondemolition on the entire payload; and the induced correction is a single group element common to all n pairs. Anchoring the parity measurement on the auxiliary reference (rather than on a second resource pair, as in the distillation protocol of [1, Thm. 4.4]) means every stabilizer element beyond the target group $\langle K_1, \ldots, K_n \rangle$ touches the auxiliary rebits (it carries an M_a, M_b, or K_{aux} factor) and dies under the free partial trace. This is why the amplifier needs no twirl and no shared randomness, and why the output is an exact product rather than a state with correct marginals.

Broadcast. Once extracted, the syndrome is a classical bit: trivially copyable and broadcastable (two outcome bits x_a, x_b are communicated in the single round; their product h is the

one-bit sufficient statistic). Being a projective-measurement outcome, the syndrome would be classical whatever the algebra; what commutativity buys is upstream of that — it is why a *non-demolition* extraction of the one hidden sign exists at all, and why one group element corrects all n pairs. The contrast is with coordination schemes whose coordinating datum is inherently quantum: there, no exact broadcast to the $n-1$ remote ports exists for $n \geq 3$ [8]; here the coordinating datum is a character value of an Abelian algebra, classical from birth. The frame-character algebra is Abelian — the very fact that makes frame characters alone incapable of producing nonclassical behavior [1, Prop. 6.3] is the fact that makes the settlement layer of this architecture classical, hence scalable. The noncommuting payload that carries the usefulness is never copied; only the Abelian syndrome is. We regard this two-sidedness — the Abelian trap and the Abelian bus are one structural fact — as the conceptual core of the paper.

8 Limits, and what we do not claim

Open Problem 13 (Quantitative real-sector monotone). Construct a monotone under rSEP-with-declared-auxiliaries that is not magnitude-blind on the frame sector: a functional that bounds the *multiplicities and asymptotic rates* of exact pairwise references extractable from a given resource-plus-auxiliary configuration. The charge-code span already pins the servable pairs and caps the distinct-pair count at n (Remark 9) but is blind beyond that; all complex-entanglement monotones vanish identically on this sector (every state involved is complex-separable). Until such a monotone exists, every rate in this paper is a lower bound in the multiplicity/rate sense, and the possibility that Σ_n permits more than n references — necessarily by multiplicity on the bus pairs, since the span monotone pins which pairs are servable — is not excluded by any known invariant.

Open Problem 14 (Single-copy activation rank of general codes). For the frame bus, the single-copy activation-rank function jumps from 0 to exactly n at one auxiliary — for *any* single-auxiliary placement the input span $C_n + \langle w \rangle$ has $\mathbb{F}_2$-dimension at most n, so at most n independent pair words lie in it, and Theorem 6 attains n (Remark 9 computes the attaining case). For an arbitrary frame code C, determine $R_C^{(1)}(k)$: the maximum number of independent pairwise references extractable from one copy with k auxiliaries. In particular: does a "rank-preserving peel" always exist in the Abelian frame sector, or do codes exist whose independent charges suffer genuine single-copy congestion?

We also record what the amplifier does *not* establish: no robustness (Remark 8); no security or secret-sharing application is claimed anywhere in this paper; and no asymptotic conversion-rate theory beyond the catalytic lower bound.

9 Related work and positioning

The locking/unlocking mechanism class is well established in entanglement theory: Smolin's four-party unlockable bound entangled state [3], its graded generalizations to any even number of qubits [4], the Dür–Cirac families [5], activation and superactivation of bound entanglement [7, 20], and the unlockable stabilizer class of Wang and Ying [6], of which the frame-code states are the frame-sector transcription [1]. Small classical keys unlocking large accessible information appear in quantum data locking [9]. Quantum catalysis originates with Jonathan and Plenio [17] (see [18] for a review); catalytic amplification phenomena appear in coherence theory [10, 11],

where a catalyst can raise a distillation rate from zero to unbounded — though those conversions are approximate (arbitrarily small but nonzero error) or asymptotic in rate, whereas ours is exact and single-copy. The closest existing analogue to our headline result is the catalytic activation of Bell nonlocality [19]: a Bell-local state plus a catalyst becomes nonlocal, deterministically, by local operations, with the catalyst returned exactly. Our result lives in a different resource sector (complexness references in real networks rather than nonlocality), yields n references from one seed with a completely flat profile, and comes with the charge-code characterization of exactly when the seed placement works. Coordinating distributed operations through shared reference structure is the subject of the reference-frame resource theory [12]; notably, that literature's default expectation is that a reference *degrades* with use [23], against which the exact in-kind return of the seed here is a sharp contrast. The adjacent single-system resource theory is imaginarity [21, 22], for which every state in this paper is free; the resourcefulness here is a network phenomenon, as in [1]. The states Σ_n themselves are the frame-sector reading of a known unlockable stabilizer class [6, 4]. Against this background, we claim as new: the amplification law $\Sigma_n \otimes \Omega \to \Omega^{\otimes n}$ with auxiliary-reference trigger (no holder coarsening) in its simultaneous, all-pairs form with the seed returned in kind (in-kind return for a single descent is already [1, Cor. 7.3]); the completely flat activation profile; and the identification of the Abelian settlement layer (nondemolition extraction; one group element correcting all pairs) as the scaling mechanism in the complexness sector. We do not claim the unlocking mechanism class, catalysis as such, exact deterministic unlocking, or "small key, large effect" as new ideas.

Reproducibility

All finite-dimensional claims in Theorem 6 and Lemma 2 were verified by exact numerical computation ($n = 2, 3, 4$; all measurement branches; all anchor choices) by two independent implementations, both included in the repository (`verify/`) together with their captured output. Theorem 10 (the no-relabelling decoupling, all anchors and branches, $n = 2, 3, 4$) is verified by `verify/engine_b_catalysis.py`; Lemma 7 (all five seed placements, $n = 2, 3, 4$) by `verify/engine_b_placement.py`; the conditioning step of Proposition 4 (both branches, $n = 2, 3$) by `verify/engine_prop_useful.py`. A machine-verified core (Lean 4, Mathlib pinned at v4.30.0, manifest commit recorded in the repository) accompanies it: fourteen kernel-checked theorems covering the $\Sigma_2 = \tau_4$ identity (as the bilinear kronecker identity, general and instantiated), the combinatorial core of the charge-code weight bound (general n: every nonzero even-parity block word has ≥ 2 nonzero blocks; the doubling to rebit weight is definitional), the $n = 2$ amplifier operator algebra (the syndrome identity $M_a M_b = K_1 K_{\text{aux}}$, both nondemolition commutations, $K_1^2 = 1$, and the sharpness lemmas), and, at $n = 2$, the seed-placement span lemma for case (i) and for case (ii)'s same-side cross word $e_{a_1} + e_{a_2}$ (the remaining placements are verified numerically, not in Lean). The Lean core covers the operator algebra underlying Theorem 6, not the theorem itself: the measurement, conditioning, and partial-trace steps are verified numerically, not in Lean. Each Lean theorem depends only on Lean's standard three axioms (`propext`, `Classical.choice`, `Quot.sound`); no `sorry`, no `native_decide`; the `#print axioms` output is committed alongside the sources.

References

[1] S. Douglas, *The Resource Theory of Complexness References in Real Quantum Networks*, v1.0.1 (2026). Zenodo (all versions): doi:10.5281/zenodo.21907062; this version: doi:10.5281/zenodo.21910329. GitHub: `Apsiape/complexness-networks`.

[2] M. Weilenmann, N. Gisin, P. Sekatski, *Partial independence suffices to rule out Real Quantum Theory experimentally*, Phys. Rev. Lett. **135**, 180201 (2025); arXiv:2502.20102.

[3] J. A. Smolin, *Four-party unlockable bound entangled state*, Phys. Rev. A **63**, 032306 (2001); arXiv:quant-ph/0001001.

[4] R. Augusiak, P. Horodecki, *Generalized Smolin states and their properties*, Phys. Rev. A **73**, 012318 (2006).

[5] W. Dür, J. I. Cirac, *Activating bound entanglement in multiparticle systems*, Phys. Rev. A **62**, 022302 (2000).

[6] G. Wang, M. Ying, *Multipartite unlockable bound entanglement in the stabilizer formalism*, Phys. Rev. A **75**, 052332 (2007).

[7] P. Horodecki, M. Horodecki, R. Horodecki, *Bound entanglement can be activated*, Phys. Rev. Lett. **82**, 1056 (1999).

[8] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, B. Schumacher, *Noncommuting mixed states cannot be broadcast*, Phys. Rev. Lett. **76**, 2818 (1996).

[9] D. P. DiVincenzo, M. Horodecki, D. W. Leung, J. A. Smolin, B. M. Terhal, *Locking classical correlations in quantum states*, Phys. Rev. Lett. **92**, 067902 (2004).

[10] J. Åberg, *Catalytic coherence*, Phys. Rev. Lett. **113**, 150402 (2014).

[11] N. Shiraishi, R. Takagi, *Arbitrary amplification of quantum coherence in asymptotic and catalytic transformation*, Phys. Rev. Lett. **132**, 180202 (2024).

[12] S. D. Bartlett, T. Rudolph, R. W. Spekkens, *Reference frames, superselection rules, and quantum information*, Rev. Mod. Phys. **79**, 555–609 (2007).

[13] S. Watanabe, *Information theoretical analysis of multivariate correlation*, IBM J. Res. Dev. **4**, 66–82 (1960).

[14] V. K. Wei, *Generalized Hamming weights for linear codes*, IEEE Trans. Inf. Theory **37**, 1412–1418 (1991).

[15] M. McKague, M. Mosca, N. Gisin, *Simulating quantum systems using real Hilbert spaces*, Phys. Rev. Lett. **102**, 020505 (2009).

[16] M.-O. Renou, D. Trillo, M. Weilenmann, T. P. Le, A. Tavakoli, N. Gisin, A. Acín, M. Navascués, *Quantum theory based on real numbers can be experimentally falsified*, Nature **600**, 625–629 (2021).

[17] D. Jonathan, M. B. Plenio, *Entanglement-assisted local manipulation of pure quantum states*, Phys. Rev. Lett. **83**, 3566–3569 (1999).

[18] P. Lipka-Bartosik, H. Wilming, N. H. Y. Ng, *Catalysis in quantum information theory*, Rev. Mod. Phys. **96**, 025005 (2024); arXiv:2306.00798.

[19] J. Bavaresco, N. Brunner, A. Girardin, P. Lipka-Bartosik, P. Sekatski, *Catalytic activation of Bell nonlocality*, Phys. Rev. Lett. **135**, 220203 (2025); arXiv:2504.02042.

[20] P. W. Shor, J. A. Smolin, A. V. Thapliyal, *Superactivation of bound entanglement*, Phys. Rev. Lett. **90**, 107901 (2003).

[21] K.-D. Wu, T. V. Kondra, S. Rana, C. M. Scandolo, G.-Y. Xiang, C.-F. Li, G.-C. Guo, A. Streltsov, *Operational resource theory of imaginarity*, Phys. Rev. Lett. **126**, 090401 (2021).

[22] T. V. Kondra, C. Datta, A. Streltsov, *Real quantum operations and state transformations*, New J. Phys. **25**, 093043 (2023); arXiv:2210.15820.

[23] S. D. Bartlett, T. Rudolph, R. W. Spekkens, P. S. Turner, *Degradation of a quantum reference frame*, New J. Phys. **8**, 58 (2006).