Perfect play in a finite nonlocal game requires infinite entanglement entropy

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Abstract

How much infinity can a finite nonlocal game require? We construct a finite two-player game with classical questions, classical answers and a Boolean winning rule. It has a perfect strategy on a tensor product of separable Hilbert spaces. Every perfect normal spatial strategy contains, under local isometries, the same pure entangled state with infinite entropy. For mixed strategies, both marginal entropies are infinite and the pure state remains a tensor factor. Thus infinite dimension can be strengthened to infinite entropy without losing spatial attainment.

The proof first constructs a rational Bell functional that extracts a state whose Schmidt multiplicities are Catalan numbers. A finite compiler then turns rational linear identities among synchronized projections into winning conditions, using a finite maximally entangled ancilla. Near the optimum, the minimum marginal entropy diverges. Explicit finite-dimensional strategies give an upper bound of order the inverse fifth root of the loss. This exponent is optimal among scalar spectral filters of the canonical state, with the canonical measurements fixed; an unrestricted matching lower bound remains open.

AI systems assisted with this work. The authors take responsibility for its claims; see "How this work was done." This preprint has not been peer reviewed.

Contents

1 Introduction **3**
1.1 Prior work and scope of the contribution . . . 4
1.2 Proof route . . . 5

2 Normal states and support calculus **5**

3 The Catalan relations and the pure tensor factor **6**
3.1 Five bipartite equations . . . 7

4 The normal flagged magic-square bridge **8**
4.1 Questions, calibration and exact bindings . . . 8
4.2 Normal bridge theorem . . . 8
4.3 A shared question fixes transported labels . . . 10

5 Rational projection equations that certify products **10**
5.1 Graphs and triangles . . . 11
5.2 Binding prescribed parameters . . . 11

6 A common Catalan table from two bridges **12**
6.1 Nine corners and the small polynomial input . . . 12
6.2 Extraction and mirrors . . . 13
6.3 Simultaneous normal realization . . . 13

7 The rational Bell functional **14**
7.1 Literal coefficient convention . . . 15

8 Rational rows as Boolean winning conditions **15**
8.1 A certified scalar matrix algebra . . . 15
8.2 Signed reflections and column graphs . . . 16
8.3 The explicit game and proof of Theorem C . . . 17

9 Entropy near the optimum **18**
9.1 Divergence at bounded entropy . . . 18
9.2 A spectral taper and its exact loss . . . 18
9.3 Restricted lower bound for all spectral filters . . . 20

10 Discussion **21**

How this work was done **21**

A Reproducibility **22**

1 Introduction

How "inherently infinite" can a finite quantum correlation be? Coladangelo and Stark posed this question and exhibited a finite correlation that requires infinite dimension [5]. Dimension does not measure the amount of entanglement: a state can have infinitely many nonzero Schmidt coefficients and finite entropy. The game Γ gives this distinction in a perfect-winning setting: it has a perfect spatial strategy, no finite-dimensional perfect strategy, and a finite-entropy perfect strategy [8, Theorems 1.1 and 13.1]. A different rung is provided by Douglas and Mghirbi: a finite 0/1 game has perfect normal commuting play in type III and none when either ambient algebra is semifinite [7, Theorem 1.1]. We ask whether a finite game can force infinite entropy at every perfect spatial strategy.

We answer this question affirmatively. The optimum is attained by a normal state on a tensor product. The conclusion therefore concerns an actual spatial strategy, rather than a limit of strategies or a commuting-operator optimum outside the tensor-product model. The game has a perfect normal spatial strategy, and perfect play extracts the same pure state with infinite entropy from every such strategy, including mixed states. The rational-row compiler gives a separate way to convert synchronized additive identities into Boolean tests.

A *normal spatial strategy* consists of local POVMs on Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ and a positive trace-class operator ρ of trace one on $\mathcal{H}_A \otimes \mathcal{H}_B$. Its probabilities are

$$p(a, b \mid x, y) = \operatorname{Tr}\bigl(\rho(E_{a|x} \otimes F_{b|y})\bigr).$$

We use entropy in bits, $S(h) = -\operatorname{Tr}(h \log_2 h)$, allowing the value $+\infty$. For pure bipartite states this is entanglement entropy. For mixed states our conclusion is infinite marginal entropy together with a pure entangled tensor factor.

Let $\mathcal{W}$ be the words generated from the empty word by appending A at any height and B only at height at least two. The letters respectively raise and lower the height by one. Write $n(w)$ for the number of A letters. There are $\mathrm{Cat}_n = (n+1)^{-1}\binom{2n}{n}$ words of size n. Define

$$\psi_{\mathrm{Cat}} = \frac{1}{\sqrt{2}} \sum_{w \in \mathcal{W}} 2^{-n(w)} |w, w\rangle. \tag{1}$$

The Catalan generating function normalizes this vector. Its reduced density has infinite entropy.

Theorem C (the game). *The explicitly specified game G has* 169 460 *questions per party, at most* 256 *answers per question, the uniform distribution on all question pairs, and a $\{0,1\}$ winning predicate. It has a perfect normal spatial strategy on $\ell^2 \otimes \ell^2$. For every perfect normal spatial strategy there are local isometries V_A, V_B and a normal state τ such that*

$$(V_A \otimes V_B)\rho(V_A \otimes V_B)^* = |\psi_{\mathrm{Cat}}\rangle\langle\psi_{\mathrm{Cat}}| \otimes \tau. \tag{2}$$

Both original marginal entropies are infinite. No finite-dimensional strategy is perfect, and finite-dimensional winning probabilities approach one.

Here and below local isometries are asserted on the state supports and extended to unused space by enlarging the target auxiliary spaces. The finite specification of G is in `game.zip`. Sections 5–8 also give finite rules generating every constraint. This is a large construction; the size is an explicit bound, not an optimality claim.

The core is the following Bell equality. The functional uses the ordinary conditional probabilities without an additional question-distribution normalization.

Theorem A (the Bell functional). *The explicitly tabulated rational Bell functional B has* 5 625 *questions per party, at most* 12 *answers, and* 73 769 *nonzero coefficients with common denominator* 2048*. For every normal spatial strategy, $B \le 0$. A normal spatial strategy attains zero, and every such equality state satisfies* (2)*. No finite-dimensional strategy attains zero; finite-dimensional values approach zero. If H is a lower-bounded local Hamiltonian with $\operatorname{Tr} e^{-\beta H} < \infty$ for some $\beta > 0$, its mean energy in either equality marginal is infinite.*

Section 8 states and proves Theorem B, which compiles finitely many rational linear identities among synchronized projections into Boolean tests. Section 9 proves Theorem D: the minimum entropy tends to infinity near the optimum, and explicit strategies give an upper power of 1/5. The matching lower power is proved only for scalar spectral filters of the canonical strategy.

1.1 Prior work and scope of the contribution

Coladangelo–Stark concern infinite-dimensional spatial correlations [5]. The game Γ supplies an earlier Boolean game with spatial perfection, no finite-dimensional perfect strategy, and finite-entropy attainment [8, Theorems 1.1 and 13.1]; its displayed pure perfect strategy attains the minimum entropy $\log_2 24 + 7/9$ bits. Neither dimension separation alone nor an infinite-rank self-test implies the every-optimum entropy assertion here. Pauwels proves finite-dimensional nonattainment for I_{3322} [23]. Coladangelo's I_{3322} theorem [4] is another dimension separation. Its attaining state has finite entropy by the tail argument in Section 10.

There is a different route to infinity outside the tensor-product model. Paddock and Schmidt self-test exact embezzlement with a commuting-operator correlation that has no tensor-product realization [21]. Harris characterizes exact embezzlement through Cuntz isometries and a quasi-free state, with a type III conclusion [12]. Beside these results, Douglas and Mghirbi give a finite 0/1 game with 243 uniformly sampled pairs that is perfect on $R_{1/2}$ and has no perfect strategy with a normal marginal on a semifinite ambient algebra of either party [7, Theorem 1.1]. This is a type-III-only requirement for normal play, rather than a spatial entropy requirement. The structural relation between embezzlement and type III algebras is developed in [26, 27]; algebraic embezzlement is studied in [16]. Exact embezzlement is impossible in tensor products even with infinite entropy [2]. Our state is spatial and is not an exact embezzler. These are different levels of the question, not priority claims about infinite dimension, type III algebras, or embezzlement.

Limit-only resource requirements occur in the three-player game of Ji, Leung and Vidick [14] and the two-player construction of Coladangelo [3]. They differ from normally attained perfection. Mančinska's flagged magic-square game supplies the finite nonmaximally entangled bridge used here [17]. We prove the needed normal-state extension, rather than importing a finite-dimensional self-test into infinite dimension.

Projection partitions and synchronous game algebras are standard [9, 13, 22]. Known reductions encode multiplicative relations and relations between game algebras [25, 10, 11]. Sums of projections also underlie the correlation self-tests of Mančinska, Prakash and Schafhauser [18] and the nonclosure constructions of Musat and Rørdam [20]. Those correlation statements do not supply the Boolean row tests below. Chen and Zhao encode real-coefficient unitary presentations by graph and column projections in synchronous correlations [1, Section 4]. Their result uses prescribed tracial correlation values. Our Boolean compiler instead enforces rational additive identities among the retained synchronized projections by zero-probability tests after finite maximally entangled amplification. To our knowledge, this extension is new as stated. The graph and Hadamard-column arguments below give the additional identification proofs.

To our knowledge, Theorem C is the first finite two-player Boolean game with a normally

attained perfect strategy for which every perfect normal spatial strategy has infinite marginal entropy. This bibliographic statement rests on the cited comparisons and targeted literature searches. It is not an exhaustive priority certificate. In particular, we claim no priority for nontracial perfect games. The known synchronous obstruction of Kim, Paulsen and Schafhauser [15, Theorem 3.10 and Corollary 3.11] explains why the nonsynchronous bridge questions are essential.

1.2 Proof route

The operator relations in Section 3 force the Catalan spectrum. Five bipartite equations force its pure tensor factor. Two calibrated flagged magic-square bridges supply density ratios. Shared opposite questions fix the gauge of duplicated bridge measurements. Graph projections and triangles then encode the required products as rational linear projection identities. Their Bell penalties prove Theorem A. The row compiler replaces those penalties by Boolean tests and proves Theorem C. Exact finite computations check the exported objects; the infinite statements follow from the written proofs.

2 Normal states and support calculus

We collect the support facts used throughout the proof. They are elementary operator facts, not additional novelty claims. A trace-class density has separable support. Thus even if the ambient spaces are nonseparable, all state-bearing arguments take place on separable spaces. We purify mixed states on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E$; no queried measurement acts on E.

Perfect matching. For opposite-party effects E, F, zero mismatch implies

$$0 = \langle E + F - 2EF \rangle = \langle (E - F)^2 + (E - E^2) + (F - F^2) \rangle. \tag{3}$$

Each positive term annihilates the purification Ω. In coefficient-operator coordinates for the cut $A \mid BE$, the first identity is $EK = K(F \otimes I_E)^T$. The range of K is dense in Alice's state support. Hence E preserves that support, and self-adjointness makes it reducing. Repeat on the $B \mid AE$ cut. On these supports the positive sharpness defects vanish: if h is injective there and $D \geq 0$, then $\mathrm{Tr}(hD) = 0$ gives $D^{1/2}h^{1/2} = 0$, and dense range gives $D = 0$. Thus matched effects restrict to projections. Multiplying the intertwining equation and its adjoint gives $[E, h_A] = 0$, and similarly on Bob. This order of reasoning establishes reduction before using projectivity.

Projective compression. If P projects onto a state support, $0 \leq E \leq I$, and $PEP = Q$ is a projection on $P\mathcal{H}$, then E reduces $P\mathcal{H}$. Indeed $\langle v, (I - E)v \rangle = 0$ for $v \in \operatorname{ran} Q$ forces $Ev = v$. For $v \in \ker Q \cap P\mathcal{H}$, positivity forces $Ev = 0$. Self-adjointness finishes the reduction argument.

Sharp marginals. A POVM with sharp coarse marginals is their commuting joint spectral measurement. Each compressed fine effect lies below the corresponding marginal projections. If $0 \leq q \leq Z$ and Z is a projection, then $q = ZqZ$. Summing over the other outcomes shows that the marginals commute and that their joint products equal the fine effects (or zero for omitted outcome patterns). If this argument starts with compressed effects, projective compression is used only after sharpness has been proved.

Forbidden events. Suppose D_A, K_A, K_B are support projections and $(K_A - K_B)\Omega = 0$. A zero event for D_A and K_B gives $D_A K_B \Omega = 0$, hence $D_A K_A \Omega = 0$. Dense Schmidt range gives $D_A K_A = 0$ on the support, and taking adjoints gives the reverse product. This proof needs no density commutation in the orthogonality step.

A full synchronized question has zero probability of unequal answers. Coarse-graining gives (3) for every answer subset. Consequently all its support effects are projections commuting with the local marginal. We always mean these support-restricted projections when using local polynomial identities.

3 The Catalan relations and the pure tensor factor

Let A, B, C be bounded operators and P, Q projections. The local relations are

$$A^*A = I, \qquad AA^* + BB^* = I - P, \qquad B^*B = I - P - Q, \tag{4}$$

$$CP = P, \qquad CQ = Q/2, \qquad 0 \le C \le I, \tag{5}$$

$$CA = AC/2, \qquad CB = 2BC. \tag{6}$$

For an injective normal density h, the accompanying modular relations are

$$hA = Ah/4, \qquad [h, B] = [h, C] = 0. \tag{7}$$

The canonical representation on $\ell^2(\mathcal{W})$ appends the legal letter, with B zero where it is not allowed. It has $C|w\rangle = 2^{-\mathrm{ht}(w)}|w\rangle$, $P = |\varnothing\rangle\langle\varnothing|$ and Q the height-one projection.

Lemma 3.1 (Normal Catalan classification)**.** *Every normal density satisfying (4)–(7) has, on its support, the form*

$$\mathcal{H} = \ell^2(\mathcal{W}) \otimes \mathcal{J}, \qquad h = h_{\mathrm{Cat}} \otimes \sigma, \qquad h_{\mathrm{Cat}} = \frac{1}{2} \sum_{w \in \mathcal{W}} 4^{-n(w)} |w\rangle\langle w|,$$

where σ is a normal density and the five operators are the canonical operators tensored with identity.

Proof. The equations $CP = P$ and $CQ = Q/2$ make P, Q reducing spectral subprojections of C and imply $PQ = 0$. The operator B is a partial isometry. The three range projections AA^*, BB^*, P are orthogonal and sum to identity. Covariance makes these range projections commute with C.

Put $E_k = 1_{\{2^{-k}\}}(C)$ and $E_\infty = 1_{\{0\}}(C)$. There is no spectrum in $(1/2, 1)$: there B is isometric and would send it above one. The same argument on $E_0 - P$ gives $E_0 = P$. On $E_1 - Q$, B would isometrically map into P, orthogonally to its range, so $E_1 = Q$. Iteration excludes every open interval $(2^{-k-1}, 2^{-k})$. Bounded functional calculus for $CB = 2BC$ justifies these spectral transports. Thus $I = E_\infty + \sum_{k \ge 0} E_k$.

The zero-counter space reduces A, B, C, h. There $P = Q = 0$ and A, B are isometries with orthogonal ranges summing to identity. If $t = \mathrm{Tr}(hE_\infty)$, trace cyclicity and (7) give range masses $t/4$ and t. Their sum is t, so $t = 0$. There is no zero-counter state mass.

Starting with $P\mathcal{H}$, apply legal words in chronological order; write W_w for the resulting isometry from the root space. Different sectors $W_w P\mathcal{H}$ are orthogonal. Different final letters have orthogonal ranges, and a nonempty sector is orthogonal to P. Matching final letters can be canceled using $A^*A = I$ or $B^*B = I - P - Q$, which is identity on each legal predecessor for B. Repeating proves the assertion. On sector w, $C = 2^{-\mathrm{ht}(w)} I$ and

$$hW_w = 4^{-n(w)} W_w h.$$

The root projection commutes with h, so each word-sector projection does too. Its density block is $4^{-n(w)}W_w h_0 W_w^*$, where $h_0 = PhP$.

The closed span of the root sectors reduces all the operators and h. Suppose its orthogonal complement carried positive mass. The compact density on that complement has a largest positive eigenvalue λ, with a nonzero finite-dimensional eigenspace L. The adjoint modular equation is $hA^* = 4A^*h$, so $A^*L = 0$. Since $P = 0$ on the complement, $BB^* = I$ on L. The space L reduces B; its restriction is therefore a unitary. It also reduces C. Multiply $CB = 2BC$ by the unitary B^* on the right and trace on L. This gives $\operatorname{Tr}_L C = 2\operatorname{Tr}_L C$. Positivity forces $C = 0$ there, contradicting the absence of zero-counter mass. Hence the word sectors exhaust the support.

For $n \geq 1$ and $0 \leq m \leq n-1$, reflection of the first bad prefix counts the words with n letters A and m letters B as

$$\binom{n+m-1}{m} - \binom{n+m-1}{m-1}.$$

Summing gives Cat_n; the empty word gives $\mathrm{Cat}_0 = 1$. The generating function $C(z) = 1 + zC(z)^2$ gives $\sum_n \mathrm{Cat}_n\, 4^{-n} = 2$. Thus $\operatorname{Tr} h_0 = 1/2$, and $\sigma = 2h_0$ gives the claimed identification. □

Its entropy is

$$S(h_{\mathrm{Cat}}) = \sum_{n\geq 0} \frac{\mathrm{Cat}_n}{2\,4^n}(1+2n) = \infty, \qquad \frac{\mathrm{Cat}_n}{4^n} \sim \frac{1}{\sqrt{\pi} n^{3/2}}. \tag{8}$$

Infinite rank alone would not give this conclusion. Tensoring with any normal density preserves the divergence.

3.1 Five bipartite equations

On a purification Ω require each party's local relations and

$$(A_A/2)\Omega = A_B^*\Omega, \qquad A_A^*\Omega = (A_B/2)\Omega, \tag{9}$$

$$B_A\Omega = B_B^*\Omega, \qquad B_A^*\Omega = B_B\Omega, \tag{10}$$

$$C_A\Omega = C_B\Omega. \tag{11}$$

Tensor identities and the identity on the environment are suppressed. Both adjoint equations and both factors 1/2 matter.

Lemma 3.2 (Pure extraction)**.** *The local relations and* (9)–(11) *imply* (2)*.*

Proof. For a pure vector with coefficient operator K, the two A equations read $A_A K = 2K\overline{A_B}$ and $A_A^* K = K A_B^T/2$. Combining either with the adjoint of the other gives $h_A A_A = A_A h_A/4$. The B equations and the C equation give the remaining commutators. The same holds on Bob. For a purification, use the two original local cuts. Paired equations ensure support reduction, without inverting K.

Lemma 3.1 identifies the word sectors on both sides. The C equation gives $P_A\Omega = P_B\Omega$ by functional calculus. Put $\Omega_0 = P_A\Omega$. Transferring adjoint letters in their actual order across the vector gives

$$(W_w^{A*} \otimes I)\Omega = 2^{-n(w)}(I \otimes W_w^B)\Omega.$$

The root projection on the transferred side selects the same word sector. Consequently

$$(P_w^A \otimes I)\Omega = 2^{-n(w)}(W_w^A \otimes W_w^B)\Omega_0.$$

The sectors exhaust the state and $\|\Omega_0\|^2 = 1/2$. Their sum is $\psi_{\mathrm{Cat}} \otimes \eta$, where $\eta = \sqrt{2}\Omega_0$ lies in the root multiplicities and the environment. Tracing out the environment proves the mixed-state assertion. □

4 The normal flagged magic-square bridge

The magic-square relations originate with Mermin and Peres [19, 24]. We use Mančinska's flagged magic-square (FMS) geometry [17]. We spell out the predicate and prove its normal extension because the published finite-dimensional statement does not suffice here. All indices start at zero.

4.1 Questions, calibration and exact bindings

Alice has R_0, R_1, R_2, T_0, T_1 and Bob has C_0, C_1, C_2, U_0, U_1. A row question has a flag answer f or a triple of product $+1$. A column question has f or a triple of product $+1, +1, -1$ according to its column. The magic square is

$$\begin{pmatrix} XI & IX & XX \\ IZ & ZI & ZZ \\ XZ & ZX & YY \end{pmatrix}. \tag{12}$$

Here the displayed matrices describe a canonical realization; the predicate only checks the corresponding signs. Select column-zero outcomes $c_0 = (+++)$, $c_1 = (+--)$ and row-zero outcomes $r_0 = (+++)$, $r_1 = (+--)$. Question T_x has answers $+, -$ and the three column-zero triples other than c_x; U_y analogously retains all row-zero triples other than r_y.

On R_i, C_j, reject flag/nonflag mismatch and otherwise check the intersecting cell; two flags are accepted. On T_x, C_0, a sign is allowed only against f or c_x; a retained triple must equal the opposite triple. On R_0, U_y use the analogous rule with r_y. On T_x, U_y, reject two signs $a, b \in \{+1, -1\}$ unless $ab = (-1)^{xy}$; all pairs with a nonsign answer are accepted. Other question pairs are accepted. For a conditional instance add an inactive answer $\perp$, accept two inactive answers, and reject exactly one inactive answer. Bind each noninactive event to its synchronized active projection and all row/column flags to the synchronized flag.

Let M be the nonflag magic support. The four column-zero projections $m_{00}, m_{01}, m_{10}, m_{11}$ correspond respectively to $(+++), (+--), (-+-), (--+)$. On Alice, m_{00} is read from the retained $(+++)$ answer of T_1 and the other three from T_0. On Bob all four come from C_0. Each of these chosen occurrences is matched to its opposite synchronized binary question. The positive nonflag ZI event is read from Alice's R_1 component 1 and Bob's C_1 component 1. The positive IX event comes from Alice's R_0 component 1 and Bob's C_1 component 0. Put

$$x = I - 2M + 2E_{ZI,+}, \quad z = I - 2M + 2E_{IX,+}, \quad t_{ab} = x^a z^b m_{00}.$$

These are reflections extended by identity outside M, and $t_{ab}^* t_{ab} = m_{00}$, $t_{ab} t_{ab}^* = m_{ab}$.

4.2 Normal bridge theorem

Lemma 4.1. *A perfect normal conditional FMS strategy has on its active supports the decomposition*

$$\mathcal{H}_{\text{active}} = \mathcal{H}_f \oplus (\mathbb{C}^4 \otimes \mathcal{J}), \qquad h = h_f \oplus (I_4 \otimes h_m).$$

The calibrated transports w_A, w_B from the flag to the selected c_0 corner satisfy $w^*w = f$, $ww^* = m_{00}$ and $hw = 2wh$. On a purification their matched components satisfy

$$\Omega_{c_0c_0} = \sqrt{2}(w_A \otimes w_B \otimes I_E)\Omega_{ff}. \tag{13}$$

The other three magic directions obey the transported versions of this equation.

Proof. First apply Section 2 to the activity, flags and checked cells. They reduce the original state supports and are sharp there. On the compressed nonflag space each compressed fine effect lies below its assigned sharp cell marginals. Thus it is their joint spectral projection. The projective-compression fact then makes the original fine effects reduce the support. Alice's row relations and Bob's column relations transfer through the dense coefficient range, in reversed product order; within a context the products commute.

For completeness, write $a = M_{00}, b = M_{01}, c = M_{10}, d = M_{11}$ for the four corner reflections of the square. Context relations give

$$[a, b] = [c, d] = [a, c] = [b, d] = 0, \qquad (ac)(bd) = -(ab)(cd).$$

Cancel the outside a, d to obtain $cb = -bc$. Commutation of ac and bd then gives $da = -ad$. The pairs (a, d) and (b, c) are anticommuting Pauli pairs commuting across pairs. Each pair exchanges its two eigenspaces unitarily. Hence they generate $M_4 \otimes I_{\mathcal{J}}$. Marginal commutation gives the stated density form. The synchronized magic units identify its coefficient block as $I_4 \otimes D$, with D positive, injective and dense-range. Write the flag coefficient block as F with the same properties.

Calibration makes Alice's sign support $S_x^A = f + \Pi_{c_x}$ and Bob's $S_y^B = f + \Pi_{r_y}$. The selected directions have overlap $\langle c_x, r_y\rangle = \sigma_{xy}/\sqrt{2}$, where $\sigma_{xy} = (-1)^{xy}$. Their compressed coefficient operator is

$$W_{xy} = \operatorname{diag}(F, \sigma_{xy}D/\sqrt{2}).$$

For compressed sign effects E_a, B_b, a forbidden pair gives $E_a^{1/2} W_{xy} (B_b^T)^{1/2} = 0$, and therefore $E_a W_{xy} B_b^T = 0$. Summing the two complete binary effects gives

$$E_a W_{xy} = W_{xy} B_{\sigma_{xy}a}^T = E_a W_{xy} B_{\sigma_{xy}a}^T.$$

Injectivity and dense range imply $B_b^2 = B_b$ and $E_a^2 = E_a$ on the sign supports. Their zero extensions give projective compressions; thus the original sign effects reduce the state supports. Their differences X_x, Y_y obey

$$X_x W_{xy} = \sigma_{xy} W_{xy} Y_y^T. \tag{14}$$

Identify the magic directions using the actual matrix units. Set $H = \operatorname{diag}(F, D/\sqrt{2})$ and $J = \operatorname{diag}(I, -I)$. The three positive equations give a common X, Y with $XH = HY^T$. The remaining one gives $XJH = -JHY^T$, hence $XJ = -JX$. Thus

$$X = \begin{pmatrix} 0 & \beta^* \\ \beta & 0 \end{pmatrix}, \qquad \beta : \mathcal{H}_f \longrightarrow \mathcal{J} \text{ unitary.}$$

Self-adjointness and $XH = HY^T$ imply $[X, H^2] = 0$. Its off-diagonal block gives $h_m\beta = 2\beta h_f$. The corresponding corner transport has the required ratio.

The actual Alice extraction is $w_A = \Pi_{c_0} X_0 f$. Bob's r_0 projection is the retained $(+++)$ answer of U_1, equivalently the sign support of U_0 minus its flag. His extraction to the same c_0 direction is

$$w_B = \sqrt{2}\Pi_{c_0}\Pi_{r_0} Y_0 f.$$

The coefficient $\sqrt{2}$ occurs in extraction, not in the predicate. To see the phase in (13) directly, put $\Omega' = (S_0^A \otimes S_0^B)\Omega$. The zero sign events give $X_0\Omega' = Y_0\Omega'$. Project onto $c_0 \otimes f$. The left side is $(w_A \otimes I)\Omega_{ff}$; the right side is

$$2^{-1/2}(I \otimes fY_0|r_0\rangle\langle c_0|)\Omega_{c_0c_0}.$$

Apply Bob's flag-to-r_0 block and then $|c_0\rangle\langle r_0| = \sqrt{2}\Pi_{c_0}\Pi_{r_0}$. This gives (13).

For a mixed state, first apply the pure-vector calculation on the cut $A \mid BE$. The synchronized opposite effects reduce the Schmidt support, and the sign effects do so after their compressed sharpness is proved. Thus every compressed product used above is the restriction of an original Bob-only product tensored with I_E. The resulting Alice identities hold on her original marginal support. Repeat on $B \mid AE$ for Bob's identities. The component calculation then gives Eq. (13) on the original purification, using $w_A \otimes w_B \otimes I_E$. Tracing E preserves the claimed pure factor. □

4.3 A shared question fixes transported labels

Lemma 4.2 (Label refinement)**.** *Retain the original FMS questions. For a synchronized projection Z, add Alice's $T_0[Z]$ with answers (a, z). Ignore z against the unchanged U_0, U_1, C_0 and use the original predicates; match z to Bob's binary Z. Add the symmetric $U_0[Z]$ against unchanged T_0, T_1, R_0. Also retain activity matching. In every perfect normal strategy the coarse signs equal the original signs and commute with Z on their state supports. Conversely commuting labels give these refinements. Distinct labels require no mutual commutation.*

Proof. Write P_A for the state support and R_A for the sign support. For an arbitrary new fine effect $Q_{a,z}$ first form $q_{a,z} = P_A Q_{a,z} P_A$. Calibration identifies its compressed retained effects and the coarse sign sum R_A. Let X' be its compressed sign difference. The unchanged opposite sign gives $X'W_{0y} = W_{0y}Y_y^T$, with exactly the same dense-range W_{0y} as for X_0. Hence $X' = X_0$. Its coarse effects are $(R_A \pm X_0)/2$, hence projections. Projective compression now makes the original coarse effects reduce P_A.

Label matching gives $P_A(\sum_a Q_{a,1})P_A = Z_A$. Positivity gives $0 \le q_{a,1} \le Z_A$ and $0 \le q_{a,0} \le I - Z_A$. Thus the compressed coarse effects commute with Z_A, and so does X_0. The symmetric argument applies on Bob. Conversely use $E_a Z, E_a(I - Z)$ when the original coarse effects commute with Z. A separate duplicated question for each label always returns the same original sign. This proves the final assertion. □

If a label is block diagonal with flag block Z_f and identical magic blocks Z_m, the commutator says $\beta Z_f = Z_m \beta$. In particular, for a graph projection of a block unitary it certifies transport of that same unitary by the calibrated w. It does not supply a fresh bridge with an independent gauge.

5 Rational projection equations that certify products

All projections in this section are synchronized. We give the finite rules used in the core presentation. The projection-partition viewpoint is standard [9]; the formulas here provide the needed actual multiplication identities. See also the graph-projection encoding in [1, Section 4].

5.1 Graphs and triangles

Let F_i, F_j be orthogonal, $S = F_i + F_j$, and introduce projections G, H, K_G, K_H with

$$G + K_G = S, \quad H + K_H = S, \quad 5H + S = 3F_i + 4G. \tag{15}$$

On S, put $Z = F_i - F_j$, $X = 2G - S$. These are reflections, and $2H - S = (3Z + 4X)/5$. Squaring forces $ZX + XZ = 0$. Thus

$$X = \begin{pmatrix} 0 & u_{ij} \\ u_{ij}^* & 0 \end{pmatrix}, \qquad u_{ij}^* u_{ij} = F_j, \quad u_{ij} u_{ij}^* = F_i.$$

The graph blocks are certified, rather than assumed. Conversely any such unitary corner map gives the four projections in (15).

For a partition $F_1 + F_2 + F_3 = I$ impose the graph tests on all three pairs and add a projection D with

$$3D = 2(G_{12} + G_{23} + G_{13}) - I. \tag{16}$$

Its block matrix has diagonal $I/3$ and off-diagonal $u_{ij}/3$. The $(1,3)$ block of $D^2 = D$ gives $2u_{13} + u_{12}u_{23} = 3u_{13}$, hence $u_{13} = u_{12}u_{23}$. Conversely this product identity makes the entire matrix a projection.

Start with a reference triangle for identity. Its scalar matrix algebra is called the *frame*. It gives scalar matrix units and $\mathcal{H} = \mathbb{C}^3 \otimes \mathcal{K}$. For every unitary label U, use graphs in all three positions. Triangles $(U, 1, U)$ and $(1, U, U)$ identify the placements. A triangle (U, V, W) then certifies $W = UV$; a triangle with product identity certifies inverse labels. Build each word from its proper prefix and final token. Finally a real rational equation $\sum_w q_w U_w = 0$ is precisely

$$2\sum_w q_w G_{12}(U_w) - \Big(\sum_w q_w\Big)(F_1 + F_2) = 0. \tag{17}$$

Its two off-diagonal blocks are the equation and its adjoint. All products therefore refer to actual common operators. Synchronization makes the extracted unitaries commute with the base marginal density.

5.2 Binding prescribed parameters

A prescribed synchronized projection E is linked to the frame by projections EF_i, NF_i with

$$EF_i + NF_i = F_i, \quad \sum_i EF_i = E, \quad \sum_i NF_i = I - E.$$

These symbols are names for projections before the equations identify their products. For every pair ij, introduce $J_{11}, J_{10}, J_{01}, J_{00}$ and impose

$$\begin{aligned} J_{11} + J_{10} &= EF_i + EF_j, & J_{00} + J_{01} &= NF_i + NF_j, \\ J_{11} + J_{01} &= G_{ij}(1), & G_{ij}(s_E) &= J_{11} + J_{00}. \end{aligned}$$

Projection sums give the indicated joint partitions. They force E to commute with the frame and identify the extracted s_E with $2e - I$, where $E = I_3 \otimes e$. In particular these parameters are the specified FMS events, not independent replacements. There are thirteen such parameters below.

These rules are a deterministic finite compiler for the finite word list. Cancellation of adjacent inverse tokens is performed before taking proper prefixes. All coefficients stay rational. Conversely a centralizer unitary assignment with the prescribed parameter projections gives every graph, complement, H, triangle and joint projection simultaneously by these formulas.

6 A common Catalan table from two bridges

6.1 Nine corners and the small polynomial input

Take two conditional FMS instances, indexed $g = 1, 2$. Use nine orthogonal base corners

$$e_0;\ m_{1,00}, m_{1,01}, m_{1,10}, m_{1,11};\ m_{2,00}, m_{2,01}, m_{2,10}, m_{2,11}.$$

Put $M_g = \sum_{ab} m_{g,ab}$, $e_1 = m_{1,00}$, $e_2 = m_{2,00}$, and impose $e_0 + M_1 + M_2 = I$. Bridge g has flag e_{g-1}, magic support M_g, and activity $e_{g-1} + M_g$. Every event binding is a zero-mismatch test in both party orientations. In particular Bob's flag/four-magic/inactive C_0 partition certifies the activity decomposition.

Let $w_g : e_{g-1} \to e_g$ be the calibrated transports. Then

$$v = w_2 w_1, \quad v^* v = e_0, \quad vv^* = e_2, \quad hv = 4vh. \tag{18}$$

The bridge masses and cover give root mass $1/25$, with weights 1, four copies of 2, and four copies of 4. Thus the root cannot vanish.

Use the thirteen parameters e_0, the eight $m_{g,ab}$, and the four positive ZI, IX events. Bind them to the scalar frame by Section 5. Introduce four global unitary axis labels equal to x_g, z_g of Section 4. For $r = 0, 1, 2$ and $k \in \{b_+, b_-, p, q, c\}$ use unitary labels $u_{r,k}$ and impose

$$(u_{r,k} - I)(I - e_r) = 0. \tag{19}$$

A unitary fixing $e_r^\perp$ reduces e_r, so $e_r u = e_r u e_r$. Define the actual polynomials

$$\begin{aligned} b_r &= e_r(u_{r,b_+} + u_{r,b_-})/2, & p_r &= e_r(I + u_{r,p})/2, \\ q_r &= e_r(I + u_{r,q})/2, & c_r &= e_r(2I + u_{r,c} + u_{r,c}^*)/4. \end{aligned} \tag{20}$$

Require $u_{0,p}^2 = u_{0,q}^2 = I$. Hence p_0, q_0 are projections, b_r are contractions, and $0 \le c_r \le e_r$.

For each g, k introduce one unitary $v_{g,k}$ and impose, with $R_{ab} = x_g^a z_g^b$,

$$v_{g,k} = I - (e_{g-1} + M_g) + e_{g-1} u_{g-1,k} + \sum_{ab} R_{ab} e_g u_{g,k} R_{ab}^*. \tag{21}$$

For each bridge, refine both FMS questions as in Lemma 4.2 for the six frame projections $F_1, F_2, F_3, G_{12}(1), G_{13}(1), G_{23}(1)$ and the five projections $G_{12}(v_{g,k})$. This means eleven labels per bridge, twenty-two refined questions per party. Frame preservation comes first. It makes $w_g = I_3 \otimes \widetilde{w}_g$ because the selected magic parameters generate the full M_4 algebra and already commute with the frame. The graph-label refinements then give

$$w_g u_{g-1,k} = u_{g,k} w_g. \tag{22}$$

Thus the repeated operators are transported by the same bridge.

Take two more global unitaries a_+, a_- and $a = (a_+ + a_-)/2$. The remaining nine equations are

$$a = e_2 a e_0, \qquad a^* a = e_0, \qquad aa^* + b_2 b_2^* + p_2 = e_2, \tag{23}$$

$$b_0^* b_0 = e_0 - p_0 - q_0, \qquad p_0 q_0 = 0, \qquad c_0 p_0 = p_0, \tag{24}$$

$$c_0 q_0 = q_0/2, \qquad c_2 a = a c_0/2, \qquad c_0 b_0 = 2 b_0 c_0. \tag{25}$$

Together there are four axis equations, fifteen support equations, ten block equations, nine equations above, two root reflection equations, and one cover: forty-one polynomial inputs.

Expand every parameter $e = (I + s_e)/2$ and apply the prefix/graph rules. They give 358 words and 4 529 rational linear projection rows, fully listed in `core.json`.

For clarity, four reductions from a longer presentation are exact. The axis right sides are reflections because their positive events lie under M_g. The omitted source/range equations for t_{ab} follow from the actual Pauli cells and the stated outcome bindings. Equation (19) removes the final e_r from every sandwich in (20). Finally $t_{ab} = R_{ab}e_g$ and (19) turn $t_{ab}u_{g,k}t^*_{ab}$ into the summand in (21). No unproved equation deletion enters the smaller list.

6.2 Extraction and mirrors

After factoring the synchronized scalar frame, work on the root corner and define

$$A = v^*a, \quad B = b_0, \quad C = c_0, \quad P = p_0, \quad Q = q_0. \tag{26}$$

These are fixed bounded corner expressions in the original effects. From (18)–(25), $A^*A = e_0$, $AA^* = e_0 - P - BB^*$, $B^*B = e_0 - P - Q$. The two counter boundary equations are explicit. Moreover

$$CA = c_0v^*a = v^*c_2a = v^*ac_0/2 = AC/2, \qquad CB = 2BC.$$

All centralizer words commute with the full density. Since $hv = 4vh$, the normalized root marginal satisfies $h_0A = Ah_0/4$. Thus every local relation of Section 3 holds for one common table.

The synchronized scalar M_3 extracts a maximally entangled qutrit. For real-coefficient base words, graph matching gives $T_A\Omega = T_B^*\Omega$; products transfer in reversed order, as their adjoints require. Let Ω_r denote the shared component in e_r. The calibrated vector equations give

$$\Omega_1 = \sqrt{2}(w_{1,A} \otimes w_{1,B})\Omega_0, \qquad \Omega_2 = 2(v_A \otimes v_B)\Omega_0.$$

Since $a = e_2ae_0$, its transfers read $a_A\Omega_0 = a_B^*\Omega_2$ and $a_A^*\Omega_2 = a_B\Omega_0$. Substitute the second component formula and apply v_A^* or v_B^*. This yields both equations in (9). The root centralizer words b_0, b_0^*, c_0 give (10)–(11).

Magic units identify each of the other three directions in a bridge. Hence all nine sectors are local images of the same root component with coefficients 1, four copies of $\sqrt{2}$, and four copies of 2. This factors the entire state into a fixed finite scaffold state and the normalized root state. It is not a postselection argument. Lemma 3.2 gives (2). Every operation acts on original local supports and leaves the purifying environment alone, so the conclusion includes mixed states.

6.3 Simultaneous normal realization

Use the canonical Catalan operators and scaffold density

$$h_{\mathrm{sc}} = \frac{1}{25}\,\mathrm{diag}(h_{\mathrm{Cat}}, 2h_{\mathrm{Cat}}, 2h_{\mathrm{Cat}}, 2h_{\mathrm{Cat}}, 2h_{\mathrm{Cat}}, 4h_{\mathrm{Cat}}, 4h_{\mathrm{Cat}}, 4h_{\mathrm{Cat}}, 4h_{\mathrm{Cat}}). \tag{27}$$

Take identity transports along the bridges. Put $a = |e_2\rangle\langle e_0| \otimes A_{\mathrm{Cat}}$, with the canonical copies for b_r, c_r, p_r, q_r. The modular ratio makes a commute with (27); all nine equations hold.

We justify the unitary encodings simultaneously. The centralizer of an injective trace-class density is a product of finite matrix blocks, one for each positive eigenvalue. In each block the polar partial isometry of any contraction T extends to a unitary W. Then

$$U_\pm = W\big(|T| \pm i\sqrt{I - |T|^2}\big), \qquad T = (U_+ + U_-)/2.$$

Direct sums give unitaries in the same centralizer. Apply this to the amplified a and to the root contraction B. This does not extend the proper unamplified isometry A_{Cat} in a finite space. Use $2P - I, 2Q - I$ for the root reflections and

$$U_c = 2C - I + 2i\sqrt{C(I - C)}, \qquad C = (2I + U_c + U_c^*)/4.$$

Transport the same choices to the other corners and extend by identity outside them. This realizes all support and block equations.

Append a maximally entangled qutrit and use the graph formulas. Each proposed label commutes with the coarse FMS effects: the frame is preserved and each block unitary has identical transported flag and magic blocks. Refined effects are their actual joint products. Distinct labels use separate measurements. Thus all rows, FMS predicates, bindings and refinements have one simultaneous normal realization.

7 The rational Bell functional

For opposite-party binary effects E_i, F_i and rational coefficients $\alpha_0, \ldots, \alpha_m$, define

$$S = \alpha_0 I + \sum_i \alpha_i E_i, \qquad T = \alpha_0 I + \sum_i \alpha_i F_i, \qquad k = 1 + \tfrac{1}{2}\sum_i \alpha_i^2,$$
$$F_\alpha = ST + k\sum_i (E_i + F_i - 2E_iF_i). \tag{28}$$

This is a Bell expression after expanding ST: constants and marginals use a one-answer anchor. It contains no same-party products. For $\Delta_i = E_i - F_i$, cross-party commutation gives the exact identity

$$F_\alpha = \frac{S^2 + T^2}{2} + \sum_i \Delta_i^2 + \frac{1}{2}\sum_{i<j}(\alpha_j\Delta_i - \alpha_i\Delta_j)^2 + k\sum_i (E_i - E_i^2 + F_i - F_i^2). \tag{29}$$

Indeed $ST = (S^2 + T^2 - (S-T)^2)/2$; expansion of the last square and the elementary Lagrange identity yield (29). No same-party commutation or idempotency is assumed in this calculation. Every displayed summand is positive for effects. Equality gives perfect matching and the row identity $S = 0$ on the reducing state support by Section 2.

Define B as minus the sum of (28) for every one of the 4 529 core rows, one binary synchrony loss per binary symbol, and the ordinary losing-event probabilities of the conditional FMS predicates, refinements, activity and event bindings. Every loss has unit weight. There are 5 592 binary symbols, ten original FMS questions and twenty-two refined questions per party, in addition to the anchor. FMS questions have six answers including inactivity, and refinements have twelve. These finite rules specify the whole functional. Exact aggregation produces the advertised coefficient counts.

Proof of Theorem A. Equation (29) and nonnegative event probabilities prove $B \le 0$ for arbitrary POVMs. At equality each loss vanishes separately. Support matching, the bridge and label lemmas, and the projection compiler therefore recover the full construction of Section 6. Lemma 3.2 gives the pure factor. The simultaneous realization there attains equality normally. Equation (8) and invariance of nonzero marginal spectra under isometries give infinite entropy on both sides.

For finite-dimensional nonattainment, the root has nonzero mass and $A^*A = I$ makes A unitary there. The range equation then forces $B = P = 0$. The initial equation gives $Q = I$, hence $C = I/2$. Covariance would say $A/2 = A/4$, a contradiction.

For approximation, let $a_n = \mathrm{Cat}_n / 4^n$, $Z_N = \sum_{n \le N} a_n$, $t_N = a_N / Z_N$, and use the canonical words with $n \le N$. Compress the word operators and make centralizer unitary completions on the finite scaffold as before. Every relation is exact except $A_N^* A_N - I = -\Pi_N$. The corresponding compiled row is

$$S_{\mathrm{iso}} = -(|1\rangle\langle 2| + |2\rangle\langle 1|) \otimes e_0 \otimes \Pi_N.$$

All event losses and synchrony losses vanish. Synchronization makes the row penalty equal to $\mathrm{Tr}(h_N S_{\mathrm{iso}}^2) = 2t_N/75$, the expectation of its square. Hence

$$B_N = -\frac{2t_N}{75} \longrightarrow 0, \qquad t_N \sim \frac{1}{2\sqrt{\pi} N^{3/2}}. \tag{30}$$

Finally the Gibbs entropy bound in natural units is $S(h) \ln 2 \le \beta \,\mathrm{Tr}(hH) + \ln \mathrm{Tr}\, e^{-\beta H}$. It follows by truncation, or the variational principle, also when the entropy is infinite. A lower bound on H makes its mean energy well-defined as an extended number. Finite mean energy would force finite entropy, giving the stated contradiction. □

7.1 Literal coefficient convention

The archive contains ordered question/answer lists, `core.json`, `format.json` and one sorted coefficient stream. Each sixteen-byte little-endian record is an unsigned 64-bit key followed by a signed 64-bit numerator. The coefficient is the numerator divided by 2048, with

$$\mathrm{key} = 2^{32}\mathrm{event}_A + \mathrm{event}_B, \qquad \mathrm{event} = 16\,\mathrm{question\ id} + \mathrm{answer\ index}.$$

Unlisted coefficients are zero. Indices start at zero. The anchor makes constants and marginals literal cross-party probabilities. No implicit affine substitution or distribution factor is needed.

8 Rational rows as Boolean winning conditions

The Bell penalties have signed coefficients. Reclassifying their negative terms as losses would not enforce their zero sets. We instead implement each rational row as a subspace orthogonality test.

Theorem B (rational-row compiler). *Let a finite strategy specification name finitely many synchronized binary projections and finitely many rational linear identities among them. There is a finite extension by Boolean tests with the following properties.*

Soundness: *in every perfect normal spatial strategy for the extended tests, the identities hold as operator identities for the named support-restricted projections on each local state support.*

Completeness: *any normal spatial strategy with these synchronized projections, satisfying the identities on its supports, extends to satisfy all the added tests perfectly after adjoining a finite maximally entangled state. The old measurements act trivially on the new factor. The rest of the old strategy, including nonsynchronous questions, is retained; if its old tests were perfect, the whole extension is perfect.*

8.1 A certified scalar matrix algebra

Choose $d_{\max} = 2^{2r}$ large enough for every row construction below. Use r synchronized magic squares, binding each context's cell marginals to synchronized binary cells. The Pauli-pair calculation in Lemma 4.1 gives two anticommuting pairs per square. Four-answer joint

questions for all pairs of axes from distinct squares force cross-square commutation. Hence their algebra is $M_{d_{\max}} \otimes I$ on the support. Synchronization makes the marginal $I_{d_{\max}}/d_{\max} \otimes h$.

Add joint questions between every named old projection and every scalar axis. Their sharp marginals force the old projections into the commutant: $E_i = I_{d_{\max}} \otimes e_i$. This step concerns only synchronized projections, never an FMS sign. Add global Z and global X measurements with all bit outcomes, binding each bit marginal to its axis. They are the actual joint spectral measurements.

For a row dimension $d = 2^n \le d_{\max}$ use the first n qubits; the rest are spectators. For each nonzero $j \in \mathbb{F}_2^n$, choose a basis $\ell_1, \dots, \ell_{n-1}$ of $j^\perp$. Add a joint context for $X^j, Z^{\ell_1}, \dots, Z^{\ell_{n-1}}$, binding its marginals to the corresponding global parity events. These commute because $\ell \cdot j = 0$. The zero-Z-syndrome outcomes with X^j sign $\pm$ are

$$R_j^{\pm} = \tfrac{1}{2}(|0\rangle \pm |j\rangle)(\langle 0| \pm \langle j|) \otimes I_{\text{spectator}}.$$

The reference $R_d = |+_d\rangle\langle +_d| \otimes I_{\text{spectator}}$ is the event that the first n global X bits are positive. It includes all spectator outcomes.

8.2 Signed reflections and column graphs

For one row $S = \alpha_0 I + \sum_i \alpha_i e_i$, let L clear all coefficient denominators and set

$$s_i = 2e_i - I, \quad c_i = L\alpha_i, \quad c_0 = 2L\alpha_0 + \sum_i L\alpha_i.$$

Then $2LS = c_0 I + \sum_i c_i s_i$. Expand this into signed reflections with integer repetitions, add a canceling $I, -I$ pair with $V_0 = I$, and pad by further such pairs to a power of four d. The list has even length because its coefficient sum is even. Thus

$$\sum_{j=0}^{d-1} V_j = 2LS, \qquad V_0 = I, \qquad V_j \in \{\pm I,\ \pm(2e_i - I)\}. \tag{31}$$

For each $j > 0$ use a six-answer joint question of e_i and the three-way scalar measurement $R_j^+, R_j^-, I - R_j^+ - R_j^-$. For $V_j = \pm I$, bind its binary bit to the anchor. Matching both marginals identifies all joint effects. Appropriate coarse events are the two-coordinate graph projections

$$G_j^{\pm} = \frac{1}{2}\begin{pmatrix} I & \pm V_j \\ \pm V_j & I \end{pmatrix}$$

on coordinates $0, j$, zero elsewhere. A negative reflection sign exchanges the two coarse events.

Add a synchronized d-answer question with projections D_t, indexed by $t \in \mathbb{F}_2^n$. Put $\chi_t(j) = (-1)^{t \cdot j}$. Forbid D_t together with G_j^- when $\chi_t(j) = 1$, or G_j^+ when $\chi_t(j) = -1$. The forbidden-event support rule gives actual orthogonality. A vector in $\operatorname{ran} D_t$ therefore has coordinates $v_j = \chi_t(j) V_j v_0$.

Define

$$J_t \xi = d^{-1/2}\big(\chi_t(j) V_j \xi\big)_{j=0}^{d-1}, \qquad P_t = J_t J_t^*.$$

Unitarity and character orthogonality give $J_t^* J_u = \delta_{tu} I$ and $\sum_t P_t = I$. Neither equation requires different V_j to commute. The kernel conditions give $D_t \le P_t$. Since both families sum to identity, $D_t = P_t$ for every t.

Finally forbid D_0 jointly with the opposite R_d. Then $D_0 R_d = 0$, equivalently $R_d J_0 = 0$. The column overlap is

$$J_0^* R_d J_0 = d^{-2}\Big(\sum_j V_j\Big)^*\Big(\sum_j V_j\Big).$$

It vanishes exactly when $S = 0$. This proves soundness on arbitrary normal supports.

For completeness append $\Phi_{d_{\max}}$, use the scalar Paulis and the actual joint products, and define the column partition by the J_t. Synchronized original projections commute with the marginal and transfer across a purification. Products transfer in reversed order. Define Bob's column effects by those transferred products, equivalently by transpose in canonical Schmidt coordinates. This order matters for noncommuting e_i. All column and synchrony tests hold. The row-zero test holds precisely because $S = 0$. These constructions act trivially on every old measurement's new factor and prove completeness, including for mixed original states.

8.3 The explicit game and proof of Theorem C

For the present 4 529 rows, $d = 16$ for 4 507 rows, 64 for twenty rows and 256 for two rows. Four magic squares certify eight scalar qubits. The old core has 5 590 projection variables; two additional activity symbols do not enter a row. There are 89 440 old-variable/axis joint questions, 333 scalar pair contexts, 69 375 slot joint questions and 4 529 column questions. Together with scalar cells, contexts, cross-square joints and global measurements these add 163 835 questions per party.

Retain the 6 916 old FMS/refinement/binding forbidden events and old binary synchrony tests. Add all the compiler rules. Accept any answer pair not forbidden by these rules and reject invalid answers. Choose questions independently and uniformly, so every pair has probability Q^{-2}, where $Q = 169\,460$. The predicate is Boolean; masks in `game.json` list the finite answer subsets used by each rule.

In a perfect strategy every pair has positive sampling probability, so each rejecting event is zero. Theorem B gives every old rational row. The restricted old strategy has $B = 0$ by (28), so Theorem A gives (2). Completeness extends the canonical normal realization by Φ_{256}. These separable infinite spaces identify locally with ℓ^2. Finite nonattainment follows from Theorem A.

The cutoff strategies also extend by Φ_{256}. Every new structural rule is exact; only one row-zero event fails. For a row,

$$p(D_0, R_d) = \frac{4L^2}{d^3} \operatorname{Tr}(h_{\mathrm{old}} S^2). \tag{32}$$

The maximally mixed scalar marginal supplies the third power of d, including spectator normalization. For the isometry row $L = 2, d = 16$ and $\operatorname{Tr}(h_{\mathrm{old}} S^2) = 2t_N/75$. Thus

$$1 - \omega_N = \frac{t_N}{9600 Q^2} \longrightarrow 0. \tag{33}$$

This completes the proof of Theorem C.

The known synchronous obstruction. Kim, Paulsen and Schafhauser prove that a synchronous game with a perfect spatial strategy has a perfect finite-dimensional strategy [15, Theorem 3.10 and Corollary 3.11]. On a normal support its synchronized projections reduce each positive finite-dimensional marginal eigenspace, where a maximally entangled strategy remains perfect. The present game retains nonsynchronous FMS and refined questions. Their transports change density eigenvalues. The full strategy cannot be restricted to a single marginal eigenspace. This observation explains the construction; it is not a new obstruction theorem.

9 Entropy near the optimum

Define

$$E_B(\varepsilon) = \inf\{S(\rho_A) : B \geq -\varepsilon\}, \qquad E_G(\varepsilon) = \inf\{S(\rho_A) : \omega(G) \geq 1 - \varepsilon\}.$$

The infima range over normal spatial strategies. Purifying onto Bob preserves Alice's marginal and all probabilities, so the same infima are obtained with pure states. They need not be attained. The corresponding assertions hold with the parties exchanged.

Theorem D (near-optimal entropy). *As $\varepsilon \downarrow 0$, both $E_B(\varepsilon)$ and $E_G(\varepsilon)$ tend to infinity. Put*

$$C_L = \frac{24}{25\sqrt{\pi}}, \quad C_S = \frac{64}{45\sqrt{\pi}}, \quad C = C_S C_L^{1/5} = 0.709793116354\ldots.$$

There are finite-dimensional strategies giving

$$E_B(\varepsilon) \leq (C + o(1))\varepsilon^{-1/5}, \qquad E_G(\varepsilon) \leq (CQ^{-2/5} + o(1))\varepsilon^{-1/5}, \quad Q = 169\,460.$$

With the infinite canonical measurements fixed, the exponent $1/5$ is optimal among arbitrary scalar spectral filters of their canonical marginal, for B and likewise for G. This restricted optimality does not assert a matching lower power for unrestricted strategies.

9.1 Divergence at bounded entropy

Here is the compactness argument, included to identify precisely what it proves. Sort the Schmidt probabilities $p_1 \geq p_2 \geq \cdots$ of any pure state of entropy at most K. Since $p_j \leq 1/j$,

$$\sum_{j>d} p_j \leq \frac{K}{\log_2(d+1)}. \tag{34}$$

Compress each POVM to the (separable) state supports; this preserves all probabilities. If the Schmidt rank is finite, extend by a fixed deterministic answer on the complement in ℓ^2. After local basis changes write each state as $\sum_j \sqrt{p_j}|j,j\rangle$ in common copies of ℓ^2. Uniformly small tails yield a norm-convergent subsequence of these vectors with unit norm. Finite partial entropy sums and lower semicontinuity show that its entropy is at most K.

Bounded effect sets on a separable Hilbert space are sequentially weak-operator compact. Extract a common subsequence for the finitely many effects. Positivity and POVM sums survive the limit. On the first d Schmidt coordinates all probabilities involve only finitely many matrix entries. The tail bound then lets d tend to infinity, uniformly over the measurements. Thus entropy-at-most-K spatial correlations form a compact set in each finite scenario.

If a sequence had vanishing loss and bounded entropy, nearly minimizing pure strategies would have a limit attaining the optimum with finite entropy. Theorems A and C forbid this. Equivalently each finite entropy budget has a strictly positive loss gap. The argument gives no evaluated modulus for that gap.

9.2 A spectral taper and its exact loss

In the canonical realization, let $r = 0, 1, 2$ denote the corner-weight exponent, with multiplicities $m_0 = 1, m_1 = m_2 = 4$. Including the qutrit, each marginal eigenvalue is $2^r 4^{-n}/150$. Define

$$J|\text{frame}, r, w\rangle = (n(w) - r/2)|\text{frame}, r, w\rangle.$$

Every synchronized core projection commutes with J, because J is affine in the logarithm of the canonical marginal. In particular the centralizer unitary completions preserve it. Take

$$f_N(j) = \begin{cases} 1 & j \le 0, \\ (1-(j/N)^2)_+ & j > 0, \end{cases} \qquad \psi_N^f = \frac{f_N(J_A)\psi_*}{\sqrt{Z_N^f}}.$$

Here ψ_* is the full canonical state. The filter has finite Schmidt rank, supported in $n \le N$. Compressing the original measurements to that support gives valid finite POVMs with exactly the same probabilities; the support need not reduce the FMS signs. Filtering bare word size would fail to preserve centralizer tests, which is why the shifts $r/2$ are retained.

Write $a_n = \mathrm{Cat}_n / 4^n$ and $f_{n,r} = f_N(n - r/2)$. Exactly,

$$Z_N^f = \frac{1}{50} \sum_{n \ge 0} a_n (f_{n,0}^2 + 8 f_{n,1}^2 + 16 f_{n,2}^2) \longrightarrow 1. \tag{35}$$

Core rows are zero operator identities, and their projections remain synchronized after filtering. All activity, flag, retained-event and label bindings remain exact. Each refined measurement remains a joint product with its single label. Only wrong sign/sign events in original or duplicated FMS pairs can lose.

Put

$$D_1(N) = \sum_n a_n [f_N(n) - f_N(n-1/2)]^2,$$
$$D_2(N) = \sum_n a_n [f_N(n-1/2) - f_N(n-1)]^2.$$

In bridge g, put $f_f = f_{n,g-1}$ and $f_m = f_{n,g}$, the filter values on its flag and magic corners. On a fixed word and frame, a forbidden sign pair in bridge g has amplitude

$$\frac{\sqrt{2^{g-1}}\, 2^{-n} (f_f - f_m)}{2\sqrt{150 Z_N^f}}.$$

Its probability, after summing the frame and words, is $2^{g-1} D_g / (200 Z_N^f)$. There are two wrong signs per pair. In each bridge there are four original pairs, twenty-two duplicated Alice pairs and twenty-two duplicated Bob pairs. Summing refined labels gives the coarse event. Consequently the *literal* Bell loss is

$$\ell_B(N) = -B(\psi_N^f) = \frac{12}{25} \frac{D_1(N) + 2D_2(N)}{Z_N^f}. \tag{36}$$

The serialized direct-event audit counts 368 potentially nonzero fine sign records; their labels sum to these 192 coarse events across both bridges. The derivation uses only scalar values of the filter on the J-spectrum. Thus (36) holds for every nonzero Borel scalar filter $f(J)$ with $\mathrm{Tr}(h_* |f(J)|^2) < \infty$, using its corresponding normalization Z and difference sums D_1, D_2; for complex filters the squared differences are $|f_f - f_m|^2$.

For $F(x) = (1-x^2)_+$ and $a_n \sim n^{-3/2}/\sqrt{\pi}$, half-step difference sums give

$$D_g(N) \sim \frac{N^{-5/2}}{4\sqrt{\pi}} \int_0^1 x^{-3/2} F'(x)^2 \, dx = \frac{2}{3\sqrt{\pi}} N^{-5/2}.$$

To justify the Riemann sum, first restrict to $[\eta, 1]$. The omitted lower interval is bounded by a constant times $\eta^{3/2} N^{-5/2}$, since $F'(x) = -2x$. The finitely many endpoint terms are $O(N^{-4})$ near zero and $O(N^{-7/2})$ at the upper edge. Therefore

$$\ell_B(N) \sim C_L N^{-5/2}.$$

For example $Z_N^f \geq 1/2$ and each half-step difference is at most n/N^2, which also give a nonasymptotic bound $\ell_B(N) \leq 5N^{-5/2}$ for $N \geq 1$.

The filtered eigenvalues give, with $0 \log 0 = 0$,

$$S_N = \log_2(150 Z_N^f) + \frac{1}{50 Z_N^f} \sum_n a_n \sum_{r=0}^{2} m_r 2^r f_{n,r}^2 [2n - r - 2\log_2 f_{n,r}]. \tag{37}$$

Only the $2n$ term contributes to the leading order. All other terms are uniformly bounded using $\sum_n a_n = 2$ and boundedness of $u^2|\log u|$. Since $Z_N^f = 1 + O(N^{-1/2})$,

$$S_N \sim \frac{\sqrt{N}}{\sqrt{\pi}} \int_0^1 x^{-1/2}(1-x^2)^2\, dx = C_S \sqrt{N}.$$

Choosing an integer $N = (C_L/\varepsilon)^{2/5}(1 + o(1))$ with sufficient positive slack proves the Bell upper bound. The $o(1)$ can decay slowly; the leading constant alone is not a finite-loss error estimate. The game extension adds eight bits. Its new row identities remain exact; only the retained old FMS sign events lose. Thus $\ell_G(N) = \ell_B(N)/Q^2$, proving the game constant. Formula (33) is for a different cutoff family and must not be used for these filtered states.

9.3 Restricted lower bound for all spectral filters

Normalize $\mathrm{Tr}(h_*|f(J)|^2) = 1$ and set $f_k = f(k/2)$ for $k = -2, -1, 0, \ldots$. Absolute values preserve entropy and normalization and can only decrease the difference energy, so assume $f_k \geq 0$. The shell masses and degeneracies are

$$\begin{aligned} \mu_{2m} &= (a_m + 16a_{m+1})/50, & d_{2m} &= 3(\mathrm{Cat}_m + 4\,\mathrm{Cat}_{m+1}), \quad m \geq 0, \\ \mu_{2m-1} &= 8a_m/50, & d_{2m-1} &= 12\,\mathrm{Cat}_m, \quad m \geq 0, \\ \mu_{-2} &= 16/50, & d_{-2} &= 12. \end{aligned}$$

Thus $\sum_k \mu_k f_k^2 = 1$ and

$$S = H((\mu_k f_k^2)_k) + \sum_k \mu_k f_k^2 \log_2 d_k \geq \tfrac{1}{2} \sum_{k \geq 0} k \mu_k f_k^2. \tag{38}$$

The elementary bounds $\mathrm{Cat}_m \geq 2^{m-1}$ for $m \geq 1$ and $a_m \geq [2(m+1)^{3/2}]^{-1}$ imply $\log_2 d_k \geq k/2$ and $\mu_k \geq [100(k+3)^{3/2}]^{-1}$ for $k \geq 0$.

For a normalized filter, (36) becomes $\ell = (12/25)D$, where

$$D = \sum_{n \geq 0} a_n [(f_{2n} - f_{2n-1})^2 + 2(f_{2n-1} - f_{2n-2})^2].$$

This is the energy of a connected weighted half-line. The sum of inverse conductances from -2 to L is at most $2(L+3)^{5/2}$. Cauchy–Schwarz along the path gives

$$|f_k - f_{-2}| \leq \sqrt{2D(L+3)^{5/2}}, \qquad -2 \leq k \leq L.$$

For small D choose $L = \lfloor (32D)^{-2/5} - 3 \rfloor$, so the right side is at most $1/4$. If $f_{-2} \geq 1/2$, then $f_k \geq 1/4$ through L. Equation (38) bounds the entropy below by

$$\frac{1}{3200} \sum_{k=1}^{L} \frac{k}{(k+3)^{3/2}} \geq \frac{\sqrt{L}}{25600} \quad (L \geq 6).$$

If $f_{-2} < 1/2$, the low-shell amplitudes are at most $3/4$, so at least $7/16$ of the normalized mass lies above L. Then $S \geq 7L/32$. Both cases give $S \geq c\ell^{-1/5}$ for an absolute positive c. For complex filters apply this argument to their absolute values; their energy is no larger. For filters $f(J) \otimes I$ of the game's canonical marginal, the same compiler extension gives $\ell_G = \ell_B/Q^2$ and $S_G = S_B + 8$. Together with the taper this proves restricted optimality for both B and G and completes Theorem D.

An unrestricted lower power remains open. Exact dense-range cancellation has no uniform operator-norm version at small marginal eigenvalues. Approximate magic blocks, compatible labels, and a simultaneous word-sector extraction would need additional estimates. We make no claim that an arbitrary near-optimal strategy is a scalar spectral deformation.

10 Discussion

The finite game forces a normal pure factor with a critical Catalan tail. At word size n there are Cat_n Schmidt probabilities equal to $4^{-n}/2$. Their total mass is of order $n^{-3/2}$, while their entropy contribution is of order $n^{-1/2}$. The masses sum to one and the entropy diverges. A finite auxiliary system implements the test but does not cause the divergence. Forthcoming work gives a second finite game, built from a context-free grammar with critically weighted counts of order $n^{-5/4}$, whose canonical entropy exponent is $1/3$ instead of $1/5$, so the canonical exponent is not universal.

The I_{3322} contrast. Coladangelo's attaining state [4, Definition 10.12 and Theorem 10.14] has finite entropy. This is a consequence of its construction, not an entropy claim made in that paper. Here is the tail argument. Its Schmidt vector λ satisfies a bounded Jacobi equation $J_\infty(c)\lambda = q_*\lambda$, with monotone labels of total variation at most two and $q_* > 1/4$. The half-line version of the estimate in its Lemma 10.10 bounds the tail restriction by $1/4 + \mathrm{TV}_N/2$. Choose N so this lies below q_*. The resolvent equation on that half-line has only a boundary source. The Combes–Thomas exponential-weight estimate [6], using bounded off-diagonal entries, gives $\sum_i e^{\mu|i|}\lambda_i^2 < \infty$ for some $\mu > 0$, on both tails. Gibbs' inequality then gives $H((\lambda_i^2)_i) < \infty$. Thus dimension nonattainment need not force entropy divergence. We do not need a claim about every I_{3322} optimum.

The main open quantitative problem is a lower entropy rate for all near-optimal strategies. The present compactness proof establishes divergence, whereas the spectral calculation gives only a restricted matching power. Smaller question and answer sets are also desirable. Our graph and row constructions leave substantial room for sharing contexts, but any reduction must preserve the same actual products and calibrated transports.

The theorem concerns normal states on tensor products. We do not classify nonnormal states or nonspatial commuting strategies. Nor do we claim dimension-independent robust self-testing, minimal alphabets, or an optimal unrestricted upper constant. These boundaries leave the distinction among dimension, entropy and type III structure visible.

How this work was done

The authors selected the questions and proof standards, directed the research, and take responsibility for every claim. AI systems from more than one developer assisted with exploration, derivation, programming and drafting.

The mathematical evidence consists of the proofs in this paper and the explicitly cited external theorems. The ancillary programs provide reproducible finite checks. Every theorem passed at least one independent check by a different model family, with its own exact computation. The two Bell evaluators and two game evaluators use separate implementations;

their provenance and scope are stated in Appendix A. Independent checks of Theorem D include exact scalar values and constants and a literal full-functional evaluation on filtered states. The latter reuses the cross-checked Bell measurement builder but independently constructs the filtered state, contracts the coefficients and compares the result with (36).

Finite exact evaluations test the exported predicates, coefficient tables, projection identities and cutoff boundary formulas. They do not prove infinite-dimensional attainment, universal entropy divergence, or a robustness theorem by sampling. The asymptotic conclusions rest on the written proofs. This preprint has not been peer reviewed.

A Reproducibility

The accompanying `verification/` directory contains both archives, two independent Bell evaluators, two game evaluators, the literal verifier checks, scalar near-optimality checks, recorded outputs and fresh execution logs. Its README gives commands, dependencies, provenance and limitations. `SHA256SUMS` identifies every bundled file apart from itself. Paths in public files are relative.

The Bell producer-side evaluator reads every literal coefficient and uses exact arithmetic in $\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$. The review evaluator reconstructs the measurements independently in $\mathbb{Q}(\zeta_{24})$, using a different field representation. Both give

$$B_0 = -2/75, \qquad B_1 = -2/375, \qquad B_2 = -2/825.$$

Each finds only the isometry row failing, the predicted full boundary matrix, and zero direct FMS/refinement loss.

The producer-side game checker reuses the producer-side Bell evaluator's matrix kernel. It is independent of the game producer, but its old measurements are not a second independent implementation. The game review supplies that implementation: it independently rebuilds every old projection, FMS/refined measurement and forbidden event, then reconstructs the new gadgets from predicate data. Both game evaluators give the exact losses

N	Local dimension	$1 - \omega_N$
0	6 912	1/275680239360000
1	13 824	1/1378401196800000
2	27 648	1/3032482632960000

These agree with (33). The public verifier is checked on 10 076 128 column-graph and spectator answer cases, together with the complete pair-rule index. Independent brute-force and changed-column controls accompany the reduced game evaluator. The checks evaluate rule classes and exact structural zeros, not a dense enumeration of every answer pair in the full question square.

The scalar checker audits the literal direct-event classes, performs thirty-two exact rank-one contractions, checks rational taper losses, and evaluates positive scalar sums through $N = 10^6$. A second engine for (36) contracts all 73 769 literal coefficients on filtered states at word cutoffs $M = 1, 2$, including the tapers and mixed-sign rational filters. It reuses the cross-checked Bell measurement builder; its state construction, contraction, aggregation and comparison are separate. The scripts and recorded and fresh outputs are bundled with hashes. Floating-point asymptotic illustrations do not replace the analytic limits or provide an unrestricted entropy lower rate.

The functional archive SHA-256 is

```
bb5c0bf32e9a7175bef73d9b156a5c6f3
e3a0487462c90a9138e82f809927aaf.
```

The game archive SHA-256 is

```
f29237d3920afe1b4ff9a3749b6893416
9d8109bbaa0fa19dbbcc34f334824d2.
```

The line breaks above split each hash after thirty-two hexadecimal characters. Neither archive was regenerated for packaging. No proof-assistant certification is claimed.

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