A finite game that normal hyperfinite strategies approach but never win*

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Abstract

We exhibit non-closedness at perfect play for normal injective hosts: an explicit finite two-player game with 0/1 scoring has a perfect commuting-operator correlation approached by finite-dimensional quantum correlations, while no perfect strategy has a normal marginal on an injective von Neumann algebra of either player. The game has 561 Alice questions, 157 Bob questions, 3,696 sampled pairs and at most 20 answers per question. Perfect play is realized on a matrix amplification of the Bellaterra critical-KMS factor of type III with parameter 1/2. After restricting to the support of the chosen normal marginal, every perfect strategy yields a modular Cuntz pair and six self-similar relations that force the canonical trace on the free product of three groups of order two. The resulting noninjective group von Neumann algebra is an expected subalgebra of that support corner, with a faithful normal state-preserving expectation. A two-sided POVM lift through a matrix ultraproduct proves approximation by finite systems. Consequently correlations admitting normal injective hosts are not closed. Under the explicit hypotheses of Halvorson's Proposition 2.31, locally normal quantum-field-theory strategies cannot win exactly. We also formulate a KMS-face criterion for complete modular presentations. Exact arithmetic certificates check every supported losing event; the forcing proof applies to arbitrary normal strategies.

1 Definitions and main results

The main phenomenon is non-closedness at perfect play. Finite-dimensional strategies, and hence normal hyperfinite strategies, approach a perfect correlation of an explicit finite game, but no strategy with a normal injective host for either player attains perfection. The perfect correlation itself belongs to C_{qa}; this distinguishes the result from commuting-versus-approximate value gaps.

A finite two-player game consists of finite question sets X, Y, finite nonempty answer sets A_x, B_y, a probability distribution π on $X \times Y$, and a predicate $V(a, b \mid x, y) \in \{0, 1\}$. For a correlation p its loss is

$$\ell_G(p) = \sum_{x,y,a,b} \pi(x, y)(1 - V(a, b \mid x, y))p(a, b \mid x, y).$$

Perfect play means $\ell_G(p) = 0$. Since all summands are nonnegative, it is equivalent to vanishing of every forbidden probability on a supported question pair. The scoring is binary; the answers need not be binary.

A commuting strategy consists of POVMs A_a^x, B_b^y on a Hilbert space and a state on their joint operator algebra, with $[A_a^x, B_b^y] = 0$. A specified *Alice host* is a von Neumann algebra M containing

*The authors developed this work with AI models from more than one developer and take responsibility for every claim. How the work was done, and what the machine checks cover, is described at the end of the paper.

her effects, with Bob's effects in the full commutant M' and the marginal state normal on M. A smaller commuting partner is allowed. The definition with Bob distinguished exchanges the parties. Normality refers to this specified ambient algebra. It is additional data, beyond the generated measurement algebra.

After passing to the support of the normal marginal, every such strategy has a faithful normal standard-form realization. Write $(M, L^2(M), J, \mathcal{P})$ for this standard form and $\xi = \xi_\phi$ for the natural-cone vector of the faithful normal state ϕ. Then

$$p(a, b \mid x, y) = \langle \xi, A_a^x B_b^y \xi \rangle, \qquad B_b^y \in M' = JMJ.$$

The reduction preserves POVMs and probabilities, including when the original state is mixed; it need not preserve products of the compressed other-party effects. A proof is given below.

An algebra is *injective* if completely positive maps into it extend across inclusions of operator systems; equivalently it is the range of a norm-one completely positive projection from $B(H)$ in a faithful representation. Injectivity passes to corners, commutants and expected von Neumann subalgebras. For separable predual, injectivity is equivalent to hyperfiniteness by Connes's theorem [3]. The algebras constructed here have separable predual. Our exclusion applies to arbitrary injective ambient hosts, without a separability premise. An expected subalgebra here is the range of a faithful normal conditional expectation.

Let C_q be the finite-dimensional spatial correlation set, $C_{qa} = \overline{C_q}$ its Euclidean closure, C_{qs} the spatial set allowing infinite-dimensional Hilbert spaces, and C_{qc} the commuting set. These sets refer to the full finite correlation array, including unsupported question pairs. Write $\mathcal{I}_A$, respectively $\mathcal{I}_B$, for arrays admitting a specified normal injective host for Alice, respectively Bob.

Theorem 1.1 (Finite-game separation)**.** *The game G_{B} of Section 3 has 561 Alice questions, 157 Bob questions, 3,696 uniformly sampled pairs and at most 20 answers per question. It has a perfect standard-form strategy on*

$$M_9 \bar{\otimes} M_2 \bar{\otimes} P,$$

where P is the Bellaterra critical-KMS factor of type $\mathrm{III}_{1/2}$. Its perfect correlation p_* belongs to C_{qa}. No perfect strategy admits a normal injective host for either player. Thus the finite-dimensional supremum is one, and $p_* \notin \mathcal{I}_A \cup \mathcal{I}_B$.

The ideal strategy and exact certificate are in Section 4; the arbitrary-normal extraction is in Section 5; the two-sided approximation is in Section 6. Section 7 finishes the exclusion and proves non-closedness of the normal injective-host sets, together with a qualified quantum-field-theory consequence.

The mechanism is finite at its input and infinite at its conclusion. The game forces two isometries with modular eigenvalue $1/2$ and six relations implementing a self-similar group action. Equal cylinder masses then determine every group moment. The resulting expected group algebra is noninjective. The KMS uniqueness mechanism is due to Laca–Raeburn–Ramagge–Whittaker [14]; its use in this explicit finite predicate supplies the separation.

Our type-game construction [4] forces type III. The present game adds a condition that type III alone does not satisfy: noninjectivity. The flagged magic-square ingredient originates with Mančinska [16]; the companion finite-energy paper [5] develops the underlying binary-scoring game. Neither that energy result nor an energy bound for the present game is used here.

1.1 Comparison and bounded search

Earlier commuting-versus-approximate value-gap games already exclude perfect strategies with normal injective hosts, since such correlations belong to C_{qa} by semidiscreteness. Ji–Natarajan–

Vidick–Wright–Yuen [12] established the existence of games with $\omega_{qa} < 1 = \omega_{qc}$; Gao–Yu–Zhi [6], Theorems 1.1–1.2, give an explicit linear-system game. The normal injective-host exclusion is our inference from semidiscreteness and these gap results, rather than a formulation attributed to their authors. The distinction here is that perfect play itself lies in C_{qa}: normal injective hosts approximate the perfect correlation but cannot attain perfection.

For completeness, let M be a normal injective host in the standard form above. Semidiscreteness gives unital completely positive maps $\alpha_i : M \to M_{n_i}$ and $\beta_i : M_{n_i} \to M$ with $\beta_i\alpha_i \to \mathrm{id}_M$ pointwise ultraweakly. The positive normal functionals $\omega_{y,b}(x) = \langle \xi, xB_b^y \xi \rangle$ sum to the same normal state ϕ for every y. The functionals $\omega_{y,b} \circ \beta_i$ therefore form finite-dimensional assemblages with common density matrix. Bob can realize them by POVMs on a purification of that matrix, while Alice uses $\alpha_i(A_a^x)$. Their correlations converge to the original full array, proving membership in C_{qa}. The same argument applies with Bob distinguished.

Slofstra's Theorem 1.1 [20] already gives a finite game with perfect C_{qa} play and no perfect spatial play. Kim–Paulsen–Schafhauser's Corollary 3.7 [13] characterizes synchronous perfect approximate play by representations in R^ω and amenable traces. Such an ultrapower is different from a normal injective host. The Boolean constraint-system framework of Paddock–Slofstra [18] provides broader algebraic reductions.

Harris's Theorem 4.9 [10] identifies injective factors arising from exact embezzlement. Paddock–Schmidt's Theorem 1.1 and Corollary 4.6 [17] give a generalized-game self-test in $C_{qa} \setminus C_{qs}$, using the embezzlement model. Chen–Zhao's Theorem 1.4 [2] encodes real finitely presented tracial algebras into binary correlations. Prescribed nonzero probabilities are data beyond the forbidden-event face of a game; their theorem does not directly encode this nontracial Cuntz presentation.

Before writing this account, we searched combinations of finite game, nonlocal game, normal, injective, hyperfinite, exact attainment, nonamenable and compiler, and inspected the primary statements just cited. For the conjunction in Theorem 1.1, no earlier version found in a bounded search is the extent of our priority conclusion. The search does not prove absence of an earlier construction.

Lemma 1.2 (Normal support reduction; [4], Lemma 1.6)**.** *Every commuting strategy whose specified marginal is normal has a faithful normal standard-form realization on its support corner, preserving all correlations.*

Proof. Take the joint GNS representation with vector ψ. For the marginal support s, $s\psi = \psi$ and the other party commutes with s. Replace Alice's effects by sAs in $M = sNs$. The map $x\xi_\phi \mapsto x\psi$ is an isometry from $L^2(M, \phi)$ onto $\overline{M\psi}$. This subspace reduces M. Compressing Bob's effects through that isometry gives positive effects summing to one and commuting with the left M action. Their matrix coefficients with ξ_ϕ preserve the joint probabilities. The support marginal is faithful and normal. □

2 Modular contraction and an expected group algebra

Use $\sigma_t^\phi(x) = \Delta_\phi^{it} x \Delta_\phi^{-it}$. An r-eigenoperator satisfies $\sigma_t^\phi(x) = r^{it}x$. Our analytic KMS convention is

$$\phi(xy) = \phi(y\sigma_{-i}^\phi(x)). \tag{1}$$

It agrees with the modular convention in Hiai's Theorem 2.14 [11]. The centralizer M_ϕ is the fixed algebra of this modular group.

Let $\Gamma = C_2 * C_2 * C_2 = \langle a, b, c \mid a^2 = b^2 = c^2 = 1 \rangle$. The Bellaterra action on binary words has recursions

$$a = \mathrm{swap}(c, c), \qquad b = (a, b), \qquad c = (b, a).$$

The generator a exchanges the first bit and has section c on both bits; b, c preserve that bit and have the indicated sections. Grigorchuk–Savchuk's Theorem 1.1 identifies the group and essential freeness of this action; their Proposition 3.9 gives the bireversible-automaton criterion [7]. Essential freeness means that each nonidentity group element fixes a Bernoulli-null subset of the binary boundary.

Theorem 2.1 (Modular contraction)**.** *Let (M, ϕ) have a faithful normal state. Suppose S_0, S_1 are Cuntz isometries with $\sigma_t^{\phi}(S_x) = 2^{-it} S_x$, and $U_a, U_b, U_c \in M_\phi$ are self-adjoint unitaries satisfying*

$$\begin{aligned} U_a S_0 &= S_1 U_c, \quad U_a S_1 = S_0 U_c, \\ U_b S_0 &= S_0 U_a, \quad U_b S_1 = S_1 U_b, \\ U_c S_0 &= S_0 U_b, \quad U_c S_1 = S_1 U_a. \end{aligned} \tag{2}$$

Then $\phi(U_g) = \delta_{g,e}$ for every $g \in \Gamma$. The algebra $W^*(U_a, U_b, U_c)$ is a copy of $L(\Gamma)$ and is the range of a faithful normal ϕ-preserving expectation from M. In particular M is noninjective.

Proof. For a word μ set $R_\mu = S_\mu S_\mu^*$. Equation (1) gives

$$\phi(S_x z S_x^*) = \tfrac{1}{2}\phi(z).$$

Thus the level-n cylinder projections partition one, belong to M_ϕ and have mass 2^{-n}. Iterating (2) gives $U_g R_\mu U_g^* = R_{g \cdot \mu}$.

For a centralizer projection R, $\phi(Rz) = \phi(RzR)$: the complementary term is zero after moving R across the state. Therefore

$$\phi(U_g) = \sum_{|\mu|=n} \phi(R_\mu U_g R_\mu), \qquad |\phi(U_g)| \le 2^{-n} \#\{\mu : g \cdot \mu = \mu\}.$$

A nonfixed diagonal block vanishes by orthogonality of the cylinders. A fixed block has absolute expectation at most $\phi(R_\mu)$, by Cauchy–Schwarz. The fixed fractions decrease to the boundary measure of $\mathrm{Fix}(g)$. Essential freeness makes this zero for $g \neq e$.

The restricted faithful normal state consequently has the regular group moments. Its GNS representation sends $U_g \xi$ to δ_g, identifying $N = W^*(U_a, U_b, U_c)$ with $L(\Gamma)$. There is no group-quotient ambiguity: collapsing a nonidentity g would make $\phi(U_g) = 1$. The algebra N is pointwise modularly fixed. Takesaki's expectation theorem, in the formulation of Hiai's Theorem 6.6 [21, 11], yields a normal ϕ-preserving expectation $M \to N$. Faithfulness follows from that of ϕ.

For completeness, $L(\Gamma)$ is noninjective. Were it injective, a conditional expectation $B(\ell^2\Gamma) \to L(\Gamma)$, followed by the regular trace and restricted to diagonal multiplication operators, would give a left-invariant mean on $\ell^\infty(\Gamma)$. Bimodularity under the left regular unitaries gives invariance. But the even-word index-two subgroup of Γ is the free group on ab, ac, and is nonamenable. Finally an expected subalgebra of an injective algebra is injective, by composition of completely positive extensions. This contradicts injectivity of M. □

This cylinder argument is the critical-KMS uniqueness mechanism of Laca–Raeburn–Ramagge–Whittaker. Their Theorem 7.3(2)–(3) [14] gives the canonical state and uniqueness when every element has finitely many sections. That hypothesis holds here: sections of a word of length k are represented by words of length at most k. Contracting is unnecessary. Their later groupoid treatment [15] gives a broader setting. The expected noninjective-subalgebra conclusion above is the consequence needed for arbitrary hosts.

3 The finite predicate

3.1 The inherited flagged game

We give the base rules explicitly to fix every convention. Let $\mathcal{R}$ be the triples of signs with product $+1$, and $\mathcal{C}_i$ those with product η_i, where $(\eta_1, \eta_2, \eta_3) = (+1, +1, -1)$. Set

$$r_0 = c_0 = (+, +, +), \qquad r_1 = c_1 = (+, -, -).$$

Alice has R_1, R_2, R_3, T_0, T_1, and Bob has C_1, C_2, C_3, U_0, U_1. A row answer is a member of $\mathcal{R}$ or a flag f; a column answer is a member of $\mathcal{C}_i$ or f. On T_x the answers are the three triples in $\mathcal{C}_1 \setminus \{c_x\}$ and two signs. On U_y they are the three triples in $\mathcal{R} \setminus \{r_y\}$ and two signs.

On (R_i, C_k) accept two flags, reject a single flag and otherwise compare the intersecting signs. On (T_x, C_1) a sign matches Bob's class $\{f, c_x\}$, while a triple must match that triple. On (R_1, U_y) reverse this calibration rule, using r_y. On (T_x, U_y) two signs s, t must satisfy $st = (-1)^{xy}$; if either answer is a triple, accept. Every remaining internal pair accepts every valid answer. All 25 internal pairs are sampled, including the unrestricted ones.

The game G_{III} of [4, Section 2] consists of nine copies indexed by $j \in \mathbb{Z}/9\mathbb{Z}$ and a Bob question Lab with answers in $\mathbb{Z}/9\mathbb{Z}$. Besides the 225 internal pairs, add

$$(R_1^j, \mathrm{Lab}): \quad f \iff \mathrm{Lab} = j,$$

$$(R_2^j, \mathrm{Lab}): \quad (+, +, +) \iff \mathrm{Lab} \in \{j+1, j+2\}.$$

There are 243 supported base pairs. All indices below are modulo nine. The flag question is named `flag` in the machine table; the symbols U_0, U_1 for base questions are distinct from the global group observables U_a, U_b, U_c.

3.2 Labels and conditional contexts

Retain all base questions and pairs. Add three binary Bob questions for the global signs U_a, U_b, U_c. For each j, add twelve binary Bob questions: nine cells O_{rs}^j $(0 \le r, s \le 2)$ and Y_a^j, Y_b^j, Y_c^j. Write $X_j = O_{00}^j$ and $Z_j = O_{11}^j$.

For each of the five Alice base questions Q in each copy add nine joint questions (Q, t). Six contexts t are the rows and columns of the auxiliary square. Their sign triples have product $+1$ on every row and the first two columns, and -1 on the last column. The other three contexts are the singleton signs Y_q^j. An answer to (Q, t) contains an ordinary valid answer to Q and a valid context tuple. Against each of the five original Bob questions of its copy, apply the original predicate to the ordinary coordinate. Inherit any Q–Lab link as well. Against each binary Bob question occurring in t, compare that coordinate of the tuple. These are all its tests.

Add three Alice questions (Lab, U_q), with answers (j, s). Against Bob's Lab compare j; against his global U_q compare s. They enforce joint measurability of the global signs and the flag partition.

A conditional context specifies a block $D \subset \mathbb{Z}/9\mathbb{Z}$ and a list of binary variables. Its answers are one off-block symbol (0) and the on-block tuples $(1, s_1, \ldots, s_k)$ satisfying the relation in Table 1. Against Lab, the block bit must equal membership in D. Against each listed binary variable compare its sign on the block, and accept the off-block answer. The twelve contexts per copy are exactly the following. No context jointly measures U_a, U_b. The two-element block is tested as a block; its two flags are not resolved in a context containing the flag-interchanging X_j.

This specifies G_{B} completely. Sample uniformly from the supported pairs. Invalid answers are rejected. Unsupported pairs have weight zero and may be assigned the all-accept predicate. The

Block	Variables and relation	Number
$\{j+1\}$	$Z_j = +1$	1
$\{j+2\}$	$Z_j = -1$	1
$\{j\}$	$Y_q^j = U_q$, $q = a, b, c$	3
$\{j+1\}$	$Y_b^j = U_a$	1
$\{j+2\}$	$Y_b^j = U_b$	1
$\{j+1\}$	$Y_c^j = U_b$	1
$\{j+2\}$	$Y_c^j = U_a$	1
$\{j+1, j+2\}$	joint Y_a^j, X_j, U_c; $Y_a^j X_j U_c = 1$	1
$\{j+1, j+2\}$	joint X_j, U_a; joint X_j, U_b	2

Table 1: Conditional contexts defining G_{B}. Joint variables are required to have an on-block joint sign tuple.

question counts are

$$|X| = 45 + 405 + 3 + 108 = 561, \qquad |Y| = 46 + 3 + 108 = 157.$$

The supported pairs consist of 243 base pairs, 2,187 inherited joint/base tests, 945 label matches, six global matches and 315 conditional tests, totaling 3,696. The largest answer set has $5 \cdot 4 = 20$ elements.

The file `anc/labelled-giii-game.json` lists every question's valid answers and every supported pair's forbidden pairs of zero-based answer indices. It records the uniform denominator 3,696. The two checker implementations compare semantic outcomes, so the ordering of answer arrays has no mathematical significance.

4 Perfect play and the exact certificate

Section 6 constructs a faithful normal pair (P, φ) with Cuntz generators T_0, T_1 of modular eigenvalue $1/2$ and centralizer involutions U_a, U_b, U_c satisfying (2). It also identifies P as the Bellaterra critical-KMS factor. For the present construction use

$$(M, \phi) = (M_9 \bar{\otimes} M_2 \bar{\otimes} P, \mathrm{tr}_9 \otimes \mathrm{tr}_2 \otimes \varphi).$$

The auxiliary M_2 is suppressed where it acts trivially. Write $F_j = e_{jj} \otimes 1$. For $1 \le k \le 4$ put $p = j + 2k - 1$ and

$$V_0 = F_j, \qquad V_k = e_{p,j} \otimes T_0^* + e_{p+1,j} \otimes T_1^*. \tag{3}$$

For each fixed copy j, these five partial isometries have initial projection F_j and orthogonal final projections summing to one. Hence $E_{kl} = V_k V_l^*$ are unital five-by-five matrix units. The four nonflag corners are centralizer matrix units; V_k for $k > 0$ has modular eigenvalue two.

For clarity, the real magic-square convention on the four-dimensional core is

$$\begin{pmatrix} X \otimes I & I \otimes X & X \otimes X \\ I \otimes Z & Z \otimes I & Z \otimes Z \\ X \otimes Z & Z \otimes X & Y \otimes Y \end{pmatrix},$$

where X, Y, Z are the Pauli matrices. Its row and column eigenspaces define the four ordinary outcomes; append the flag vector f. The calibrated rays are

$$c_0 = (1, 0, 1, 0)/\sqrt{2}, \quad c_1 = (0, 1, 0, 1)/\sqrt{2},$$

$$r_0 = (1,1,1,1)/2, \quad r_1 = (1,-1,1,-1)/2.$$

Use the two rays $(f \pm c_x)/\sqrt{2}$ on T_x and $(f \pm r_y)/\sqrt{2}$ on U_y, with the three residual magic rays. Embed each real five-by-five projector Q as $\sum Q_{kl} E_{kl}$. This defines all base effects. Bob's base effects are their standard-form mirrors; his flag effects are JF_jJ.

For a corner operator l define

$$L_j(l) = \sum_{k=0}^{4} V_k (F_j l) V_k^*.$$

Set

$$\begin{aligned} X_j &= L_j(T_0 T_1^* + T_1 T_0^*), \\ Z_j &= L_j(T_0 T_0^* - T_1 T_1^*), \qquad Y_q^j = L_j(U_q). \end{aligned} \tag{4}$$

These are centralizer involutions commuting with the copy's five-by-five matrix units and thus with all its base effects. Let $a = X_j$, $b = X_{\mathrm{aux}}$, $c = Z_{\mathrm{aux}}$ and $d = Z_j$. Complete the auxiliary square by

$$\begin{pmatrix} a & b & ab \\ c & d & cd \\ ac & bd & acbd \end{pmatrix}. \tag{5}$$

The two Pauli pairs anticommute within pairs and commute across pairs, giving the required row and column products. Every label is degree zero for the modular action.

On $C_j = F_{j+1} + F_{j+2}$ these operators satisfy

$$X_j = e_{j+1,j+2} + e_{j+2,j+1}, \qquad Z_j = F_{j+1} - F_{j+2},$$

with the equalities interpreted after restriction to C_j. Also $Y_q^j F_j = U_q F_j$, and

$$\begin{aligned} Y_b^j C_j &= F_{j+1} U_a + F_{j+2} U_b, \\ Y_c^j C_j &= F_{j+1} U_b + F_{j+2} U_a, \\ Y_a^j C_j &= X_j U_c C_j. \end{aligned}$$

The conditional variables commute on their tested blocks. Alice uses the products of the commuting spectral projections specified by each context, and the off-block complement. For (Q,t) multiply each base effect by the context spectral projection. Bob uses JBJ for the corresponding self-adjoint left effect B. This gives every ideal question in the finite table.

Proposition 4.1 (Exact winning certificate)**.** *The stated operators define a perfect strategy for* G_{B}*. The ancillary exact checker certifies this using finite algebraic identities over* $\mathbb{Q}(\sqrt{2})$*.*

Proof. Every effect in this model is analytic. For a self-adjoint analytic B, Tomita's identity gives

$$JBJ\xi = \sigma^{\phi}_{-i/2}(B)\xi. \tag{6}$$

A monomial $T_\mu U_g T_\nu^*$ of degree $d = |\mu| - |\nu|$ acquires the coefficient $2^{-d/2}$ in this half-translation. Therefore it suffices for every forbidden event to prove

$$A\sigma^{\phi}_{-i/2}(B) = 0. \tag{7}$$

Indeed its expectation on ξ is exactly the forbidden probability.

The program `compile_and_check.py` constructs all effects, proves self-adjointness, idempotence and completeness of every question, and proves (7) for every supported forbidden event. Its

coefficients are pairs of rational numbers representing $\mathbb{Q}(\sqrt{2})$. Its monomials are matrix entries times $T_\mu U_g T_\nu^*$, using the two prefix-comparison multiplication identities in Laca–Raeburn–Ramagge–Whittaker's Lemma 3.4 [14]. Group words cancel only adjacent equal generators. To certify zero, it expands annihilation suffixes to a common length using the Cuntz partition and cancels identical monomials. Each reported zero thus has a finite derivation from the stated relations; completeness of the zero test is not required.

Cyclic flag translation reduces 718 question checks to 86 POVM templates and 3,696 pair checks to 672 pair templates. All 90,630 forbidden-event occurrences are covered by 19,014 representative losing-product reductions. The same program checks the extracted Cuntz identities and all six recursions. These verified identities, together with the existence of (P, φ), prove perfect play. □

The independent predicate checker transcribes the human rules without importing the compiler or its operator engine. It compares all answer sets and all 3,696 supported pairs, and detects a deliberately deleted forbidden event. A separate ideal checker uses adjacent generator rewriting and independent outer-product formulas. On the recorded 512-pair sample it proves all 12,537 losing identities and checks the 485 measurements that occur there. The sample is reproducible from its recorded seed; it is not a second exhaustive ideal check. An action diagnostic additionally performs 91,840 finite comparisons. These finite certificates establish the ideal realization. They do not establish the arbitrary-strategy extraction in the next section.

5 Forcing in every perfect normal strategy

We work after Lemma 1.2, in faithful normal standard form. The following precise base-game facts are the imports from [4, Lemmas 2.1 and 2.2, and Section 2].

Lemma 5.1 (Checked classes and the base bridge)**.** *A perfectly matched coarse partition is a sharp centralizer partition, with its unique standard commutant mirror. Fine effects whose marginals are sharp are the corresponding joint spectral projections. In a perfect G_{III} strategy, the flag effects F_j form a centralizer partition of one. Each copy has a centralizer M_4 on $1 - F_j$ and a partial isometry w_j satisfying*

$$w_j^* w_j = F_j, \quad w_j w_j^* = C_j := F_{j+1} + F_{j+2}, \quad \sigma_t^{\phi}(w_j) = 2^{it} w_j. \tag{8}$$

In a copy write $f = F_j$, choose a minimal core projection q and centralizer maps u_x from q to c_x. If D_x is the T_x sign difference, its pullback is

$$X_x = (f + u_x^*) D_x (f + u_x) \in (f + q) M (f + q).$$

The positive (T_x, U_0) bridge cell gives $X_x H\xi = Y_0 H\xi$, where $H = f + q/\sqrt{2}$ and Y_0 is the pullback of the original Bob sign difference. The extracted bridge is a core translate of

$$\beta = q(f + u_x^*) D_x (f + u_x) f. \tag{9}$$

Here the identity involving H is the positive-cell calculation in the proof of the cited modular-bridge lemma. It uses calibrated sign supports and rejected sign events, before any assumption of sharp individual T_x signs. Thus the same calculation applies to a calibrated variant of that question. We spell out its consequence because it is essential to the labelling argument.

For the checked-class assertion, let E_c, F_d be matching coarse effects on the two sides. Zero mismatched events give $E_c F_d \xi = 0$ for $c \neq d$. Summation gives $E_c \xi = F_c \xi$. It follows that $E_c E_d \xi = 0$ for $c \neq d$, and separation makes the partition sharp. A self-adjoint commutant mirror implies

$\phi(E_c z) = \phi(z E_c)$, so it is centralizer and its mirror is $J E_c J$. Positive fine effects lie below their sharp marginals; summing them identifies their joint spectral projections.

The base square then gives a centralizer M_4 with arbitrary multiplicity. Its matrix units belong to the algebra generated by its rays. For the bridge, calibration fixes sign support $f + c_x$. Pullback yields commuting left and right contractions on the standard $(f + q)$ corner. The four parity cells have joint-support vectors $(f + (-1)^{xy} q/\sqrt{2})\xi$. The three positive cells identify the common reflection; the negative cell makes it anticommute with $f - q$. Its off-diagonal part is (9), with initial projection f, final projection q and modular eigenvalue two. The base flag links identify its designated range with C_j, giving (8). These are exactly the bridge conclusions used below.

5.1 Common unlabelled marginals

Every occurrence of a particular label matches the same binary Bob question. Lemma 5.1 therefore makes it one sharp centralizer involution, independent of its Alice context. A fine joint effect lies below its sharp label projection. Consequently its unlabelled marginal commutes with every label in that context.

For a labelled row question, its comparisons to the three original Bob columns fix the flag and all three cell-sign marginals. Those are the original sharp marginals. The checked-class assertion identifies every fine unlabelled row effect with the original joint spectral projection. Hence every labelled row has its original unlabelled POVM.

Consider a labelled T_x in context t. Calibration against the common original Bob C_1 fixes the residual effects and the sum of the two sign effects, $f + c_x$. Write their difference as D_x^t and pull it back:

$$X_x^t = (f + u_x^*) D_x^t (f + u_x).$$

The inherited positive (T_x, U_0) cell is present for this variant and the original question. The calibrated core, the original Bob question and the vector $H\xi$ are the same in both calculations. Thus the positive-cell identity of Lemma 5.1 gives

$$X_x^t H\xi = Y_0 H\xi = X_x H\xi.$$

On the standard corner $e = f + q$, the centralizer element H is invertible. The vector $H\xi$ separates eMe. Subtraction therefore gives $X_x^t = X_x$. Since $f + u_x$ is a partial isometry from e onto $f + c_x$, pushing forward recovers $D_x^t = D_x$. The fixed sign sum and residual effects identify the entire unlabelled T_x POVM. This proves common unlabelled marginals for all nine variants of every base question.

5.2 Commutation with the bridge and the block equations

Each of the twelve labels occurs in a context attached to every base question. The common-marginal result consequently makes each label commute with every original Alice effect in its copy. It commutes with the core matrix units, with D_x and with (9), hence with its core translates w_j. This conclusion does not assert commutation of incompatible labels with one another.

The auxiliary square relations now hold as operator identities on the full identity. Their internal magic-square calculation gives $X_j Z_j = -Z_j X_j$. For example, writing the first two rows as a, b, ab and c, d, cd, the first two columns give ac, bd, and the last row and column give $acbd = -abcd$. Together with the row/column commutations these imply $cb = -bc$ and $ad = -da$. Here $a = X_j, d = Z_j$.

The three (Lab, U_q) questions make U_q centralizer involutions commuting with every F_j. To justify conditional contexts without assuming sharp original effects, let their on-block effects be

E_s. Matching the block bit to Lab gives $\sum_s E_s = F_D$. If a matched centralizer sign projection P_t is incompatible with s, the losing event gives

$$E_s J P_t J \xi = 0.$$

Since $JP_tJ\xi = P_t\xi$, separation yields $E_sP_t = 0$. Thus each E_s lies under all its compatible sign projections. Summing gives $[F_D, P_t] = 0$ and identifies the restricted joint spectral projections. Missing tuples impose exactly their stated relation on F_D. There is no conclusion off that block.

In particular X_j, Z_j commute with C_j by their commutation with w_j, and the two Z contexts give

$$Z_j C_j = F_{j+1} - F_{j+2}, \qquad X_j F_{j+2} X_j = F_{j+1}. \tag{10}$$

The remaining contexts give

$$\begin{aligned} Y_q^j F_j &= U_q F_j, \\ Y_b^j C_j &= F_{j+1} U_a + F_{j+2} U_b, \\ Y_c^j C_j &= F_{j+1} U_b + F_{j+2} U_a, \\ Y_a^j C_j &= X_j U_c C_j, \qquad [X_j, U_q] C_j = 0 \quad (q = a, b, c). \end{aligned} \tag{11}$$

For $q = c$ the commutation is furnished by the active commuting triple; the other two are the explicit joint contexts.

5.3 The extracted Cuntz pair

Put $p = j + 1$, $r = j + 2$ and

$$e_{j,0} = F_p w_j, \qquad e_{j,1} = X_j F_r w_j, \qquad S_x = \sum_j e_{j,x}^*. \tag{12}$$

The bridge and flag-swap identities imply

$$e_{j,0} e_{j,0}^* = e_{j,1} e_{j,1}^* = F_p, \qquad e_{j,0} e_{j,1}^* = 0, \qquad \sum_x e_{j,x}^* e_{j,x} = F_j.$$

For distinct j their initial supports F_j and their common final supports F_{j+1} are disjoint. Expanding the finite sums therefore gives

$$S_x^* S_y = \delta_{xy} \mathbf{1}, \qquad S_0 S_0^* + S_1 S_1^* = \mathbf{1}.$$

Every S_x has modular eigenvalue $1/2$, since the flags and labels are centralizer and the w_j have eigenvalue two.

Commuting Y_b^j, Y_c^j through w_j, and using (11), gives

$$e_{j,x} U_q = U_{q|x} e_{j,x} \quad (q = b, c),$$

where $b|0 = a$, $b|1 = b$, $c|0 = b$, $c|1 = a$. For $x = 1$, the imposed X_j/U_a and X_j/U_b commutations move the section observable to the left. For the active generator the same transport, the flag swap and $Y_a^j = X_j U_c$ on C_j give

$$e_{j,0} U_a = U_c e_{j,1}, \qquad e_{j,1} U_a = U_c e_{j,0}.$$

Taking adjoints and summing in j gives all six equations (2) for the global U_q and S_x. Theorem 2.1 applies inside the faithful support host.

Remark 5.2 (The limit of label commutation)**.** Requiring every label to commute with every already-labelled fine effect would be inconsistent: summing the fine effects of a context containing Z_j would make X_j commute with Z_j. Their anticommutation forbids this. The property proved and used is commutation with the unlabelled base marginals and the extracted bridge. It permits all the incompatible square contexts.

6 The Bellaterra model and finite-dimensional approximation

6.1 An expected matrix-ultraproduct model

Let H_L have orthonormal basis the binary words of lengths at most L. Let t_x prepend x, killing the top level, and let u_q be the Bellaterra permutation on each level. Put

$$\rho_L \delta_w = \frac{2^{-|w|}}{L+1} \delta_w, \qquad \phi_L(z) = \mathrm{Tr}(\rho_L z).$$

Each level has state mass $1/(L+1)$. Writing Π_k for its projection, direct action on basis words gives

$$\begin{aligned} \rho_L t_x &= \tfrac{1}{2} t_x \rho_L, & [\rho_L, u_q] &= 0, \\ u_q t_x &= t_{q \cdot x} u_{q|x}, & u_q^2 &= 1, \\ t_x^* t_y &= \delta_{xy}(1 - \Pi_L), & \sum_x t_x t_x^* &= 1 - \Pi_0. \end{aligned} \tag{13}$$

These are exact finite-level identities, not approximate operator-norm relations.

Fix a free ultrafilter ω and let

$$\mathcal{O} = (B(H_L), \phi_L)^\omega$$

be the Ocneanu ultraproduct. We use the following numbered imports from Ando–Haagerup [1]. Proposition 3.2 makes its state faithful and normal. Lemma 4.13 puts bounded sequences with a common bounded modular spectral window into the normalizer algebra. Theorem 4.1 identifies the modular group coordinatewise. Proposition 3.15 characterizes normalizer representatives by commutation with the vector-state support in the Groh–Raynaud algebra. Corollaries 3.27–3.28 identify its standard form on the double-support compression.

The shifts and permutations in (13) have fixed modular spectral support. Their images T_x, U_q therefore exist in $\mathcal{O}$, with eigenvalues $1/2$ and one. Both boundary projections have state mass $1/(L+1)$; they vanish in the strong-star ideal. Their images satisfy the exact Cuntz and Bellaterra relations. The algebra

$$P = W^*(T_0, T_1, U_a, U_b, U_c) \subset \mathcal{O}$$

is globally modularly invariant. Takesaki's theorem gives a faithful normal state-preserving expectation $\mathcal{O} \to P$.

Theorem 2.1 and the KMS identity determine every normal-form moment:

$$\varphi(T_\mu U_g T_\nu^*) = \delta_{\mu,\nu}\, 2^{-|\mu|} \delta_{g,e}. \tag{14}$$

Different degrees vanish by modular invariance. For equal lengths, move T_μ using (1) and apply $T_\nu^* T_\mu = \delta_{\nu,\mu}$. The multiplication rules in [14, Lemma 3.4] show that these words span a dense star algebra. The state is thus fully identified.

To identify the factor type, the Bellaterra action is countable and essentially free. In Yoshida's Definition 2.6 [22], a nongeneric point for g must be fixed by g. The identity is generic everywhere; for all other elements the fixed set is null. Countability gives full measure to the set generic for every element. Yoshida's Theorem 3.13 identifies the unique critical state on the maximal self-similar algebra, and Theorem 4.1 makes its GNS closure a type $\mathrm{III}_{1/2}$ factor. Formula (14) identifies our (P, φ) with that state and closure. The type computation is additional information; noninjectivity already follows from Theorem 2.1.

6.2 POVM lifting on both sides

Theorem 6.1 (Two-sided approximation)**.** *For every fixed k and every finite pair of POVM families, the standard-form correlation of $(M_k \bar{\otimes} P, \mathrm{tr}_k \otimes \varphi)$ belongs to C_{qa}. In particular the full perfect correlation of Section 4 belongs to C_{qa}.*

Proof. First work in $\mathcal{O}$. Let $\mathcal{Q}$ be the C^*-algebra of bounded normalizer sequences and $\pi : \mathcal{Q} \to \mathcal{O}$ its quotient map. A finite POVM (c_i) in $\mathcal{O}$ has positive lifts $d_i \in \mathcal{Q}$: lift $c_i^{1/2}$ and take the product of its adjoint with itself. Put

$$s = \sum_i d_i, \qquad r = \max(s, \tfrac{1}{2}\mathbf{1}), \qquad f_i = r^{-1/2} d_i r^{-1/2}.$$

Continuous functional calculus stays in $\mathcal{Q}$. Since r commutes with s and dominates it, $h = 1 - \sum_i f_i \geq 0$ coordinatewise. Also $\pi(s) = \pi(r) = 1$, so $\pi(h) = 0$. Replace f_1 by $f_1 + h$. We have obtained coordinatewise positive effects summing exactly to one, with images c_i.

Apply this to every Alice POVM and also to each left POVM $c_{y,b} = J_{\mathcal{O}} B_b^y J_{\mathcal{O}}$ associated with Bob's commutant POVM. Denote the coordinate lifts by $a_{x,a,L}, c_{y,b,L}$. Both families are normalizer representatives; normalization on one side does not constrain the other.

Let N be the Groh–Raynaud algebra on the Hilbert ultraproduct, $\Omega = (\rho_L^{1/2})_\omega$, p the support of its vector state in N, and J_ω the coordinatewise modular conjugation. Put $q = pJ_\omega p J_\omega$. Proposition 3.15 says each normalizer representative x commutes with p. It also commutes with $J_\omega p J_\omega \in N'$. Thus it commutes with q, as does $J_\omega x J_\omega$. The projection q reduces both lifted families and $q\Omega = \Omega$.

Corollaries 3.27–3.28 identify the standard form of $\mathcal{O}$ with this compression. Since the compression is reducing on both sides, its projections can be removed from the product, giving exactly

$$\langle \xi_{\mathcal{O}}, a J_{\mathcal{O}} c J_{\mathcal{O}} \xi_{\mathcal{O}} \rangle = \lim_\omega \langle \rho_L^{1/2}, a_L J_L c_L J_L \rho_L^{1/2} \rangle. \tag{15}$$

Removing arbitrary nonreducing compressions would not give this identity. Each coordinate on the right is a finite-dimensional spatial strategy: $L^2(B(H_L)) = H_L \otimes \overline{H_L}$, with left and right matrix actions. For any accuracy the finitely many probability constraints select an intersection of finitely many ultrafilter-large sets. Choosing one coordinate in that intersection approximates the entire correlation array. This proves membership in C_{qa}.

For the passage from P to $\mathcal{O}$, use the expectation and [4, Lemma 3.1]: $K = \overline{P\xi_{\mathcal{O}}}$ identifies with $L^2(P, \varphi)$, and modular invariance gives $J_{\mathcal{O}}|_K = J_P$. Embedding the left effects in $\mathcal{O}$ and mirroring the right effects with $J_{\mathcal{O}}$ preserves every probability. Finally tensor the coordinates and expectation with M_k. The same argument proves the assertion for every fixed matrix amplification. □

This approximation includes unsupported pairs. It is stronger than convergence of the winning score alone. The finite Toeplitz models supply existence and approximation; they are not used as finite cutoffs for the exact zero identities in Proposition 4.1.

7 Exclusion, non-closedness and local quantum fields

Proof of Theorem 1.1. The predicate and counts are established in Section 3. Sections 4 and 6 give its perfect standard-form correlation $p_* \in C_{qa}$ on $M_9 \bar{\otimes} M_2 \bar{\otimes} P$.

If Alice's specified ambient host is injective and her marginal normal, Lemma 1.2 passes to an injective faithful support corner. Section 5 gives the hypotheses of Theorem 2.1 in that corner. Its expected $L(\Gamma)$ contradicts injectivity.

Now suppose only Bob's specified host B is injective and his marginal normal. Apply Lemma 1.2 with Bob distinguished. On the standard form of its injective support corner B_s, Alice's compressed POVMs belong to the full commutant B_s'. This commutant is the standard opposite of B_s, hence injective. The natural-cone vector is cyclic and separating on both sides, so its restriction to B_s' is faithful and normal. Regard B_s' as Alice's host and apply the preceding argument. All probabilities were preserved by the support reduction. This proves exclusion for either distinguished party.

Every finite-dimensional strategy has a normal type-I, hence injective, host. The convergence to p_* proves that their winning probabilities have supremum one. No individual injective-host strategy attains it. □

Corollary 7.1 (Non-closedness)**.** *The sets $\mathcal{I}_A$, $\mathcal{I}_B$, their union and their intersection are not closed. Also $p_* \in C_{qa} \setminus C_{qs}$.*

Proof. Choose $p_n \in C_q$ converging to p_*. Each belongs to both normal injective-host sets, whereas the limit belongs to neither. Every spatial strategy, including an infinite-dimensional one, is normal on the ambient type-I algebra $B(H_A) \otimes 1$, using its vector state or density operator. It therefore cannot realize p_*. □

This proof requires no identification of the entire closure of $\mathcal{I}_A$ or $\mathcal{I}_B$. The assertion concerns exact attainment. Since the same sequence has loss tending to zero, there is no uniform positive loss bound for injective hosts.

Corollary 7.2 (A specified quantum-field-theory setting)**.** *Let two spacelike separated regions carry Alice's and Bob's measurements. If one specified local von Neumann algebra is injective and the strategy's marginal is normal there, the strategy cannot win G_{B} perfectly. In particular this applies to locally normal strategies for double cones in the setting of Halvorson's Proposition 2.31 [9].*

Proof. Microcausality puts the other region's effects in the specified local commutant, giving a forbidden normal injective host. The named sufficient setting assumes microcausality, the spectrum condition, the funnel property and construction from a Wightman theory satisfying asymptotic scale invariance in Halvorson's Definition 2.30. Proposition 2.31 concludes that each double-cone algebra is the hyperfinite type III_1 factor tensored with its abelian center. That tensor product is injective, so the preceding argument applies. □

The asymptotic scale-invariance assumption has specific content. In Definition 2.30 a generating Wightman field with vanishing vacuum one-point function admits a monotone normalization $N(\lambda)$ and rescaled test functions $f_\lambda(x) = N(\lambda) f(\lambda^{-1} x)$. Its two-point expectations converge as $\lambda \to 0$, nontrivially for some test function, while both $\|\Phi(f_\lambda)^* \Phi(f_\lambda)\Omega\|$ and $\|\Phi(f_\lambda)\Phi(f_\lambda)^*\Omega\|$ stay bounded. The corollary uses that numbered formulation; the historical attribution to Buchholz–D'Antoni–Fredenhagen is through Halvorson's statement. It asserts no conclusion for all quantum-field-theory nets, singular local states or noninjective local algebras.

8 A nontracial dividing line

The mechanism suggests a general criterion, provided a presentation recovers both parties. An abstract presentation of Alice's extracted algebra alone is insufficient for this purpose.

Definition 8.1 (Complete modular presentation)**.** For a finite game, a complete modular presentation consists of a unital C^*-dynamical algebra $(\mathcal{B}, \gamma)$ and analytic POVM families $(a_{x,a})$, $(b_{y,b})$ in $\mathcal{B}$,

with the following properties. Every perfect normal strategy, after faithful support reduction, yields a unital representation $\pi : \mathcal{B} \to M$ with

$$\pi(\gamma_t(z)) = \sigma_t^{\phi}(\pi(z)), \qquad p(a, b \mid x, y) = \langle \xi, \pi(a_{x,a}) J \pi(b_{y,b}) J \xi \rangle.$$

Conversely the prescribed standard-form families define the game strategy in every admissible KMS-GNS representation. Admissible states are the KMS states with the sign convention (1), whose GNS vector is separating and whose modular group implements γ. Equivalently one may use their usual finite-temperature KMS-GNS formulation, with that sign convention.

The recovery of the displayed probabilities is part of completeness. Merely recognizing the relations among Alice's measurements does not imply it. The forward representation is unital in the faithful support host, so its von Neumann closure is globally modularly invariant.

For such states define the zero-loss subset K_G by the finite equations

$$\psi(a_{x,a}\gamma_{-i/2}(b_{y,b})) = 0 \quad \text{whenever } \pi(x, y) > 0, V(a, b \mid x, y) = 0. \tag{16}$$

The left sides are affine functionals of ψ. In the standard form they are probabilities by (6); hence they are real and nonnegative on the admissible KMS state space. Their common zero set is therefore a face. Write $\pi_\psi(\mathcal{B})''$ for the GNS closure.

Theorem 8.2 (The KMS-face criterion)**.** *For a complete modular presentation, a perfect normal strategy exists exactly when K_G is nonempty. A perfect strategy with a normal injective host exists exactly when K_G contains a state whose GNS closure is injective.*

Proof. A perfect normal strategy yields $\psi = \phi \circ \pi$. Modular covariance and the faithful normal modular KMS characterization [11, Theorem 2.14] place this state in the admissible class. Completeness recovers both families, so its zero forbidden probabilities are exactly (16). The algebra $N = \pi(\mathcal{B})''$ is modularly invariant in M, and Takesaki's theorem gives a state-preserving expectation $M \to N$. The faithful normal restriction identifies N with the GNS closure of ψ. If M is injective, so is N.

Conversely, take $\psi \in K_G$ and its prescribed GNS standard form. The two analytic POVM families and their mirrors have the probabilities in (16). All forbidden probabilities vanish, so they win. If the GNS closure is injective, this standard form itself is the required normal injective host. □

This is a criterion under a completeness hypothesis, not a claim that every finite game has such a presentation.

Proposition 8.3 (Essentially free self-similar models)**.** *For a countable essentially free self-similar action on an alphabet of size $d \geq 2$, its canonical critical-KMS GNS algebra is injective if and only if the group is amenable. For a game forcing the corresponding modular self-similar relations, the same equivalence for perfect injective play holds provided the canonical state is a perfect model of the game's complete presentation.*

Proof. Replace 2^{-n} by d^{-n} in the cylinder proof of Theorem 2.1. It gives an expected regular group von Neumann algebra in every faithful normal realization. A nonamenable group therefore excludes injective hosts. For the amenable direction, essential freeness and countability imply full-measure genericity exactly as in Section 6. Yoshida's Proposition 4.2 [22] makes the canonical GNS algebra AFD, hence injective; his Theorem 4.1 gives its type. The perfect-model premise puts this state in K_G, and Theorem 8.2 supplies the strategy. Extra game constraints could otherwise make K_G empty, so the premise is necessary. □

Remark 8.4 (Nuclear presentations)**.** The usual nuclearity–injectivity implication makes the GNS closures of a nuclear presentation injective. A further universality argument is often available by selecting a perfect factor fiber and transferring its strategy to the hyperfinite type III_1 factor. The relevant absorption statement is Haagerup–Musat's Lemma 3.8 [8]: for a hyperfinite *factor* N with separable predual, $N\bar{\otimes}R_\infty \cong R_\infty$. It does not assert this for arbitrary algebras with center. We record this additional universality route only as a sketch; neither a general factor-selection theorem for modular presentations nor that route is needed for G_{B}.

9 Open problems and how this work was done

A smaller predicate is a natural next target. The present count reflects the use of the same nine label contexts on all five base questions and the explicit conditional blocks. Which of these tests can be shared without losing the common-marginal argument?

A quantitative question is the price of nearly perfect injective play. The present result gives no robustness rate. One could seek bounds in terms of dimension, entropy, modular spectral concentration or approximation complexity, together with matching constructions.

The tracial analogue remains separate. Paulsen's Problems 2.14 and 2.16 [19] distinguish uniformly amenable trace correlations and ask about amenability after passing to a game quotient. Our modular-cylinder method does not resolve those problems. It uses a nontracial state and a forced Cuntz scaling that a finite trace cannot support.

More generally, which nonamenable von Neumann algebras can a finite game force with expectation? Can the present label transport be made robust, or strengthened to self-testing? The result does not identify a unique perfect correlation or classify every perfect host. Its quantum-field-theory consequence is the conditional one in Corollary 7.2; no energy or preparation bound is derived.

How this work was done

The authors developed the construction and its proofs with assistance from AI models from more than one developer. They take responsibility for every claim. The analytic inputs are identified where used, including the base support and bridge lemmas, critical-KMS uniqueness, essential freeness, modular expectations and matrix-ultraproduct standard forms.

The ancillary programs provide two kinds of evidence. The exhaustive exact checker proves finite algebraic zero identities for the specified ideal operators on every supported forbidden event, together with the POVM and extraction identities. The independent predicate program reconstructs the complete finite rules, and a separately implemented operator program checks the recorded random sample by a different rewriting method. Recorded logs state their coverage and resource measurements. These checks use exact rational-pair arithmetic; they do not infer infinite-dimensional identities from finite-depth numerical matrices.

The argument for arbitrary normal strategies is the analytic proof in Section 5. The approximation proof is the two-sided normalization and reducing-support argument in Theorem 6.1. Neither conclusion follows from a computational pass alone. The source distribution contains the game, programs, recorded outputs, a runner that stops on a failed check, and an LF-encoded SHA-256 manifest. Run `python -u anc/run_checks.py -output-dir CHECK_DIRECTORY` from the paper directory, choosing a fresh directory outside the ancillary source tree. Assertions must remain enabled.

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