Fault-tolerant quantum computation on a line from one fixed translation-invariant rule

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Abstract

One fixed finite-range, translation-invariant, time-periodic quantum circuit on a one-dimensional ring samples the computational-basis output of any width-n, depth-D Toffoli/Hadamard circuit on the all-zero input within total variation error ε. The ring size and number of periods are at most $A(n + D + \log_2(1/\varepsilon))^A$ for a fixed A, and the per-site register is fixed. The polynomial-time computable product preparation puts the program in classical basis registers and initializes all quantum registers to zero; a deterministic polynomial-time classical decoder uses only n, D and a sufficient integer precision request. Independent wire depolarization or whole-site replacement acts after every elementary layer, including on classical control, below a fixed positive threshold. Theorem D extends the accuracy and polynomial resource bounds to sufficiently small general local fault-path noise, including coherent errors and correlated bath interactions satisfying a fault-path superset bound. Its threshold is positive but astronomically small. The proof combines exact recovery on certified supports, multiscale dense-witness counting for overlapping long tests, and the established fault-path superset method. Noiseless recursive encoding and recovery give the corresponding reference-entangled input/output guarantee. An elementary register-bank corollary replaces the periodic word by one fixed channel at every step, preserves noise after every original layer, and stores a reference-entangled qubit with fidelity at least 15/16 through a horizon linear in ring size. Its ideal onsite bank shift is a noiseless timing structure; noise within a decomposition of that shift and times beyond the horizon are not covered. The construction follows the one-dimensional route proposed by Dünnweber, Styliaris and Trivedi for their two-dimensional construction, which already treats general local fault-path noise with polylogarithmic overhead. Our overhead is polynomial, with very large constants and an initialized synchronous hierarchy.

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1 One rule as a quantum computer

A fixed local rule can contain a universal processor while leaving the desired computation to its initial state. Under continuing noise this requires more than protecting quantum data: addresses, program symbols, clocks, routing decisions and every bit of classical correction work must also survive. We construct such a rule in one spatial dimension. Its local register, gate range and elementary period are fixed once. Larger computations and smaller errors change only the length of the ring, the product preparation and the stopping time.

Here and below the error of a probability law is total variation distance. The quantum endpoint result uses one half of the trace norm, including a reference. A period means one complete application of the fixed elementary circuit. The elementary gates include reset to zero, and noise follows every layer, also on idle and reset wires. Section 2 specifies the channels exactly.

Theorem A (Product preparation and classical readout)**.** *There exist a single finite-range, translation-invariant, time-periodic quantum circuit on rings, a finite number Q_{site} of qubits per site, and constants $p_0 > 0$ and $A \geq 1$ with the following property. For every $n \geq 1$, $D \geq 0$, every depth-D Toffoli/Hadamard circuit C on n qubits and every $0 < \varepsilon < 1$, use a sufficient integer precision request h as specified in Section 2.1. There is a polynomial-time computable product preparation in which only classical basis registers carry nonzero data and every quantum register is zero. The classical state describes C and h.*

For every $0 < p \leq p_0$, apply independent single-qubit depolarization of rate p to every wire after every elementary layer. After t periods on L sites, measure all qubits in the computational basis. A deterministic polynomial-time classical decoder depending only on n, D, h gives the measured output law of $C|0\rangle^{\otimes n}$ within total variation distance $2^{-h} \leq \varepsilon$. In particular the decoder does not read or evaluate the gate list of C. Both resources satisfy

$$L, t \leq A\big(n + D + \log_2(1/\varepsilon)\big)^A. \tag{1.1}$$

The hardware, preparation, decoder and stopping time do not depend on p. Explicit dependence of p_0 and A on the fixed construction is given in (6.4) and (7.3).

Dünnweber, Styliaris and Trivedi [15, Theorem 1] use a fixed local update channel in an alternating two-dimensional brickwork schedule, with noisy control and an initial-state program. For a sufficiently correct hierarchical input they prove a positive local fault-path threshold, using $O(N \operatorname{polylog}(ND/\delta))$ qudits and $O(D \operatorname{polylog}(ND/\delta))$ steps after translation-invariant logical compilation. Their Section VI describes encoded preparation in polylogarithmic depth. Theorem A instead supplies a computable basis-product start on a line, with polynomial space and time and the independent noise models of Section 2.1. Theorem D below extends our rule to their local fault-path noise convention, including coherent errors and bath correlations obeying the specified superset bound. Their polylogarithmic overhead remains stronger than our polynomial bound. They explicitly suggested embedding a quantum code in Gács's one-dimensional hierarchy as a route to the one-dimensional case. This suggestion already appears in their January 2026 manuscript and is retained in its later versions. We follow that route.

One-dimensional fault-tolerant quantum computation with local gates and no intermediate measurements or classical processing already appears in Aharonov and Ben-Or's threshold theorem [2, Theorem 13]. Gottesman [22] gives low-dimensional local-gate thresholds, with next-to-nearest-neighbor gates in one dimension, and suggests interspersing reliable classical regions of the Gács type. Those constructions have polylogarithmic overhead, but they compile a target circuit into a spacetime gate list. Here the spacetime rule is fixed and periodic, and only its initial state contains the target program. The interpretation of the Aharonov–Ben-Or construction with classical correction compiled into a fully local noisy quantum circuit is also discussed by Selub, Bhardwaj and Lake [32, Section 1.2], citing Gottesman's textbook manuscript [23]. Shtanko and Sharma [34] simulate local target circuits in every dimension without intermediate measurements, using a contracting nonunital channel with a unique nonmaximally mixed fixed point; the quantum refrigerator of Ben-Or, Gottesman and Hassidim [6] is a related reset-generating precedent. Our depolarizing noise is unital, and our elementary circuit has explicit resets.

Pattison, Haah and Preskill [37, Theorems 4.5 and 5.7] prove universal fault-tolerant sampling on a planar device with a fixed depth-three periodic circuit away from a constant number of controlled qubits. Their processor receives an instruction stream during the run and uses a noiseless global clock. Here the target program is supplied entirely at time zero as part of a product preparation on a one-dimensional ring. Their manuscript was submitted on 18 November 2026; this is concurrent and independent work. Our rule also uses a fixed time-periodic schedule. The distinctions are one spatial dimension, a program supplied entirely in the initial state, and all classical control state, including the hierarchy's clocks, on noisy wires; like theirs, our layer schedule is applied noiselessly. Corollary 10.1 packages our periodic word into identical discrete updates by adding finitely many register banks. This is an elementary consequence of Theorem A; its onsite bank shift acts on all $D_{\mathrm{layer}}Q_{\mathrm{site}}$ qubits at once and remains a noiseless timing structure. It does not remove the timing assumption in this comparison.

The classical foundation is Gács's initialized synchronous reliable automaton [18], rather than the stronger self-organizing asynchronous form [20]. Tsirelson's local reliable memory [11], Gács's two-dimensional construction [19], Gács–Reif's three-dimensional real-time array [21], and Gray's account [24] place the hierarchy in its classical context. We use the explicit classical construction of Douglas and Mghirbi [13] and the one-dimensional local lemmas of their quantum memory [12]. The exact imported statements and all hypotheses used here are recorded in Sections 2.2–2.3.

Balasubramanian, Davydova and Lake [5, Section 1.3(3)] formulate a quantum analogue of Gács's automaton as an open direction. Their two-dimensional toric-code memory has a reliable read-only instruction set and is not translation-invariant in spacetime. Drechsler, Oreg and Aharonov [14] relate CSS quantum memories to paired classical processes. For the 4D toric code they prove a Toom-rule memory threshold under local Pauli noise for a synchronous channel and for a periodically driven Lindbladian. They ask whether their framework helps in one dimension and leave a general time-independent lift from synchronous classical rules open. Our rule is discrete time with one fixed elementary period. Selub, Bhardwaj and Lake [32, Sections 1.2–1.3] discuss local quantum computation with reliable classical processing and suggest Gács's self-simulation as a possible route to removing that assumption. Gidney and Bergamaschi [17, Theorem 1.2 and Section 11] obtain constant quantum space overhead with nonlocal noiseless classical control and quasi-polylogarithmic depth overhead in the range of their computation theorem, for compiled local target circuits. They ask for constant spatial overhead together with translation invariance. Our result supplies the fixed translation-invariant rule, with polynomial overhead. It does not establish their combined target.

The proof has three additional interfaces. Repeated classical bits are generally not quantum codewords, so a teleportation normalizer must first select the desired logical basis state. All parent registers, including program and control, must be encoded on the children's compiler tracks; the children's own processor runs on disjoint factors. Finally, a total classical decoder must agree with quantum recovery even on a bad intermediate block. Sections 3–5 prove these channel statements. Section 6 obtains rank decay strong enough for a resource bound polynomial in logarithmic precision. Section 7 pays for the entire physical past cone on a finite ring, and Section 8 proves Theorems A–C. Section 9 proves Theorem D: the same circuit tolerates arbitrary sufficiently small local CPTP perturbations, and more generally local fault-path noise, with the same requested accuracy and polynomial resources. Its new ingredient is the dense-witness count for the one-dimensional hierarchy with overlapping long tests. The summation method is that of Aliferis, Gottesman and Preskill [42], with related local-bath threshold methods of Terhal and Burkard [43] and Aharonov, Kitaev and Preskill [44]; Dünnweber, Styliaris and Trivedi [15, Definition 3] already use the same noise model in two dimensions. Section 10 proves the fixed-channel corollary and its reference-entangled storage guarantee through a horizon linear in ring size. Neither its ideal bank shift nor its finite-horizon guarantee supplies a time-independent continuous-time implementation or a guarantee after that horizon.

2 Circuit model and published inputs

2.1 Layers, noise and precision

A site contains $Q_{\mathrm{site}} < \infty$ named qubit factors. An elementary layer is a translated tensor product of disjoint gates of arity at most three, at fixed integer displacements. A finite word of such layers repeats forever. Translation invariance is by one site, with internal port factors distinguishing directions. The elementary inventory is

$$I,\ X,\ H,\ \mathrm{CNOT},\ \mathrm{Toffoli},\ R_0,\ D_Z, \qquad R_0(\rho) = |0\rangle\langle 0|\,\mathrm{Tr}\,\rho,$$

where D_Z dephases in the computational basis. A discard on an existing slot is a reset. SWAP, Fredkin, Z and CZ denote fixed decompositions into this inventory. For example $Z = HXH$ and $\mathrm{CZ} = H_{\mathrm{target}}\,\mathrm{CNOT}\,H_{\mathrm{target}}$. Classical control is implemented by these gates on named qubits. There is no intermediate measurement with external feedback.

After every elementary layer, on every wire q, independently apply

$$\mathcal{N}_{p,q}(\rho) = (1-p)\rho + p\,\frac{I_q}{2} \otimes \mathrm{Tr}_q\,\rho. \tag{2.1}$$

The tensor ordering is the original ordering of factors. A reset is followed by this same noise. An idle qubit is also noisy. A primitive key is a wire and elementary layer, with an independent Bernoulli flag of parameter p. A flagged replacement can be expanded in arbitrary local operators with the exterior as coefficients. This representation makes the deterministic fault statements below applicable conditional on flags; it does not assert independence of resulting logical faults.

For the second noise model, each site x and layer has one independent flag, and the replacement is

$$\mathcal{N}^{\mathrm{site}}_{p,x}(\rho) = (1-p)\rho + p\,\frac{I_x}{2^{Q_{\mathrm{site}}}} \otimes \mathrm{Tr}_x\,\rho. \tag{2.2}$$

These are the two conventions of the quantum memory [12, Section 2.2]. The whole-site channel discards correlations with the exterior on a hit and replaces all factors by a product maximally mixed state.

The physical request is an integer $h \geq 1$ demanding error at most 2^{-h}. For a rational ε given by numerator and denominator take $h = \lceil \log_2(1/\varepsilon) \rceil$, computed exactly by integer shifts and comparisons. An integer supplied on its own denotes the demand 2^{-h}; for a stated tolerance use this canonical choice or the binary-name convention below. For a binary name of ε, read through its leading one and use its position h'. Then

$$2^{-h'} \leq \varepsilon, \qquad \lceil \log_2(1/\varepsilon) \rceil \leq h' \leq \lceil \log_2(1/\varepsilon) \rceil + 1.$$

Only a nonterminating dyadic name can incur the extra one. Preparation and decoding use the same name and hence the same sufficient request; we write h for either choice. This is the chosen input interface. It makes no assertion that a sufficient request cannot be extracted in other approximation models. An unspecified approximation-only real is not an input to these algorithms. The numerical guarantee holds for every real tolerance by choosing a sufficient integer. We use $\mathrm{TV}(P, Q) = \frac{1}{2} \sum_z |P(z) - Q(z)|$ and quantum distance $\frac{1}{2} \|\rho - \sigma\|_1$.

2.2 The classical structural lane

We fix the classical construction [13] at version 2.1.0 and the quantum memory [12] at version 3.1.0. Appendix A records the exact section and lemma locators, hypotheses and matching uses. No global storage-lifetime theorem is substituted for the local statements below. The native structural lane, denoted $\mathcal{R}$, uses a fixed certified never-halting family literal, for example a machine whose transition graph has no halting state. The chosen literal is a constant of the fixed rule, rather than an externally supplied reliable control wire. Every physical factor that represents its changing state is subject to (2.1).

At rank j, a home has physical width P^j and a tick has duration T^j, with

$$P = 5\,2^w, \qquad T \geq \lfloor 3^w / w^2 \rfloor, \qquad T \geq P \geq 4.$$

The classical word width is $2^j \ell_0$. Only rank zero is physical. The finitely many inequalities specifying a sufficiently large w are those of [13, Section 3 and Appendix A], together with the additional explicit inequalities below. They are decidable polynomial/exponential inequalities and are eventually true.

Here are the precise classical statements used. Sparsity means Definition 4.1 of [13]: a set is 0-sparse if empty, and is j-sparse if in every translate of $w_R[0, P^j) \times [0, T^j)$ one can remove a translate of $3w_R[0, P^{j-1}) \times [0, T^{j-1})$ leaving a $(j-1)$-sparse set. Restricted sets are zero outside their restriction. Lemma 4.2 of [13] supplies the deterministic disjoint-child mesh implication: nonsparsity in a doubled rank-$(j+1)$ box forces nonsparsity in two disjoint doubled rank-j candidate boxes, with at most $8(P+2)^2(T+2)^2$ pair choices. We use this deterministic implication, not independence of native flags.

Proposition 5.3 of [13] applies to protected sparse data histories with ordinary unhalted lower ranks and true read-only operands. It gives the proper-live/proper-dead classification, service of a proper-live input, and exact native output on a computing tile from its served input triple; a wholly $(j-1)$-sparse rank-j test has empty rank-j exceptional column. The bound stated after Definition 5.1 of [13] limits served-observation lookahead to $T^j - 1$. Lemma 5.18 of [13] gives service of a proper endpoint under its live-hold hypotheses. Lemma 5.24 of [13] states that, in a fully protected $(j-1)$-sparse rank-j two-period test, six consecutive served positive inputs of a common tribe, whose prescribed two central transitions are live with the next residues, give exact service of all lower sites in the two central homes through $[0, 2T)$, with the endpoint $2T$ excluded. The six positive inputs, the common tribe, the two live outputs and the requisite work/collar are all supplied here by the initialized clean never-halting tower and by the recursively selected protected tests. No assertion is made about arbitrary high-rank noisy clocks.

Lemma 3.1 of [13] identifies the clean full row, including ordinary address, zero age, exact retained packet and scratch defaults. The proof of its Lemma 6.4 constructs an aligned all-live time-zero boundary: repeatedly divide the coordinate by P using floor division until it is 0 or -1; the next serialized digit is an endpoint whose entire cell is independent of the upper live state. Substitution then computes every finite boundary interval. By [13, Section 3.5], the immutable family literal is promoted by extending its word with a blank half and taking one bounded native tape step; its local operations are bounded masks, one-bit shifts and Boolean operations. Section 7 accounts for the cost of this computation.

We use the scalar serialization of [12, Lemma 13.6]. Each bounded parent scalar bit occupies the low word bit of its own live central-area letter; its other word bits are zero. Parent words use typed low/high halves; live/dead discriminants, field order and fixed endpoint padding remain those of [13, Section 3.3]. The same serialization is used by encoders, total parsers and evaluators. Lemma 13.6 checks the classical uses separately: whole-letter transport and vote, tag/padding validation, polynomial scalar scans, exact ballot rows and endpoint independence. Thus it applies to this unchanged structural lane. Required age, address and tag bits and the retained word's lowest bit have computable positions independent of rank. The middle of five packet copies is used. During an epoch the packet is the current parent state at that epoch's bottom; retained letters do not change before the ballot. A clean child LOAD therefore prepares the required clean parent bit, not a bit sampled from an actual noisy parent projection [12, Lemma 13.3].

Dependent primitive-key witnesses. We use [12, Lemma 12.1] with the final circuit's enumerated structural stencils. A native flag at (x,t) is the union of at most $s_R \geq 1$ independent elementary flags within spatial and macrodate radii r_x, r_t; negative-time native flags are zero. A protected rank-j test has width $< w_R P^j$ and height $< w_R T^j$ and asks for $(j-1)$-sparsity. The lemma gives

$$\Pr(\text{test fails}) \leq z_{R,j}(p) = [C_R p^{1/J_R}]^{2^{j-1}}, \qquad j \geq 1, \tag{2.3}$$

with the explicit constants

$$\begin{aligned} D_R &= 8(P+2)^2(T+2)^2, \quad N_R = 4(P+3)(T+3), \\ A_R &= 4w_R^2 s_R, \quad J_R = (2w_R + 4r_x + 3)(2w_R + 4r_t + 3), \\ C_R &= N_R D_R A_R^{1/J_R}, \qquad e_{R0} = A_R^{-1}(2N_R D_R)^{-J_R}. \end{aligned} \tag{2.4}$$

Here $w_R = w$ is the fixed native structural sparsity parameter, chosen large enough that the protected two-period tests have the stated width and height bounds. The finite collar and work inequalities determining this choice are fixed before noise is chosen. The independent-key assumption is checked on elementary flags, not on compiled native errors. The proof restricts each leaf to all ancestor boxes, chooses at least a $1/J_R$ fraction of disjoint stencil supports, and sums witness trees. Section 6 checks the modified period, and the proof of Theorem C gives the site-key form of the same argument.

For [12, Lemma 12.5] use the homogeneous never-halting literal, the exact aligned boundary, the total parser of [12, Lemma 12.4], and the scalar layout just specified. Failure of a full physical native symbol or one of its actual control-read bits then implies failure of a deterministic protected test. For the imported decoded-symbol projection, at each rank there are at most N_{clk} tests with key radii $C_{\mathrm{clk}}P^j, C_{\mathrm{clk}}T^j$, where

$$\begin{aligned} W_* &= \left\lceil \frac{2T + 18P}{P-1} \right\rceil + 2, \quad N_{\mathrm{clk}} = W_* + 8, \\ C_{\mathrm{det}} &= 100(w_R + W_* + T + P + 1), \quad C_{\mathrm{clk}} = C_{\mathrm{det}} + r_x + r_t + 2. \end{aligned} \tag{2.5}$$

The recursion queries date $\lfloor s/T \rfloor - 1$ and interval width at most $W/P + 2T/P + 18$. Its width is bounded by W_* and its date eventually becomes negative. Negative served observations finish before physical zero by the lookahead bound. Under the successful tests the first ballot gives an exact full bottom row, and protected exact evolution gives the desired clean symbol, including scratch; nominal acceptance alone is insufficient. These are the hypotheses and full-symbol conclusion needed here. We keep N_{clk} as this imported projection budget; the additional raw-read test for the completed word is counted separately in Section 6, without claiming unused room in $W_* + 8$. Lemma 12.6 of [12] permits ring use only when the output's entire actual past cone is in an injectively embedded interval with that initialized row. Section 7 constructs such an interval for the whole computation.

2.3 Code, gadgets, deletion and cuts

Fix the particular punctured Reed–Muller CSS family of [12, Lemma 5.1]:

$$N = 2^{2s+1} - 1, \qquad q = 2^s - 1, \qquad [[N, 1, 2q + 1]]. \tag{2.6}$$

Write $\mathcal{C}$ for the punctured self-dual evaluation code, $\mathcal{D}$ for its shortened subcode and e for the all-ones word. Then $\mathcal{D} = \mathcal{C}^{\perp}$, $\mathcal{C} = \mathcal{D} \oplus \langle e \rangle$, e has odd weight, and $d(\mathcal{C}) \geq 2q + 1$. The logical words are

$$|d_L\rangle = |\mathcal{D}|^{-1/2} \sum_{x \in \mathcal{D}} |x + de\rangle, \qquad d \in \{0, 1\}.$$

One recovery corrects the entire operator span of weight-at-most-q errors, including reference coefficients. Finite exact real encoding and total syndrome tables exist. Define $a(w)$ to be the coset in $\mathcal{C}/\mathcal{D}$ of its unique distance-at-most-q codeword, or zero when there is none. Transversal $X, Z, H, \mathrm{CNOT}, \mathrm{CZ}$ have their logical actions. These statements concern this code family, not arbitrary codes of the same distance.

The complete gadget statement is [12, Lemma 5.2]: coherent error correction, deterministic verified zero preparation and complete real-gate bodies exist with one-dimensional routing. Fault-free full error correction maps every input into the code. A correctable input with f coordinate deviations and a complete body with j faulty native gate/wait/reset locations gives at most $8j$ output deviations whenever

$$8(f + j) \leq q - 8. \tag{2.7}$$

The assertion is coefficientwise with arbitrary reference and environment. Its proof shows that the verified preparation V_q with $k \leq q$ faults has output in the span of $F|0_L\rangle$ with $\mathrm{wt}(F) \leq k$. A transversal H prepares plus; transversal CNOT then prepares Bell halves. These exact preparations and this real gate inventory are used below. The code and all preparation circuits are fixed before choosing w; their very large sizes enter the finite counts.

For a fault budget $b \geq 1$, [12, Lemma 13.1] assumes $q - 8 \geq 160b + 64$, $b + 1$ disjoint fresh full error-correction passes on every slot, and a deletion yielding at most b effective native faults. It gives the intended instruction on inputs with at most $9b$ deviations, at most $9b$ output deviations, and a common fixed recursive recovery. Arbitrary inputs first normalize to some logical subsystem; its choice is absorbed into a preceding named bad output before interslot gates. All statements are joint with bad blocks, syndromes and references, with the fixed port-permutation images of named outputs. We use its literal statement for the first complete operation after the leading passes. For later operations of our longer word, Section 5 checks (2.7) anew and repeats the joint argument; the old lemma alone is not a statement about our longer list.

The deletion statement [12, Lemma 13.2] requires fresh private decisions before every contact, exclusive dummies, complete paired shuttles, the closed tape-end rule and terminal

endpoint exchanges followed by no data-dependent contact. It bounds insertions by $K = 100000(S+1)^2$ per named child tick, independently of route, tape, list length and rank. Our lists and additional processor ports are different, so Section 5 proves their deletion and lifetime bounds explicitly, including the extra spectator factors and physical initialization pulses. The end-cell rule is built into every level, not imposed only on a clean execution.

The native before-exchange naming and union-cover geometry are [12, Lemma 13.4]: a parent is named if the union of three adjacent basic covers has more than $m = 19600$ distinct named children. With $h_{\mathrm{cov}} = 10$ and $b = K(m+1)$ a good parent has at most b effective faults, descendant radii $10P^i, 10T^i$, and a bad parent has two named children whose complete descendant footprints have max-norm gap at least $8P^i$. There are at most $(20P+5)(20T+5)$ children in a cover. Our routes preserve these covers; a proof is included below to specify their exact use.

The indexed generator and timetable bounds of [12, Lemmas 13.5 and 13.7] are used only with newly enumerated finite lists. They require the fixed private tape machine, tagged calls, rank-independent scalar bit positions, and the end-cell routine included in the tile length. They give exponent 32, padding exponent 35, and coefficient $2^{80}D_{\mathrm{desc}}^{64}$ from the finite syntactic scan. Section 5 spells out the construction, scan, phase counts and search for the expanded lists. No nominal noise rate, erasure selector, random past, higher-dimensional calendar or forgetting bound is an input to this proof. The one-way classical interface and the fixed recursive recovery are the only relevant interfaces of the memory besides the stated local lemmas.

3 Product loading and classical readout

Write $\mathcal{C}_d = \mathcal{D}+de$. A block has classical input value d with f deviations when its computational-basis support lies in the span of words within Hamming distance f of $\mathcal{C}_d$. This permits arbitrary phases and purifications. Repeating a bit d in all N coordinates satisfies this condition with $f = 0$, although such a product word generally lies outside the quantum-correctable span. Quantum deviations mean the operator span of errors supported on the stated number of coordinates applied to code support.

Lemma 3.1 (Private teleportation normalizer)**.** *There is a fixed native pass with the following properties, including arbitrary reference and environment coefficients. A clean pass maps every input into the code and acts as identity on encoded logical data. It takes classical input value d with $f \le q$ deviations to $|d_L\rangle$. After j faulty native locations, if $f + 4j \le q$, quantum-correctable input retains its logical subsystem with at most $4j$ output deviations. Under the same inequality classical input value d retains classical bit support within $4j$ of $\mathcal{C}_d$, without any condition on its incoming phase syndrome.*

Proof. Prepare fresh verified plus and zero blocks, and apply transversal CNOT from plus to zero, producing Bell halves A, B. Apply transversal CNOT with incoming block X as control and A as target, then transversal H on X. Denote their computational-basis words by u on A and v on X. Before private calculations, copy each raw word coordinatewise into a separate record for every output coordinate. Compute the total function a privately on each copy. At output coordinate r, apply X to B_r controlled by its private $a(u)$ and Z controlled by its private $a(v)$. Keep the original raw records until disposal. Fixed swaps move B to the output slots. A private computation is discarded after its one output coordinate, or reversibly uncomputed. Copies, switches, waits, resets, swaps and disposal are ordinary locations; no branch changes a schedule.

In a clean pass the prepared B has code support. Conditional on any raw words it still has code support for every input, because the other Bell half alone interacted with X. Clean

frame bits are identical at all coordinates, so the corrections are uniform transversal logical Paulis and every input is normalized.

For logical input $|l_L\rangle$ and Bell logical value c, the branch amplitude multiplying $|c_L\rangle$ is a fixed positive normalization times

$$\sum_{\substack{x\in\mathcal{C}_l\\ x+u\in\mathcal{C}_c}} (-1)^{v\cdot x}. \tag{3.1}$$

It vanishes unless $u \in \mathcal{C}_{l+c}$ and $v \in \mathcal{D}^\perp = \mathcal{C}$. In the remaining case it equals $|\mathcal{D}|(-1)^{la(v)}$: e is odd and perpendicular to $\mathcal{D}$. Thus each nonzero uncorrected logical branch is a common scalar times $X_L^{a(u)} Z_L^{a(v)}$ on the input. Undoing this frame changes only a branch scalar. This is an operator identity on the logical qubit, hence holds with any reference.

For classical input, expand X in basis words x. If the Bell logical value is c, then $u = x + y$ with $y \in \mathcal{C}_c$. For x within f of $\mathcal{C}_d$, u is within f of $\mathcal{C}_{d+c}$. When $f \le q$, the bit-frame decision is $d + c$. After that frame B has support on $\mathcal{C}_d$, regardless of the phase frame. In a clean pass it also has code support, whose intersection with this basis support is precisely the line $\mathbb{C}|d_L\rangle$. The argument is termwise for arbitrary superpositions of the allowed words and arbitrary joint coefficients.

We now count faults after routing and compiler decisions have been deleted. Include preparation faults in j. The verified preparation statement in Section 2.3 says that a preparation with $k \le q$ faults has terms with at most k affected coordinates. Faults on the transversal H for plus add only their own coordinates. Expand all other faults in local matrix or Pauli bases, retaining their shared environment coefficients. Up to the private frame calculations, the couplings, raw copies, swaps and Hadamards outside preparation are Clifford and coordinatewise. A fault on Bell coupling, input coupling or Hadamard remains at its one index across the affected blocks. A wait touches one token. A raw-copy fault may alter any subset of copies at that index but cannot alter another index. Treat the records as quantum basis controls until disposal, or expand at the raw-record cut in computational-basis matrix units. Keeping the original records makes their phases branch coefficients.

Each private decoder has its own input copy and touches just its declared output coordinate. A corrupt private computation cannot reach another decoder's input or target. A faulty private calculation, frame application or final swap leaves at most its declared output coordinate arbitrary. Subsequent copying or disposal faults expand on their named factors. Every private work factor is reset before use, so old garbage does not add a channel to another target. Verified-preparation deviations, shifts of raw words and arbitrary output targets together occupy at most $4j$ indices in each term. This is a conservative count: every listed fault can be charged to its own coordinate index, but no sharper count is needed. An arbitrarily long fanout only repeats the same raw index. Geometric shuttles do not enter this count because they have already been deleted into the native location budget.

For a quantum input with f deviations, expand the incoming deviations in Paulis. Through Bell extraction such a Pauli shifts u, v at its supported indices and supplies phases. Under $f + 4j \le q$, every intact private decoder chooses the corresponding unshifted ideal logical frame by uniqueness of bounded-distance decoding. The output in that term is the intended logical state with an operator on at most $4j$ coordinates. To see this for coherent faults, project on the true raw words, use the logical identity (3.1), and expand each affected private output in a single-qubit matrix basis. Scalar or environment coefficients can depend on the raw words and fault term, but the clean branch on the logical subsystem is proportional to identity. The conclusion therefore holds on the entire correctable span jointly with the reference.

For classical input, only the u calculation is required to select d. Every shift in an intact u decoder costs at most $4j$ indices, and $f + 4j \le q$ still forces its answer to be $d + c$ for every contributing x, y, c. Different phase-frame decisions at all positions merely change basis coefficients. Bell errors and arbitrary output targets occupy at most the charged $4j$

indices. The output support is consequently within $4j$ of $\mathcal{C}_d$ irrespective of the incoming phase syndrome. Taking sums of fault paths preserves this subspace assertion. □

Lemma 3.2 (Aligned loading at every rank)**.** *Fix $b \geq 1$ and $q - 8 \geq 160b + 64$. Before the ordinary $b+1$ leading full error-correction passes of every epoch, perform $b+1$ private teleportation passes on each slot, finishing all normalizers before an interslot body. With at most b effective native faults in a good parent, a repeated classical bit becomes the required encoded basis value. Correctable quantum input with at most $9b$ incoming deviations retains its logical value. Arbitrary input normalizes to some logical subsystem before interslot gates. These assertions hold jointly and can be iterated from the physical product boundary through every reduction rank. Every physical quantum input can be zero.*

Proof. Count touching faults on fresh work, data, waits between lists and final transport in the total b. Among $b+1$ passes on a slot one has no touching fault. Before that pass, the classical-support statement of Lemma 3.1, applied successively, leaves at most $4b+b$ deviations, including at most b wait deviations. The inequality needed at every stage is $f + 4j \leq 9b \leq q$. The clean pass puts the slot on the desired line $\mathbb{C}|d_L\rangle$. Later teleportations use the quantum clause with at most $4b + b$ deviations and preserve the chosen logical state.

For ordinary correctable quantum input, allow $9b$ incoming deviations and b waits. Each pass meets $f + 4j \leq 14b \leq q$; after the chain at most $5b$ deviations remain. For arbitrary input, a clean pass has code support for some logical subsystem, and later passes obey the same quantum estimate. The clean pass on each slot may occur at a different date; all of them finish before interslot bodies.

These are joint channel assertions. To make their meaning explicit, choose the fixed recovery coordinates W of (5.11). On a valid logical matrix unit with syndrome coefficient Y, a clean private pass has

$$(W \otimes I)\mathcal{T}\Big(W^\dagger(|a\rangle\langle a'| \otimes Y)W\Big)\,(W^\dagger \otimes I) = |a\rangle\langle a'| \otimes \Lambda_{\mathcal{T}}(Y),$$

where identities on exterior factors are understood and $\Lambda_{\mathcal{T}}$ is independent of a, a'. This follows by multiplying the Bell-branch scalar for a by its conjugate for a': the corrected branch is the same scalar times identity. For classical input value d the logical factor on the right is instead $|d\rangle\langle d|$, with the coefficient obtained from the arbitrary incoming allowed basis-word matrix units. On arbitrary input the clean pass is $V\mathcal{N}(\,\cdot\,)V^\dagger$ for a total selecting channel $\mathcal{N}$ on that block only. Its absorption into a preceding named map is the explicit composition (5.14). Section 5 proves (5.13) for full semantic operations, traces port support and gives the rank-zero and both date-zero induction steps. Ordinary error correction and bodies, with subsequent waits counted in b, return its $9b$ cut allowance.

To initialize, replace every top basis bit by N identical bits at the declared child coordinates for all S parent slots. Use fixed zero defaults on unassigned work, and repeat down to the physical row. This is a classical substitution, with no quantum encoding. A good rank-one initial epoch meets Lemma 3.1's classical-support hypothesis. A bad epoch is an arbitrary operation on that same specified input, including replacement. After one rank reduction the specified time-zero row is again exactly the repeated-bit row: a good initial instruction has its intended action on it, whereas a bad initial instruction is named at its native site and port images. At positive dates use the correctable-or-arbitrary invariant. The fault-free equivalence of Lemma 5.1 holds on arbitrary old port contents and excluded scratch, including after a wholly bad preceding epoch. With named faults present, deletion gives that compiled list with at most b charged native faults in a good parent; Lemma 3.1 and the preceding normalization argument apply with that budget. Induct on rank, using this date-zero case anew for each reduced circuit. No assumption says all physical input blocks are good. For

recursively encoded quantum input the induction instead uses the quantum clause from the start.

Finally keep a separate classical pulse bit for every physical quantum slot. Prepare the slot in zero and the pulse bit in its prescribed repeated input value. At the start of every physical period apply a CNOT from that bit to its slot, then reset the pulse bit unconditionally to zero. Noiselessly the first pulse loads the basis value and every later pulse is the identity. A wrong later pulse requires a flag since its most recent reset, within a stencil of at most two periods. Include faults on the initial copy, reset, following noise and storage in the direct physical tests. No upper-rank instruction reads a pulse bit. Work pulses may all be zero. Thus all physical quantum slots start in zero while only classical basis registers contain the nonzero input description. □

Lemma 3.3 (Total classical measurement decoder)**.** *There is a total fixed rule $g : \{0,1\}^N \to \{0,1\}$ such that measuring a block in the computational basis and applying g has exactly the same effects as applying the fixed CSS recovery and measuring its logical qubit in Z, on the entire Hilbert space. Bottom-up application gives the full joint law of recursively recovered logical output measurements, including entangled and bad intermediate blocks.*

Proof. For each bit syndrome and each phase syndrome separately, choose a least-weight representative with lexicographic tie break. Two X parts of weight at most q with the same syndrome differ by a word of $\mathcal{C}$ of weight at most $2q$, so they are equal by $d(\mathcal{C}) \geq 2q+1$. The same holds for Z parts. Each syndrome sector of the low-weight error span therefore contains one Pauli up to phase, and this total table corrects the entire span. It is an admissible instance of the total recovery tables of [12, Lemma 5.1]. Fix this table as the recovery at every recursive cut of [12, Lemma 13.1].

On a measured word w, take the bit syndrome of the Z checks, subtract its chosen leader, and return the coset in $\mathcal{C}/\mathcal{D}$. This is g; inside the bounded-distance region it agrees with a. For a fixed bit syndrome and logical answer, the corresponding corrected classical subset is a translate of $\mathcal{D}$. Its diagonal projector commutes with the X stabilizers, which translate by $\mathcal{D}$, with the Z stabilizers, and with the diagonal phase correction. Thus in the recovered logical-Z POVM the phase-syndrome measurement drops out on summation. The bit correction translates the subset by its chosen leader. On code support these are exactly the two logical-Z projectors; on the full space the same summation gives the diagonal effects $\sum_{w:g(w)=d} |w\rangle\langle w|$. This proves the claimed full-space effect identity.

For several blocks take tensor products of these effects. Their equality as operators gives the joint measurement law even for entangled blocks. Replacing each next-level recovered measurement by its diagonal physical effect and inducting from the top down is precisely bottom-up application of g to the measured words. The identity applies even to bad children, so no correctability premise is inserted at an intermediate measurement cut. Read only declared block coordinates. Tracing out garbage is equivalent to ignoring its measured bits. All tables have constant finite size, and the recovery convention is the one used at every reduction cut. □

4 A fixed synchronous quantum processor

Lemma 4.1 (A moving instruction stream)**.** *There is a fixed on-site word U and a fixed exchange E on finitely many opposite port pairs such that a programmed basis row of B commands performs any width-n, depth-D Toffoli/Hadamard circuit in V native ticks, where*

$$B = (7n+8)(nD+1), \qquad V = 2B. \tag{4.1}$$

The stationary output qubits are at sites $1, \ldots, n$*. The program, the head and all auxiliary inputs are computational-basis values. The same clean evolution acts on arbitrary input entangled with a reference. Output addresses and final time depend only on* n, D*.*

Proof. A native site holds a stationary data qubit, a stream symbol, and a possible head with three quantum buffers and a finite tag. A command moves the head by one, exchanges buffer $r \in \{1, 2, 3\}$ with the stationary qubit, applies H to buffer one, applies Toffoli to the three buffers in order, or waits. All commands except moves keep the head stationary. The head starts ready at site zero. A ready tick reads and executes the local symbol, followed by one refractory tick doing nothing before becoming ready. Symbols shift right by one each tick and are never erased. The refractory tick prevents a just-moved rightward head from reading the same symbol again. On valid states the tag depends only on the classical command history. Moving a buffer means transporting its quantum factor without measurement.

For each port pair let E swap the right factor at x with the left factor at $x + 1$ on every bond. Different bonds use different internal factors, so the layer is a translated disjoint tensor product. Encode stream symbols with a fixed number of such pairs. Swapping left and right locally before E makes the old left field shift right while the other field shifts left. Read the old left field.

Represent a head by a stationary record and two directional port records, each with empty, ready or refractory tags. At the start gather an occupied arriving record into the empty stationary record by controlled swaps. Compute controls on private copies of the old tags and instruction bits. For a valid one-head row exactly one record is occupied. Reset vacated records and their unused data; perform the stationary update; then keep the head stationary or send it into an empty prepared outgoing port. An arriving refractory head stays through the tick and becomes ready. A ready stay command sets the refractory tag. A site with no head leaves its data alone. Reset temporary controls after use.

A finite truth table with a fixed priority on malformed tags defines every branch. A conditional H scatters its operand by controlled swaps to an exclusive dummy, applies unconditional H and gathers. Conditional exchanges and Toffoli use fixed reversible Boolean gates with scratch. The elementary word, with resets, defines a channel on superposed tags as well; its asserted clean semantics needs only valid basis tags and no logical dephasing. All alphabets and records are finite, so this word has constant size. Enumerate internal slots and choose a lexicographically ordered direct truth-table compilation, serializing its gates. Fix one such word once. Let d be the total number of port pairs, including one additional compiler pair. The remaining pairs are user-processor pairs.

For each target gate, start from home zero, visit its at most three distinct operand sites among $1, \ldots, n$, and swap their stationary data into the appropriate buffers. Apply H or Toffoli on the buffers, revisit the operands to swap them back, and return home. Every operand returns to its own site and every buffer returns to zero. Seven traversals of length at most n and seven stationary commands suffice. Order gates by layer, then least operand, retaining the declared control/target order. There are at most nD gates. Pad by waits to the B commands in (4.1).

Let x_j be the head's ready position for command j at time $2j$, for $0 \le j < B$. It is computed from the motion list. Place command j initially at stream site $x_j - 2j$. These addresses are distinct: consecutive differences are $x_{j+1} - x_j - 2 \le -1$ because $|x_{j+1} - x_j| \le 1$. At time $2j$ the rightward stream puts exactly the intended symbol under the ready head. Induct over commands. During the intervening tick the head is refractory, so no accidental second read occurs. At time $V = 2B$ the stationary data contain exactly the output of C. All operations on data are the intended unitary gates or quantum swaps, so the induction is an operator statement and preserves an arbitrary reference. The auxiliary inputs, motion chart and output sites are fixed by the classical program or by n, D as asserted. □

The logical program is protected in the recursive construction. It is not left on an unencoded physical tape. A parent has disjoint compiler slots and user-processor slots. Every coordinate of every parent slot, including user symbols, tags, buffers and ports, is stored on the children's compiler data tracks. The children's own copies of U run on a distinct tensor factor and need no indication of the selected top rank.

5 Recursive compilation with noisy local control

5.1 Inventory and finite contact lists

A native tick performs an indexed compiler primitive g on compiler factors, the fixed word U on disjoint user factors, and then E on all d pairs. The old structural fields select g; a LOAD installs the actual old retained low bit. The independent user word is present at every site, date and rank. All N coordinates of all S parent slots are encoded on ordinary child compiler tracks. No child user factor is borrowed for an encoded parent factor or a private compiler computation. Only child compiler ports route simulated instructions. Thus child user traffic factors from the ideal compiler evolution even when parent user slots are being protected.

Choose S by enumerating the fixed native storage, tape-cell and two neighbor-record tracks, compiler ports, data/contact/dummy and transport tracks, and the fixed user inventory of Lemma 4.1. Arrays, private copies and ancillas of growing length use additional child sites. Fix

$$h_{\mathrm{cov}} = 10, \quad m = 19600, \quad K = 100000(S+1)^2, \quad b = K(m+1), \tag{5.1}$$

and choose the least $s \geq 1$ in (2.6) with

$$q - 8 \geq 160b + 64. \tag{5.2}$$

These are constants before w is chosen.

Prepend the $b+1$ teleportation passes of Lemma 3.2 to the $b+1$ ordinary leading error-correction passes on each slot, completing all before the body. Their primitive contacts are unconditional. For every possible parent compiler opcode and slot choice, including both LOAD values, form a finite candidate list. Its semantic steps are that primitive followed by the fixed primitives of U on their named slots. At every semantic step use a complete error-correction/body on active slots and a complete error-correction/idle on every other slot. Serialize complete operations with explicit waits. Reset each consumed work slot at its declared use.

For each native contact in every candidate, make a separate private invocation. Clear its tape and work, install its finite literal and fixed input formats, fetch the required parent bits through fixed routes, compute the candidate-selection bit, return it, scatter at most three operands conditionally into exclusive dummies, execute the unconditional primitive, gather, undo routes, and dispose of the private work. The candidate's opcode is independent of user state. No private decision reads a quantum program symbol: user instruction controls are tokens of the formal simulated circuit. Each candidate executes the actual U word rather than selecting an input-circuit gate directly.

All intersite private transfers use complete paired shuttles on the single compiler pair. No private stage reads a user port pair or any user factor. Neighbor-record tape exchanges use separate even- and odd-bond substeps. The two tape endpoints omit their outward neighbor exchange. In that substep they reset their outward copy slots and prepare the fixed blank record, consisting of blank symbol and no head, by R_0 and a fixed X pattern. During a tape-update tile, an end cell's tape and neighbor-record routines never couple to its outward port. Separately scheduled installation, fetch, decision-return and operand routes use ports only through complete paired shuttles; their transient use of a tape endpoint's port does not

supply an old port factor to the private decision logic. Physical-only pulse gates remain those of Lemma 3.2 and are covered by the physical direct tests. This rule applies to every compiled private tape at every rank, including the physical one. It removes a read itself, including reads of garbage deposited after an earlier reset, for example by a faulty physical pulse in an outer compiler port.

At the end exchange all d pairs of parent port blocks. For each pair put its left block left of its right block within a home. For one coordinate run the forward adjacent-transposition path from a right endpoint to the matching left endpoint in the next home, followed by the reverse prefix. Only the endpoints transpose. For that pair and coordinate consecutive boundary paths are disjoint. Serialize coordinates and pairs, and clear dedicated transport at its declared boundaries. No data-dependent contact follows this terminal exchange.

Lemma 5.1 (Deletion for the expanded compiler)**.** *For these lists a fault-free epoch with clean oracle reads has exactly its declared compiled action on arbitrary input, old ports and excluded scratch, including arbitrary reference entanglement. A finite set F of named child ticks, located before E with their outgoing images, deletes to at most $K|F|$ arbitrary joint native gate/wait/reset fault locations. The bound is independent of rank, tape length, route length, code size and list length. Extra child user factors and their user-port images add no compiler location charge.*

Proof. On one bond write the ordinary and facing-port factors as (A, p, q, B). Swapping ordinary factors into the facing ports and applying E gives (p, B, A, q). Swap back at both ends and apply E again. The resulting tuple is

$$(A, p, q, B) \longmapsto (B, q, p, A). \tag{5.3}$$

This is $\mathrm{SWAP}(A, B) \otimes \mathrm{SWAP}(p, q)$ on the entire input algebra, tensored with identity on any reference. A word of complete paired shuttles is the specified ordinary permutation times a separate old-port permutation. Fixed route and inverse therefore cancel away from their declared contact.

The tape endpoints read their locally prepared blank neighbor. Every other factor a private invocation reads was reset in its first phase, prepared by installation, or written by a clean LOAD. Complete shuttles keep old port factors in ports. A fault-free private tape consequently computes exactly the reset run of the installed fixed machine on the fetched input. The same channel identity holds after tensoring with the reference identity; it does not assert statistical independence of data from an initially entangled reference. At an inactive contact scatter and gather leave occupied operands alone while the primitive acts on reset dummies. At an active contact they give precisely the selected primitive, also when it is a reset or discard. Work is disposed of before reuse. The selected candidate is therefore its declared list, all other candidates are identities on formal operands, and the child user circuit factors out.

For terminal exchanges let $l_r(i) < r_r(i)$ be one pair's coordinate positions in home i. The intervals $[r_r(i), l_r(i+1)]$ are disjoint. Their forward adjacent-transposition word and reversed prefix transpose just the endpoints and restore every intermediate ordinary transport factor. Moving endpoints into and out of the dedicated transport slots gives exactly the intended parent E. Old microscopic ports still undergo only their separate permutation. Tracing/resetting work composes the same channel on both sides of this equality. This proves fault-free equivalence for arbitrary old ports and excluded scratch, also after a wholly bad preceding epoch.

For faults expand each named child's joint map in local matrix bases on compiler factors, leaving user factors, other faults, the reference and environment as common operator coefficients. Between named ticks child U and user E act on disjoint factors; they can change

those coefficients but cannot create a compiler contact. Follow a touched factor according to its role.

A formal-circuit token, including intrinsic Bell records, gadget work or terminal transport, moves only by fixed permutations until its next formal native use. Commute its error through those permutations and insert it immediately before that use, or at a wait or disposal if there is no later use. Do not commute through an active formal gate. Different insertion dates retain the same environment coefficient.

A private tape, copy, input, fetch, return, control or dummy factor belongs to the invocation pending at that date. Its immutable hardware routes and dates can expose at most the one declared contact of arity at most three. Replace its faulty evolution by its induced map on those operands. Clear and disposal remove all read private state before reuse. Residue in a compiler port cannot become an endpoint tape-neighbor input, by the end-cell rule; other clean private transfers have the product permutation (5.3). Residue in intermediate fetch or return slots is only permuted. The bit eventually read is separately prepared by LOAD, installation or the current private computation. A later fault that couples residue to a fresh live factor is itself a named tick. Thus a faulty invocation reaches no later private invocation through an uncharged path.

A scatter, primitive or gather fault also reaches only its at most three operands and disposable work. An operand stranded in a dummy and then reset is part of that arbitrary operand channel. An unused compiler-port factor stays on unused ports through clean complete shuttles. If its own tick is a faulty shuttle, charge the ordinary participating tokens at that tick by the preceding cases.

Finally a child user factor, or one of its $2(d-1)$ user-port images under E, has no clean joint gate with compiler factors. Its effect on a later joint fault stays in that fault's environment coefficient; that later named tick pays its own charge. This spectator case has zero additional compiler charge. Parent user slots encoded on child compiler tracks are formal protected tokens and belong to the first case, not this spectator case.

Count directly. There are at most S formal-factor insertions. The pending private computations incident to a named tick cost at most two contacts of three locations each; add three more locations for the pending operands even though they were already counted. Thus

$$S + 9 \leq 10(S + 1) \leq K. \tag{5.4}$$

One may instead overcount all three post-E sites, nine pending decision lifetimes and an extra hypothetical lifetime: pre/post placement, incidences and rounding by a factor 12 give

$$12(3S + 30) = 36S + 360 \leq 360(S + 1) \leq K.$$

The end-cell rule actually excludes that extra lifetime. The purpose of the count is its independence of the number of later contacts. Longer lists create later fresh invocations, not surviving private lifetimes. Intrinsic Bell-record fanouts are formal gates handled by Lemma 3.1, not private opcode decisions. Coincident insertions only reduce the union, and common coefficients preserve joint faults.

During private work an adjacent home's arbitrary outward port cannot reach compiler data, and by the end-cell rule no tape-update routine reads the outward port of its tape interval. During a terminal route only the designated endpoint wire crosses that boundary. Write the forward/reverse permutation with the boundary wire as an unspecified input/output: every intermediate internal wire connects back to itself, independently of the received endpoint state. The paired boundary crossings are two crossings of two ticks each per coordinate. They expose only the named outgoing/incoming endpoint and disposable work. No later data-dependent contact reads transport. Entire adjacent homes, their transmitted ports and neighboring basic covers contain all such incidences. Extra child user traffic remains spectator.

Hence the same charge applies at home boundaries and after an arbitrary preceding epoch. Summing over F proves the result. □

Lemma 5.2 (Named covers and separated descendants)**.** *Let $Q_{i+1}(z)$ cover the complete body of a parent instruction, its terminal paths, all shuttle microgates, and the immediate pre/post-E port neighbors. Enlarge to whole adjacent homes and the same sites in the preceding and following parent ticks. Put*

$$\begin{aligned} C_{i+1}(z) &= \bigcup_{\delta=-1}^{1} Q_{i+1}(z+\delta), \\ N_{i+1}(z) &= \mathbf{1}\big[\#\{y \in C_{i+1}(z) : N_i(y) = 1\} > m\big]. \end{aligned} \tag{5.5}$$

Here $z+\delta$ changes the spatial parent index and repeated membership counts once. Then a good parent has fewer than b effective native faults. A rank-i complete descendant footprint has radii at most $10P^i, 10T^i$. A bad parent contains two named children with complete-footprint max-norm gap at least $8P^i$. The child count is at most

$$A_0 = (20P+5)(20T+5). \tag{5.6}$$

Proof. If a basic cover $Q(z)$ has more than m named children, it is contained in $C(z-1), C(z), C(z+1)$; all three parents are named. This conservatively names the overloaded body and both transmitted ports. A good parent's union contains at most m named children. Lemma 5.1 costs at most K locations per such child, including waits and terminal routes, so the total is $Km < b = K(m+1)$. One counts named children, including a child named because of neighboring overload, rather than counting only overloaded children and later multiplying them. Outgoing images are traced to their named before-E origins.

At one scale the home, terminal paths, port neighbors and radius-one union have direct halfwidth at most $5P^j$ and the temporal collar has direct halfheight at most $2T^j$. Complete descendant enlargements obey

$$5\sum_{\ell=0}^{j-1} P^{j-\ell} \le \tfrac{20}{3}P^j < 10P^j, \qquad 2\sum_{\ell=0}^{j-1} T^{j-\ell} \le \tfrac{8}{3}T^j < 10T^j.$$

Counting child centers, whose spacings are P^i, T^i, in this rectangle with spare boundary cells gives (5.6).

If no pair had complete-footprint gap at least $8P^i$, fix one child. A spatial or temporal index distance greater than $2h_{\mathrm{cov}} + 10$ from it would leave such a gap after subtracting the two footprint radii and endpoint rounding; temporally use $T^i \ge P^i$. All child centers would therefore lie in at most

$$(4h_{\mathrm{cov}} + 21)^2 = 61^2 = 3721 < 19600 = m$$

index positions. A bad parent contains more than m distinct named children, a contradiction. The separation is between complete collared descendant footprints, which is the separation used later for primitive-key independence. □

5.2 Indexed execution and the fixed timetable

Lemma 5.3 (Uniform indexed generation and available time)**.** *The expanded finite lists admit a fixed binary tape machine U_{cpu} and a fixed indexed generator with the same native slot inventory at every rank. A computable choice of w makes every parent tick fit in $P = 5\,2^w$ child sites and the unchanged structural period T. The following formulas specify that choice completely up to finite native enumeration.*

Let N_0 be the sum of all unconditional pass lengths over all S slots, including every teleportation/Bell pass and every leading ordinary error-correction pass. Let N_1 be the sum of primitive contact counts of the expanded candidate lists. Let W_{fixed} count designated encoded blocks, maximum simultaneous catalogue work, scratch, exclusive contact/dummy slots, transport, Bell records and guards. Let λ_{tile} be a common multiple of the finite native lengths of a tape-update tile, parallel clear, literal installation, paired shuttle, scatter/contact/gather and disposal, enlarged to bound each local routine including endpoint blank-record preparation.

Fix a nonliteral input-format coefficient a_0 including a candidate tag of $\lceil \log_2(N_1+1) \rceil + 3$ bits. If v is the supplied program text, put $a_{\mathrm{in}} = a_0 + 4|v|$. There is a syntactically computable $D_{\mathrm{desc}} \geq 2$, defined in the proof, for which

$$C_{v,c} = 2^{80} D_{\mathrm{desc}}^{64}, \qquad 2C_{v,c}(1+w)^{32} + 2a_{\mathrm{in}}(1+w) + 4 \leq w^{35} \tag{5.7}$$

ensures completion and return of every valid private call. Define

$$\begin{aligned} F(w) = \lambda_{\mathrm{tile}} \big[& 10 + 32N_0(P+1) \\ & + N_1\{32 + 16a_{\mathrm{in}}(1+w)(P+1) + w^{35} + 32(P+1)\} \\ & + 32dN(P+1)\big], \\ G(w) = W_{\mathrm{fixed}} & + 2w^{35} + a_{\mathrm{in}}(1+w) + 8. \end{aligned} \tag{5.8}$$

Choose the least w satisfying (5.7), all classical structural and packet-position inequalities, $T \geq P \geq 4$, and

$$G(w) \leq P, \qquad F(w) \leq \lfloor 3^w / w^2 \rfloor \leq T. \tag{5.9}$$

This search terminates. Reserve compiler age zero for idle; U and E still act then.

Proof. An index program L takes (v, w, a, s) and locates a phase by comparisons and subtractions against the fixed phase list. For a route the active bond is its elapsed hop number, reversed on return. For a repeated fetch use quotient and remainder or scan the $O(w)$ scalar-coordinate formulas. For a tape-update tile test membership in its interval and time residue modulo the fixed tile length. Installing the pair (v, v) reads bits of the supplied literal. No index query evaluates tape contents. Parent address, age, low bit and tags are the rank-independent positions specified in Section 2.2.

An untagged call returns $d_w(a, s) = L(v, w, a, s)$. A tagged call installs candidate index i, fetched bit ξ and a leading format bit. It returns one if the untagged opcode is that candidate and, for LOAD, ξ equals its prepared value; otherwise zero. The tag is hardwired at that contact and its bit length was included in a_0. The private processor returns this selection bit, not an assumed decoded opcode of some other input format. On malformed results use a fixed total default. There is at most one active private call per home.

For clarity give the finite coefficient prescription underlying [12, Lemma 13.5] for these new lists. Generate the straight-line/loop arithmetic description of the indexed phase list, with standard binary addition, subtraction, multiplication and division. Include all finite catalogue and packet-coordinate formulas and the explicit structural period formula. Variable loops range only over the supplied literal, the $O(w)$ scalar list, 35 multiplications for w^{35}, and w multiplications for 3^w. Such loops nest at most twice; apart from the power loops a carried integer changes only by addition or selection. Let $|\mathrm{desc}|$ count operations and loop headers, let $\lambda_{\max}$ be the largest literal bit length, let $c_T(1+w)$ bound the structural period's bit length, and let $c_{\mathrm{scal}}(1+w)$ bound the scalar-list length. Set

$$\begin{aligned} c_w &= c_{\mathrm{scal}} + 35 + 2, \\ c_{\mathrm{ops}} &= |\mathrm{desc}|^{1+\delta_{\mathrm{fix}}} c_w^2, \\ c_{\mathrm{bit}} &= 2^{|\mathrm{desc}|}(6 + 35 + c_T + \lambda_{\max} + c_{\mathrm{ops}}), \end{aligned}$$

where δ_{fix} is the nesting depth of loops over fixed lists. Define D_{desc} as the maximum of $2, c_{\mathrm{bit}}, c_{\mathrm{ops}}, a_0 + 4|v|$, and two plus the sum of the lengths of these descriptions and v, all their positive integer constants and fixed exponents, 35, the formal tape/track and scratch-integer counts, finite catalogue lengths, and finite transition-table sizes of the arithmetic, one-tape simulation and interpreter templates used below. This is a terminating computation on finite strings before w.

The description scan gives bit length at most $D_{\mathrm{desc}}(1 + w)$ for every intermediate and at most $D_{\mathrm{desc}}^3(1 + w)^2$ arithmetic calls or bit-list operations. Indeed $2^w, 3^w, w^{35}$, addresses, dates and literal constants have linear-in-w bit length. A static arithmetic operation doubles the preceding bound plus two; at most $|\mathrm{desc}|$ such enlargements give the factor $2^{|\mathrm{desc}|}$. A carried loop sum or selection gains at most the logarithm of its iteration count, dominated by $(c_{\mathrm{ops}} + 3)(1 + w)$. The loop count is at most $c_{\mathrm{ops}}(1 + w)^2$ by the two nesting bounds. Input length is at most $a_{\mathrm{in}}(1+w)$ with $a_{\mathrm{in}} \le D_{\mathrm{desc}}^2$. These checks include the expanded finite U lists, all Bell preparations and d terminal pairs. They change coefficients, not the absolute degree.

On B-bit integers, schoolbook arithmetic costs at most $16(B + 1)^2$ bit operations per multiplication or division and at most $4(B+1)^2$ for addition, subtraction or comparison. With head return and at most D_{desc} work tapes, a call costs at most $68D_{\mathrm{desc}}^2(1 + w)^2$ transitions. The call count, scans of at most $D_{\mathrm{desc}}(1 + w)$ cells and input/output cost therefore total at most $72D_{\mathrm{desc}}^5(1 + w)^4$, bounded by

$$H(w) = 64D_{\mathrm{desc}}^8(1 + w)^6.$$

Store the finitely many work tapes as consecutive tracks of one tape with marked heads and sweep the visited segment for each simulated step. This one-tape program uses at most

$$R(w) = 128D_{\mathrm{desc}}H(w)^2$$

steps. Choose once a binary universal interpreter that stores the target transition table and marked target tape, scans that table and sweeps the visited segment for each target step. Its per-step cost is at most $32D_{\mathrm{desc}}^2(R + \text{input length} + 1)$, so a complete run is at most $128D_{\mathrm{desc}}^2R^2$. Substitution gives

$$t_U \le 2^{45}D_{\mathrm{desc}}^{36}(1 + w)^{24} \le C_{v,c}(1 + w)^{32}.$$

Delimiters, table sizes and halt/output conventions are in D_{desc}. A halted head parks its output and idles. A home marker allows return of the head/output in at most its preceding run time plus its input length plus two, giving (5.7). A head moves at most one cell per step.

Choose the padding exponent 35 first. Define v to be the one-tape text of this indexed program with its fixed catalogue. It does not contain D_{desc} or $C_{v,c}$; those are analysis bounds. The coefficient $a_{\mathrm{in}}(v) = a_0 + 4|v|$ is computed from the length of its supplied literal when locating phases, not embedded after measuring the program's own text. A private call on $(v, v, w, a', s', i, \xi)$ therefore computes the selection for exactly the indexed schedule on the parent. This proves self-description without circular runtime choices. The ratio $w^{35}/(1 + w)^{32}$ diverges, so the completion inequality is eventually true.

It remains to charge the phase lengths. An ordinary adjacent transposition is a two-tick shuttle. Routing up to three operands within a home to a contact and back uses at most $6P$ transpositions, or $12P$ ticks. Loads, local gates, guards, buffers and resets fit in $32(P + 1)$ tile units per unconditional contact, yielding the N_0 term. For each candidate contact first reset tape, head/state, input, return, dummies and guards; reserve 32 tile units for this and disposal. Each installed/fetched bit uses its fixed source and at most a linear route to its input and back. The two-tick realization, local copy and common-tile rounding fit in $16(P+1)$ units per bit, giving $16a_{\mathrm{in}}(1 + w)(P + 1)$. Run exactly w^{35} tape-update tiles with complete

paired record shuttles and the end-cell blank rule. Return the answer and operands, perform scatter/contact/gather and undo the route within $32(P+1)$ further tile units. No inactive candidate saves or changes time.

For each pair and code coordinate the terminal segment has fewer than $2P$ edges; the forward/reverse-prefix word uses fewer than $4P$ adjacent transpositions, each two ticks. Endpoint moves, transport clearance and padding fit in $32(P+1)$ tile units. Boundaries for that coordinate run simultaneously, and the serialized cost is $32dN(P+1)$, without a factor for the number of homes. Age-zero idle, clearance and harmless separators use the initial ten units. These counts prove F in (5.8); they include all waits and resets. The guarded tape interval of $2w^{35} + a_{\mathrm{in}}(1+w) + 8$ positions covers both possible head excursions, the input and end guards. All other positions are counted in W_{fixed}, proving G.

All coefficients precede w. Both F and G are bounded by a fixed computable constant times $(P+1)(1+w)^{36}$. Hence

$$\frac{F(w)}{3^w/w^2} \le C(2/3)^w(1+w)^{38} \longrightarrow 0, \qquad \frac{G(w)}{5\,2^w} \longrightarrow 0.$$

The classical requirements and packet checks are eventually true as well. Sequential integer search therefore terminates. The actual structural schedule already has $T \ge \lfloor 3^w/w^2 \rfloor$ from its prescribed Shrink calls; it is never lengthened in native time. Fill unused dates with compiler idles. A child tick and site have the same inventory at every rank; doubling classical word width does not change a low-bit LOAD address. The same F, G therefore hold at every rank.

The choice order is S, d, then K, m, b, then the code and finite lists, then U_{cpu}, the exponent 35, a_0, W_{fixed} and λ_{tile}, then v, then $D_{\mathrm{desc}}, C_{v,c}, a_{\mathrm{in}}(v)$, then w, P, T, and finally physical stencils and thresholds. Finite local implementations are enumerated before their tile, literal, search and stencil constants. The formulas remain fixed, but their instantiated values are computed from the completed schedule. □

Lemma 5.4 (The joint one-rank channel)**.** *For the declared covers, cut types and fault budgets of Lemmas 3.2 and 5.1–5.3, and the structural-read oracle invariant that every unmarked physical read and every good reduced* LOAD *supplies the corresponding bit of the clean retained letter at the epoch bottom, the expanded compiler on good parents reduces to $g \otimes U$ followed by E, with at most $9b$ residual coordinate deviations per slot. At named bad parents it permits an arbitrary joint operation on the named slots and their fixed port images. The same fixed recovery and input/output type can be used through any finite hierarchy. At date zero the product-loading boundary of Lemma 3.2 is valid anew at every reduced rank.*

Proof. By Lemmas 5.1 and 5.2, a good parent has at most $Km < b$ effective native faults in its declared list, including waits and terminal transport. The teleportation chain normalizes arbitrary input or loads the required basis value as in Lemma 3.2. There are then $b+1$ ordinary full error-correction passes on every slot with fresh/reset work. The first complete operation satisfies the literal hypotheses of [12, Lemma 13.1]: the particular code, (5.2), disjoint passes, deletion budget and declared common cut are all present.

For every later complete semantic operation, use [12, Lemma 5.2] separately. The preceding output has at most $8b$ residual coordinates, or $9b$ with waits. Allow another b waits before the count, so $f \le 10b$ and $j \le b$. Then

$$8(f+j) \le 88b \le 160b + 64 \le q - 8. \tag{5.10}$$

The next operation leaves at most $8j \le 8b$ deviations. Terminal waits add at most b, restoring $9b$. This induction applies to every primitive of U as its own complete error-correction/body, including logical resets of head work. A block entering partway through U is an ordinary input to that complete operation; no assumption treats all of U as one primitive.

Recovery coordinates and the joint identity. Let $V : \mathbb{C}^2 \to (\mathbb{C}^2)^{\otimes N}$ be the fixed encoder. For every full stabilizer syndrome σ choose the Pauli leader T_σ of the total recovery in Lemma 3.3, including its fixed choices outside the correctable region. The unitary change of coordinates W is defined by

$$WT_\sigma V|a\rangle = |a\rangle_L \otimes |\sigma\rangle_S, \qquad a = 0, 1.$$

The syndrome sectors are orthogonal and exhaust the block Hilbert space, so this defines W everywhere, not merely on code support. The physical recovery is $\mathrm{Tr}_S \circ \mathrm{Ad}_W$. For every operator F in the weight-at-most-q error span there is a syndrome vector v_F, independent of a, such that

$$WFV|a\rangle = |a\rangle \otimes v_F. \tag{5.11}$$

Indeed expand F in Paulis. In each syndrome its low-weight representative differs from T_σ by a scalar stabilizer on the code: the product has weight at most $2q < 2q + 1$ and cannot be a nontrivial logical Pauli. This proves the formula for a Pauli and then by linearity for its whole span. The tensor-product version holds with every exterior operator as coefficient. For $0 \le f \le q$, let S_f be the span of the vectors obtained from errors of weight at most f. A good complete operation needs only $L \otimes S_{10b}$ at its incoming cut, by the preceding wait allowance; its output is in $L \otimes S_{9b}$. Both lie in the recoverable space $L \otimes S_q$. Syndrome coefficients are arbitrary within the specified incoming space; all syndrome sectors are allowed without restriction on named bad slots.

Fix one complete semantic operation, including its ECs and charged waits, and first take a unitary native body G. Use W on each input/output slot. Let B collect every exterior bad slot and let Z collect reference, unused factors and retained environment. Expand a joint faulty Stinespring operator in local matrix bases on its charged factors, and label the resulting terms by α. The coefficientwise conclusion of [12, Lemma 5.2, final proof paragraph], applied with (5.10), says that each term sends a correctable input to its ideal logical image with correctable output errors, with a coefficient independent of every logical amplitude. In the above coordinates this is precisely

$$\widehat{K}_\alpha(|\boldsymbol{a}\rangle \otimes |s\rangle \otimes \eta) = G|\boldsymbol{a}\rangle \otimes A_\alpha(|s\rangle \otimes \eta), \quad |s\rangle \in S_{10b}^{\otimes r}, \quad \eta \in B \otimes Z. \tag{5.12}$$

Here r is the number of participating slots and $\widehat{K}_\alpha$ includes the fixed input/output coordinate changes; A_α may act jointly on all syndromes, bad slots and environment. One can read A_α directly by expanding the residual operators F and using (5.11): their vectors v_F replace the residual physical errors. Independence of $\boldsymbol{a}$ is the coefficientwise gadget assertion, not an inference from correctable output support alone.

Retain coherent cross terms before tracing fault records. For arbitrary logical matrix units, syndrome matrix units in the allowed span, and arbitrary $Y \in \mathcal{B}(B \otimes Z)$, (5.12) gives the operator identity

$$\begin{aligned} &\widehat{K}_\alpha(|\boldsymbol{a}\rangle\langle\boldsymbol{a}'| \otimes |s\rangle\langle s'| \otimes Y)\widehat{K}_\beta^\dagger \\ &\quad = G|\boldsymbol{a}\rangle\langle\boldsymbol{a}'|G^\dagger \otimes A_\alpha(|s\rangle\langle s'| \otimes Y)A_\beta^\dagger. \end{aligned} \tag{5.13}$$

This includes $\boldsymbol{a} \neq \boldsymbol{a}'$ and $s \neq s'$ and does not diagonalize Y. Summing the prescribed coherent coefficients, or summing Kraus branches after tracing their records, gives

$$\widehat{\mathcal{G}}(X) = (\mathrm{Ad}_G \otimes \Lambda)(X) \quad \text{on } \mathcal{B}(L^{\otimes r} \otimes S_{10b}^{\otimes r} \otimes B \otimes Z).$$

The retained coefficient map Λ is independent of the logical matrix indices. The actual full circuit is trace preserving; individual fault-expansion terms need not be. Thus no termwise

normalization or independence of faults is asserted. Tracing syndromes proves the logical channel equality with an arbitrary reference.

For a native preparation/reset, take fresh verified output work and discard the old logical input as the ideal instruction specifies. Equivalently dilate the ideal reset by $|a\rangle \mapsto |0\rangle \otimes |a\rangle_{\text{discard}}$, apply the same coefficientwise argument while retaining this prescribed discard, and trace it at disposal. On logical matrix units the trace is $\mathrm{Tr}(|a\rangle\langle a'|) = \delta_{a,a'}$, giving the reset channel in place of Ad_G. Its new output is the required logical basis value. The discard is not subsequently read by any instruction. The other slots obey (5.13) unchanged. This treats the nonunitary native inventory as well as gates.

Arbitrary inputs and support of a named map. The identity just proved concerns valid correctable inputs. On a completely arbitrary block a clean private normalizer instead has Kraus operators Vn_γ, for some maps n_γ from that entire block to its logical qubit, since its output has code support. Put $\mathcal{N}(X) = \sum_\gamma n_\gamma X n_\gamma^\dagger$. This is a total channel, with no claimed prescribed logical value outside the correctable input type. On a correctable input, the Bell identity of Lemma 3.1 gives $n_\gamma W^\dagger(|a\rangle \otimes |s\rangle) = c_{\gamma s}|a\rangle$; hence its version of (5.13) is identity on the logical subsystem. On the classical input-value line it gives the fixed output $|d_L\rangle$, coefficientwise, as proved there.

Suppose the incoming arbitrary block is a preceding named output on a set B_0, with joint map $\mathcal{F}_{B_0 Z}$. Before any subsequent interslot gate its normalization is therefore absorbed as the concrete composition

$$(\mathrm{Id}_{L_{\text{good}}} \otimes \mathcal{N}_{B_0} \otimes \mathrm{Id}_Z)(\mathrm{Id}_{L_{\text{good}}} \otimes \mathcal{F}_{B_0 Z}) = \mathrm{Id}_{L_{\text{good}}} \otimes [(\mathcal{N}_{B_0} \otimes \mathrm{Id}_Z) \circ \mathcal{F}_{B_0 Z}]. \tag{5.14}$$

Any syndrome coefficients of good slots are included in Z; (5.13) has already factored out their logical factors. Thus (5.14) is an equality on all matrix units even when those factors are entangled. The composition reads no additional good logical factor. The clean pass on each slot can have a different date, because all such passes precede the interslot body. Faults before that slot's clean pass belong to its selecting map; the subsequent passes obey the $5b$, then $9b$, allowance of Lemma 3.2. This proves the arbitrary-input case separately from the valid-code case. An intended gate with an operand selected this way is the intended gate on that actual selected logical input.

A named bad native tick is an arbitrary joint channel on that site's declared slots, placed before E. If B_0 denotes those named factors, its post-exchange support is exactly $E(B_0)$:

$$\mathrm{Ad}_E \circ (\mathcal{F}_{B_0 Z} \otimes \mathrm{Id}_{B_0^c}) = (\mathcal{F}^E_{E(B_0) Z} \otimes \mathrm{Id}_{E(B_0)^c}) \circ \mathrm{Ad}_E \,.$$

Every ordinary slot stays at its site; each right or left port factor goes to its facing factor at the neighboring site. The terminal routes implement just this permutation by Lemma 5.1, restoring intermediate transport. Hence there is no unnamed intermediate output in the support. An arbitrary bad child compiler/user joint map retains both sets of factors in its common coefficient; the spectator argument in deletion introduces no additional compiler contact.

Rank zero and the induction. At rank zero condition on primitive flags. Off the direct stencils all physical gates, resets, waits, exchanges and pulse actions in the base query are their declared operations on the actual input. Lemma 6.1 and (6.2) certify every actual unmarked raw opcode/low-bit read. On marked queries allow the entire joint operation on the named site slots before E, with its port images. This is the cut assertion at the physical inventory, with no encoding assumption. The flag set is independent of all quantum inputs.

Assume this joint cut assertion for rank i, as an equality on matrix units with arbitrary syndrome, bad-slot and reference coefficients. Deleting a rank-$(i+1)$ good parent's child

routines gives at most $Km < b$ faulty native locations by Lemmas 5.1 and 5.2. At positive dates every unmarked incoming slot is in the $9b$-correctable type, and every remaining slot is a named arbitrary output. Normalize the latter by (5.14); apply (5.13) to each complete semantic operation in its circuit order, using (5.10) each time; and carry the final $9b$ errors through the one fixed recovery by (5.11). Keep named maps in their circuit order rather than commuting them through an intended entangling gate. Apply the displayed port identity at the terminal exchange. These steps give exactly the same joint cut assertion at rank $i + 1$. A bad parent is itself an arbitrary named native map on that same inventory. Since at most $9b \le q$ child-coordinate operators remain in a good output block, their union span is corrected by that fixed table; the table never depends on the history.

There are two date-zero cases. For the product mode the recursive substitution places each classical bit on the repeated word $|d\rangle^{\otimes N}$, with arbitrary joint coefficients allowed on already named outputs. Lemma 3.1's classical clause and the $b + 1$ private passes select $|d_L\rangle$ before any interslot body. Reducing one rank therefore gives exactly the prescribed repeated-bit input row for the next reduction, with named maps where needed; this uses no higher-rank correctability premise for a product word. Physical pulses implement precisely that row from zero quantum slots. For the quantum-endpoint mode the recursive encoder instead supplies V at every initial block; its logical matrix units satisfy (5.11) with zero syndrome. The quantum clause of the normalizer preserves those units and their arbitrary reference coefficients, and the physical pulse values are zero. Repeating these respective date-zero arguments at each reduction rank, followed by the positive-date induction just given, proves the assertion for both input types through any finite hierarchy.

At the base, direct physical stencils and the structural tests certify actual raw opcode reads; Lemma 6.1 verifies that assertion for this completed period. At rank $i + 1$, each good reduced child LOAD supplies the low bit of its clean rank-i retained letter. The packet linkage of Section 2.2 supplies exactly the parent age/address/tags and retained bit at their fixed positions, referring to the parent state at the epoch's bottom. A bad child LOAD is already a named fault and its influence is charged to its one pending contact by deletion. The indexed private program therefore selects the intended clean reference g, independently of actual noisy parent projections. Child user evolution is a disjoint spectator; the encoded parent user factors are protected compiler tokens.

This proves the induction invariant: each good reduced rectangle implements its clean native instruction on its actual logical inputs, with common correctable output, while arbitrary joint faults occur only at named sites and their E images. Its output inventory is the same as its input inventory. For the time-zero step apply Lemma 3.2 at every rank. For encoded quantum input use its quantum clause throughout. This establishes all asserted reductions on a finite hierarchy without assuming accuracy at an unproved higher-rank oracle.

Finally implement the physical indexed dispatch by one-way controls and exclusive dummies, add the fixed U, physical-only pulses, unchanged structural transition and E, and decompose SWAP/Fredkin/CZ into the elementary inventory. Every layer is a translated disjoint tensor product. Include one isolated classical description lane, idle and noisy at every layer, in the same inventory. The resulting finite word has a fixed layer count D_{layer} and fixed finite register Q_{site}, obtained by enumerating all named factors of this completed implementation. Noise follows every layer, including waits. This is the fixed hardware used in the theorem. □

6 Primitive-key probability bounds

Lemma 6.1 (Structural interface for the completed period)**.** *For the completed physical word of Lemma 5.4, the structural lane has the four conclusions of [12, Lemma 12.4]: its decoded*

positive history is a native history off a finite-stencil flag field; every incorrect raw control read has a protected-test certificate; its dephased marginal is a one-way autonomous classical chain; and the initialized history joins the clean negative tower with no negative-time flags. The stencil dimensions in (2.4) and (2.5) are enumerated on this completed word.

Proof. Fix the native finite alphabet Alph_R, its injective canonical bit representation enc_R, and the total parser dec_R mapping unused strings to the dead symbol. The Boolean evaluator on arbitrary incoming banks is exactly

$$\mathrm{enc}_R(f_R(\mathrm{dec}_R(w_-), \mathrm{dec}_R(w_0), \mathrm{dec}_R(w_+)))\,,$$

with total defaults on unused intermediate Boolean strings. Evaluate on private initialized targets with distinct old/new banks and separate one-way destination copies. The compiler, U, Bell records and pulses do not target an autonomous bank or its private evaluator targets. Their added layers are waits for this classical computation. In the absence of flags in its complete implementing stencil, the evaluator gives the total native transition on legal or malformed inputs.

Enumerate that stencil on the completed elementary word: each incoming bank's storage from its preceding completed write through its last relevant read, one-way fanout copies and their storage, every parser/evaluator gate, old/new transfers, all resets and noise immediately after them, private targets until read, and each raw source bit through its actual quantum-control read. Include incoming period $t-1$ and outgoing period t to cover both the transition into native date t and all reads at that date. At zero keep only nonnegative elementary keys. All displacements and the layer count are fixed, so its maximal key count s_R and integer radii r_x, r_t are finite computable constants. Added layers only add the required wait keys; no stencil from a shorter old period is reused without them. Any decoded disagreement from the native transition thus requires a stencil flag. Corruption into a different legal word and into a non-codeword are both covered.

The physical processor reads exactly old age, address, retained low bit and required tags, each through a private copy. U, pulse bits and Bell records read no structural bank. For a base query $q = (x,t)$, enumerate these actual reads by their named source factor and elementary read layer. Let $\mathcal{L}(q)$ be this finite list. For $\ell \in \mathcal{L}(q)$ let $\Sigma_\ell(q)$ consist of the source's implementing write, its storage since that write, the one-way copy gates and storage, and the actual control gate with its following noise. Include the incoming/outgoing periods and clip negative elementary times exactly as above. The enumeration defining $\Sigma_R(q)$ takes the union over this entire list, not a separate native flag for each bit or copy. Hence

$$\bigcup_{\ell\in\mathcal{L}(q)} \Sigma_\ell(q) \subseteq \Sigma_R(q), \qquad E_R(q) = \mathbf{1}[\text{a flag in } \Sigma_R(q)]. \tag{6.1}$$

A source for an exchanged port is charged at its before-E origin; the opposite origin has its own base query. Private opcode-decision arithmetic and its transported result are in the direct test, not additional reads of a structural bank. This exhausts the read categories in the declared physical word.

Here is one dominating protected rank-one test, with its dimensions explicit. Put $m_R = \lfloor w_R/5 \rfloor$, $k = \lfloor x/P \rfloor$ and $h_0 = \lfloor t/T \rfloor$. Take the W_0 of [13, Lemma 5.24] with central homes $k, k+1$ and work period h_0, including its seed, work and lower-call collars. Denote it $W_{\mathrm{raw}}(q)$. It contains q. That lemma's actual stencil inclusion is

$$W_{\mathrm{raw}}(q) \subset [k - m_R - 3, k + m_R + 5)P \times [h_0 - m_R - 2, h_0 + m_R + 1)T.$$

Its width is at most $(2m_R+8)P < w_R P$ and its height at most $(2m_R+3)T < w_R T$; the structural choice of w_R satisfies $w_R \geq 20$, since the classical source fixes $w \geq 10^{10}$ globally.

The complete primitive support is the union of $\Sigma_R(y)$ over nonnegative native points y of this test. It has at most

$$(2m_R + 8)(2m_R + 3)PTs_R$$

keys. Relative to q, its spatial and macrodate radii are at most $(m_R + 6)P + r_x$ and $(m_R + 4)T + r_t$, respectively, both bounded by $C_{\mathrm{clk}}P, C_{\mathrm{clk}}T$ in (2.5). Clipping at physical zero only decreases these bounds. Define $F_{\mathrm{raw}}(q)$ to be failure of 0-sparsity of E_R restricted to this test.

If the parsed full symbol is wrong, the initialized projection's tests certify that event. If the parsed symbol agrees with the clean one but any actual raw bank/copy/control bit disagrees with its canonical value, a flagged write/storage/copy/read key is necessary. By (6.1) this implies $E_R(q) = 1$, and $q \in W_{\mathrm{raw}}(q)$ implies $F_{\mathrm{raw}}(q)$. Thus, pointwise in the primitive flags,

$$\{\text{some wrong raw read at } q\} \subseteq \bigcup_{j\geq 1} \bigcup_{\nu \leq N_{\mathrm{clk}}} F_{j,\nu}(q) \;\cup\; F_{\mathrm{raw}}(q). \tag{6.2}$$

All reads in $\mathcal{L}(q)$ use the same added test. Consequently the largest number of extra tests after this deterministic covering is $B_{\mathrm{raw}} = 1$. No replacement of an imported test or spare capacity in N_{clk} is assumed. The dimension bounds permit (2.3) at $j = 1$ for this test. The parser is not presumed to govern malformed raw quantum controls.

At each period boundary the retained banks are dephased, with joint state $\sum_a |a\rangle\langle a| \otimes \rho_a$. Their truth-table updates are basis permutations or total classical channels. A bank is only a source for a one-way control into private work. Tracing targets after a controlled unitary or a copy-and-discard does not change the bank probabilities, even with arbitrary ρ_a. A flagged depolarization of a basis-valued bit is a replacement by a private independent fair bit. Induction over elementary layers therefore gives exactly an autonomous classical stochastic chain. No quantum factor, private result, Bell record or pulse writes back to its banks. The current construction has no terminal erase acting on them. Private read copies are consumed by disposable work and their faults are in the same read/dispatch or direct stencils.

The prepared structural row is the canonical representation of the exact aligned boundary of [13, Lemma 6.4]. Glue its clean negative tower to the actual decoded nonnegative history. The row agrees at the join, negative updates are exact, and every positive native-transition failure has the stencil certificate just proved. This is the data-history domain of classical Proposition 5.3. No negative physical innovation is introduced. Thus all four clauses are proved for the present word, and the dependent-key and projection lemmas apply with its newly enumerated stencil constants. □

For each physical site/tick take a direct test consisting of all primitive keys that can spoil its instruction or incident ports during that tick, including inputs since immediately preceding resets, old control reads and the two-period pulse lifetime. Name both origins of a faulty exchange. A direct test without a flag certifies the intended action on the actual incoming quantum state when the structural reads are correct. Let s_H be its maximal key count and let $r_H \geq 1$ bound its spatial and macrodate radius. Displacement/copy gates of the structural lane have the full stencils of Lemma 6.1. In particular the noise following an elementary layer is assigned to that layer's period, including the last layer before readout.

Fix the following constants after this physical enumeration, taking maxima of key counts and radii for the wire and whole-site conventions:

$$\begin{gathered} B_{\mathrm{raw}} = 1, \qquad N_t = N_{\mathrm{clk}} + B_{\mathrm{raw}} + 1, \quad C_* = \max(1, C_{\mathrm{clk}}, r_H), \\ A_p = A_0^2, \quad a = \frac{1}{2A_p}, \quad D_* = 1058 N_t P T, \\ \lambda = 1 + \frac{1}{2T}, \qquad \rho = 2^{\log_T P}, \\ M = \max\left\{ \left\lceil \log_2 \frac{4N_t}{a} \right\rceil, 2C_* \left[\left\lceil \log_2 \frac{2D_*}{a} \right\rceil + T\lceil \log_2(PT) \rceil + 2 \right] \right\}. \end{gathered} \tag{6.3}$$

All radii are rounded upward to integers. The common threshold is

$$p_0 = \min\left\{ \frac{1}{2}, \frac{2^{-M}}{\max(1, s_H)}, \; e_{R0}, \; A_R^{-1} \left(\frac{2^{-M}}{N_R D_R} \right)^{J_R} \right\} > 0. \tag{6.4}$$

The last term already implies the e_{R0} condition because $2^{-M} \le 1/2$; it is displayed to retain the witness hypothesis explicitly. The enlarged N_t is used in D_*, both terms of M, and hence in p_0; no earlier numerical threshold is retained. All constants depend only on the completed fixed circuit. They are not chosen anew at each rank.

Lemma 6.2 (Decay of named-site probabilities)**.** *Initialize the structural lane on the line as above. Mark a base query if its direct test or one of its protected structural tests fails, and use (5.5) at higher ranks. For every $0 \le p \le p_0$, if u_i is the supremum probability of a named rank-i query, then*

$$u_i \le a\, 2^{-\lambda^i} \qquad (i \ge 0). \tag{6.5}$$

The marking event is a function only of primitive flags, independent of the circuit program and quantum input.

Proof. Independent elementary wire flags and private fair bits for classical replacements realize (2.1) exactly. Lemma 6.1 checks every hypothesis of the independent-key and initialized structural projection imports on the present word, with the unchanged literal, total parser, one-way banks and date-zero row. Negative native flags are zero. Include all keys since a preceding write through its actual read, rather than treating a stored classical bit as freshly noisy only when read. Every wrong read is therefore certified. Tests depend on flags alone.

By (6.4), direct tests have probability at most 2^{-M}. By (2.3) and the last threshold term, a rank-j structural test has probability at most $2^{-M2^{j-1}}$. At each query there are at most $N_t = N_{\mathrm{clk}} + B_{\mathrm{raw}} + 1$ tests per rank $j \ge 1$: the N_{clk} imported projection tests, the one dominating raw-read test only at rank one, and the direct test only at rank one. The added test has the probability and support bounds proved in (6.2); thus all these tests satisfy the common rank bounds, with full primitive supports inside radius $C_* T^j$ since $T \ge P$. A query can be marked by remote tests, but the certification is pointwise: one need not paint a failed test's entire rectangle into the base error set. Extend negative time by empty physical-error tests for geometric counting. Summing the doubly decreasing tail gives

$$\begin{aligned} u_0 &\le 2N_t 2^{-M}, \\ u_{i+1} &\le A_p u_i^2 + D_* (PT)^i 2^{-M\rho^i/(2C_*)}. \end{aligned} \tag{6.6}$$

We prove the second inequality explicitly, as native marks themselves are dependent.

A named parent has two named children whose complete descendant footprints have gap at least $8P^i$ by Lemma 5.2. There are at most A_p ordered pairs. For each child delete all base tests of ranks j with $C_* T^j > P^i$, and recompute its increasing naming event from the

remaining tests. Its probability is at most u_i by monotonicity. Its primitive keys now lie within P^i of its original query footprint. The two truncated events have disjoint complete key supports by the footprint gap, hence are independent. Overlaps among smaller tests within either one do not matter. This gives the square term.

The parent footprint has at most

$$(20P^{i+1}+3)(20T^{i+1}+3) \le 529(PT)^{i+1}$$

base query points. If a child was named before truncation but not after, some removed base test in that footprint failed. At the least deleted rank,

$$2^{j-1} \ge \tfrac{1}{2}(P^i/C_*)^{\log_T 2} \ge \frac{\rho^i}{2C_*}.$$

The deleted-rank tail is at most twice its first probability, since $M \ge 1$ and exponents double. Union over base points and at most N_t tests of each rank gives exactly the second term of (6.6), with $D_* = 2 \cdot 529 N_t PT$. No independent copy of a repeated primitive key is used.

We verify the induction with the displayed constants. For $T \ge P \ge 4$, using natural logarithms for this comparison,

$$\rho = \exp(\log 2 \, \log P / \log T) \ge 1 + 1/T,$$

because $T \log 2 \log 4 \ge \log T$. Hence $\rho^i \ge 1 + i/T$ and $\rho^i \ge \lambda^i$. In the tail exponent, the three summands in the square brackets of $M/(2C_*)$ pay respectively for $\log_2(2D_*/a)$, for $i \log_2(PT)$, and for λ^{i+1}: use $T\rho^i \ge i$ and $2\rho^i \ge \lambda^{i+1}$ since $\lambda \le 2$. Thus the tail term is at most $(a/2)2^{-\lambda^{i+1}}$.

If $u_i \le a2^{-\lambda^i}$, the square term is at most the same quantity, since $A_p a = 1/2$ and $2\lambda^i \ge \lambda^{i+1}$. The first definition of M gives $u_0 \le a/2 = a2^{-\lambda^0}$. Induction proves (6.5) at every rank. Merely exponential decay in the rank would not give the required polynomial dependence on logarithmic precision. □

7 Finite rings, preparation cost and precision

Lemma 7.1 (A finite chart for the whole computation)**.** *For $n \ge 1, D \ge 0$ and the common sufficient request h, let V be (4.1) and set*

$$\begin{aligned} W = (V+1)(n+2V+9), \quad H_{\mathrm{req}} = h + \lceil \log_2 W \rceil + 1, \\ k = \min\{j \ge 1 : \lambda^j \ge H_{\mathrm{req}}\}. \end{aligned} \tag{7.1}$$

Let $R \ge 1$ be one plus the sum over elementary layers of the greatest pairwise site distance within a gate of that layer, taking zero for an on-site layer and including controller, pulse and user gates. Put

$$\begin{aligned} t = VT^k, \qquad m_c = n + 2V + 10 + \lceil Rt/P^k \rceil, \\ L = (2m_c+1)P^k + 2(h+1). \end{aligned} \tag{7.2}$$

Under the independent wire noise of equation (2.1) for $0 \le p \le p_0$, with p_0 as in equation (6.4), there is a product initialization on this ring giving total variation error at most 2^{-h}. It and the decoder are computable in polynomial time in their finite input and output sizes. Their geometry, selected output slots and time depend only on n, D, h. With

$$g = \lceil \log_\lambda T \rceil, \qquad A = \lceil (300 + 150R)T25^g \rceil + g + 10, \tag{7.3}$$

the resource bound (1.1) holds. The same bad-event bound is valid in trace distance after noiseless recursive recovery for a recursively encoded input with reference.

Proof. On every top date $0, \dots, V$, use the interval $[-V-3, n+V+3]$. It contains the full ideal top past cone for stationary output sites $1, \dots, n$, including origins of exchanged ports and all needed initial stream positions. There are at most W queries, with date V included for safety. The noise after the last elementary layer of physical period $VT^k - 1$ belongs to that period's direct test, hence to top date $V-1$; there is no untested terminal noise layer. Lemma 6.2 and a union bound give failure probability at most

$$Wa2^{-\lambda^k} \le 2^{-h}. \tag{7.4}$$

On the complementary event Lemma 5.4 implements the ideal top channel throughout the cone, Lemma 3.2 supplies its basis input, and Lemma 3.3 gives the full output law. This is conditional on primitive flag histories, not on postselected logical measurement outcomes. Conditional on marks, the allowed errors are arbitrary joint channels and the good-gadget statements are full channel statements. The good event is independent of program and quantum input. Mixing the bad histories with weight at most (7.4) therefore costs at most that weight in total variation, or in trace distance with a reference after recovery.

A real physical dependency path moves at most R sites per period, by its definition as a sum over elementary layers. Embed complete rank-k homes with indices $-m_c, \dots, m_c$ consecutively in the ring starting at physical zero, translating line coordinates by $m_c P^k$. The output blocks lie in this interval at distance at least Rt from either edge. Every actual ancestor of their coordinate measurements through all t periods, including autonomous banks, private control storage and physical pulses, is therefore in the injective chart. Initialize those complete homes from the line row at zero. This is the precise cone hypothesis of [12, Lemma 12.6]; no clean full-ring hierarchy is asserted.

Outside the chart put fixed filler. Reserve an isolated classical lane, idle and noisy at every layer, to record the sufficient request in a unary suffix of length $h+1$; the extra $2(h+1)$ sites in (7.2) provide this space. In the same ignored lane within the chart record the input circuit itself, with its layer boundaries. Use n entries per layer, padding by idle, two opcode bits and three $\lceil \log_2(n+1) \rceil$-bit operand fields. Its length is at most

$$nD\big(2 + 3\lceil \log_2(n+1) \rceil\big) \le nD(2+3n) \le V.$$

Here $\lceil \log_2(n+1) \rceil \le n$ and $2 + 3n \le 2(7n+8)$ for $n \ge 1$. The chart has at least V sites, separate from the request suffix. Thus the initial basis word describes the original circuit as well as its operational stream. The ignored record is noisy from time zero and never used for later computation. Its idle wires belong to the completed physical inventory. Only the encoded operational stream is used by the processor. The physical circuit is fixed independently of the length of this record.

Couple ring primitive keys in the actual output past cone to a single independent line extension with matching initial row there. Induction over elementary gates gives identical output marginals. Analyze the line on finite portions: follow the finitely many rank reductions and enlarge until the descendant covers used are included. High-rank certificates may examine more keys in the independent line extension; they bound an already coupled output and introduce no extra ring randomness. The same ring key is never assigned two independent lifts. This pays the full physical horizon VT^k, with many top homes and many top ticks; it does not iterate a one-tick memory-fidelity theorem.

We give the initialization cost rather than only invoking computability. For a physical coordinate $|x| \le L$, the aligned boundary algorithm repeatedly takes $\lfloor x/P \rfloor$, reaches 0 or -1, includes the next endpoint digit and substitutes requested child symbols back down. There are at most $\lceil \log_P(L+1) \rceil + 2$ levels. Endpoint independence concerns the whole encoded cell. A level-j word has 2^j times the fixed width ℓ_0, and storing entire words over the chain costs at most $8\ell_0(L+1)$ bits. There are a fixed number of word fields and fixed scalar formats.

Defaults, address, age and selected scalar, half-word or padding are therefore computed by polynomial-time bit copies and integer division.

The family literal is also needed. At physical rank it is the fixed never-halting machine's initial configuration. Each promotion extends by a blank half and performs one bounded true tape step using masks, one-bit shifts and Boolean operations. For rank $r \le \lceil \log_P(L+1) \rceil + 2$, compute exactly those r promotion steps on width $2^r \ell_0$. This costs polynomial time in L. No negative history is simulated or sampled.

For quantum basis loading, recursively put the N identical copies at the designated coordinates and fixed child-work defaults. The tree of homes down to the physical row has at most $2L$ nodes because $P \ge 4$, with constant work per native slot and logarithmic-length coordinate arithmetic. Computing the stream and physical pulse values is polynomial as well. For decoding, all single-block tables are fixed constants and bottom-up measured-word processing takes at most the same number of home visits. The decoder ignores every gate of C, the operational stream and the ignored record. It reads only the geometry and output data slots determined by n, D, h. None of these algorithms uses p.

To check the single displayed polynomial, put $X = \max(1, n + D) + \log_2(1/\varepsilon)$, so $X \ge 1$ and the maximum is $n + D$ for positive width. For every admitted precision input, $h \le X + 1$. From (4.1),

$$B \le 30X^3, \quad V \le 60X^3, \quad W \le 61 \cdot 130X^6.$$

Since $\log_2(61 \cdot 130) < 13$ and $\log_2 X \le X$,

$$\lceil \log_2 W \rceil \le 6X + 14, \qquad H_{\mathrm{req}} \le 7X + 16 \le 23X \le 25X.$$

The minimal rank in (7.1) gives $T^k \le T H_{\mathrm{req}}^g$ and $P^k \le T^k$. Hence

$$t \le 60T25^g X^{3+g}.$$

The ceiling in m_c gives

$$\begin{aligned} L &\le (2n + 4V + 23)P^k + 2Rt + 2(h+1) \\ &\le (270 + 120R)T25^g X^{3+g} + 6X, \end{aligned}$$

where $2(h+1) \le 2X + 4 \le 6X$. The integer A in (7.3) dominates these coefficients and exponents, proving (1.1). A period has the fixed D_{layer} elementary layers, so elementary-layer time is at most $D_{\mathrm{layer}} A X^A$ if that unit is desired. The site count corresponds to $Q_{\mathrm{site}} L$ qubits, with fixed finite Q_{site}. □

8 Proof of the computation theorem and quantum extensions

Proof of Theorem A. Choose the processor of Lemma 4.1 and the slot inventory, code and budgets of (5.1)–(5.2). Use the closed tape-end rule for every private call, the finite native implementation and the terminating timetable search of Lemma 5.3. Keep the unchanged initialized never-halting structural transition, with the same scalar serialization and old/new one-way banks. Include the isolated input-description lane as an additional idle noisy factor. The complete finite elementary word fixes $Q_{\mathrm{site}}, D_{\mathrm{layer}}, R$, all physical stencil dimensions, and hence the strictly positive common p_0 in (6.4).

Given n, D, h and C, calculate B, V, k, t, m_c, L by (4.1), (7.1) and (7.2). Initialize the exact structural row on the chart, the operational stream and known head/work values by repeated classical substitution, and the physical pulse bits of Lemma 3.2. Every designated quantum register starts in zero. Put the circuit and integer request in the classical description lane as in Lemma 7.1. These are product basis preparations computable in polynomial time.

For $p \le p_0$, Lemmas 6.1 and 6.2 bound the flag-defined bad top event by (7.4). Off this event, Lemmas 3.2 and 5.4 give the intended top channel at every needed tick, and Lemma 4.1 gives $C|0\rangle^{\otimes n}$ at the declared stationary outputs. The actual ring has this line output marginal by the full-horizon cone coupling of Lemma 7.1. Measure every physical qubit and apply the fixed total CSS rule of Lemma 3.3 bottom-up on the selected blocks, ignoring every other measured bit. Its full-space effect identity gives the joint output law, including any bad intermediate blocks. The decoder uses n, D, h and the constant geometry only. Mixing erroneous flag histories costs at most $2^{-h} \le \varepsilon$. Lemma 7.1 gives polynomial computation of the maps and (1.1) with the explicit A of (7.3). Nothing depends on the actual p. □

Theorem B (Quantum inputs and endpoints)**.** *Use the hardware, threshold p_0, noise, request h, chart and stopping time t of Theorem A, for the same n, D, C, ε. Assume noiseless endpoints: in place of the product loading of the designated quantum slots, a noiseless recursive encoder prepares them, and after the t noisy periods a noiseless recursive recovery is applied. For every $0 < p \le p_0$ and every n-qubit input ρ_{AR} with arbitrary reference R, placed on the top stationary data slots before encoding, the recovered output satisfies*

$$\frac{1}{2}\left\|\rho_{AR}^{\mathrm{out}} - (C \otimes I_R)\rho_{AR}(C^\dagger \otimes I_R)\right\|_1 \le 2^{-h} \le \varepsilon. \tag{8.1}$$

The bound in equation (1.1) holds for the ring size L and the number t of noisy periods between the noiseless endpoints. The endpoint encoder and recovery are independent of p. The recovery is a quantum operation and need not be a classical measurement decoder.

Proof. Place ρ_{AR} on the top stationary data slots $1, \dots, n$ and prescribe the same top program, head, ports and workspace as before. Apply the exact noiseless CSS encoder recursively to every designated slot, including the known program and head, with fresh child workspace. Embed this entire encoded input and the structural row in the chart of Lemma 7.1. Initialize every physical pulse bit to zero. Its clean action is now identity even at the first period. Faults at later resets, storage or reads still have the physical direct certificates, so no threshold or stencil is removed.

At every rank the boundary is an ordinary quantum-correctable input. Use the quantum clause of Lemmas 3.1 and 3.2, rather than their classical-basis clause. Their Bell calculation is an operator identity with a reference, and the joint proof of Lemma 5.4 preserves that reference through each good complete operation and each recursive cut. Apply the fixed unrestricted recursive recovery at time t to the n data blocks and take their marginal. On the good top event, this gives exactly the intended joint output by Lemma 4.1.

The good event depends only on primitive flags and is independent of ρ_{AR}. Its complement has probability at most 2^{-h} by the same union bound and line-to-ring coupling. The unconditional recovered state is the ideal state with weight at least $1 - 2^{-h}$ plus arbitrary bad-history states. Trace distance between states is at most one and is convex, giving (8.1). The time, chart, rank and fixed code are unchanged; the same A bounds the resources. This proves the reference-entangled input/output assertion. □

Theorem C (Independent whole-site replacement)**.** *Theorems A and B hold on exactly the same hardware under independent whole-site replacement (2.2), using the common p_0 of (6.4), the same A, and the same preparation, timing and endpoint maps.*

Proof. Use one independent Bernoulli key per site and elementary layer. Name every affected stencil origin, including both incident exchange origins when required. A hit discards every incoming factor at that site and prepares a product maximally mixed register. Realize its classical bits by private independent fair bits. Conditional on the hit set, the retained banks still evolve by a classical stochastic rule with one-way copies. The product output of replacement

prevents a quantum input from feeding back into their probabilities. Quantum and classical errors at the same hit can be correlated; the joint named-error model of Lemma 5.4 explicitly permits this.

We verify the primitive-key witness bound instead of replacing the wire rate by an unjustified native iid rate. The deterministic mesh implication is the same classical Lemma 4.2 used in Section 2.2. In a depth-$j-1$ witness tree there are 2^{j-1} leaves. Restrict each leaf to its doubled rank-zero box intersected with every ancestor and the initial test. Its nonempty region has at most $4w_R^2$ native points. Distinct leaves have disjoint regions because they separate in disjoint sibling boxes. Join two leaves if the unions of their site-key stencils intersect. A neighboring region supplies a distinct lattice point within the first leaf's box enlarged by $2r_x, 2r_t$. There are fewer than J_R such points, so the conflict degree is at most $J_R - 1$. A deterministic greedy choice retains at least $2^{j-1}/J_R$ leaves with disjoint site-key supports.

A retained leaf requires some hit among at most $A_R = 4w_R^2 s_R$ distinct site keys and has probability at most $A_R p$. Those retained leaf events are independent. There are at most $D_R^{2^{j-1}-1}$ trees per root and at most N_R root boxes. For $A_R p \le 1$ this gives

$$\Pr(\text{test fails}) \le N_R D_R^{2^{j-1}-1} (A_R p)^{2^{j-1}/J_R} \le [C_R p^{1/J_R}]^{2^{j-1}},$$

precisely (2.3). All constants were taken as maxima for both primitive-key conventions, and $p_0 \le e_{R0}$ ensures the displayed hypothesis. Removing negative-time keys only decreases the increasing events.

Every other independence assertion in Lemma 6.2 is between disjoint complete spatial and temporal key supports after truncation. Such supports also have disjoint site keys. The same recurrence, constants and rank decay therefore follow. A direct test costs at most $s_H p$ under the site-key convention by its definition. Lemma 6.1 still holds: off hit stencils the same total classical transition is exact, malformed/raw reads have the same tests, and the date-zero join is unchanged. The common threshold pays for these bounds without altering a gate.

Conditional hits can act jointly on compiler and user factors, but their expansion with common environment coefficients is exactly the arbitrary joint fault expansion used in deletion. The spectator case adds no clean contact between them. Thus the same good-parent budget, loading, complete-operation margins, total measurement effects and full-horizon chart all apply. The good-event probability is input independent and at least $1 - 2^{-h}$. Its classical output mixture proves Theorem A under this noise, and its recovered joint quantum mixture proves Theorem B in trace distance. This proves the assertion with a single hardware choice and a single positive threshold. □

The following sections add two consequences within explicit scopes. Theorem D keeps this physical word and polynomial resources under general sufficiently weak local fault-path noise, including coherent perturbations. Corollary 10.1 implements its elementary layers by identical banked updates and gives reference-entangled storage through a horizon linear in ring size. Its ideal onsite bank shift remains a noiseless timing structure, and no storage claim is made after that horizon.

9 General local noise on the same circuit

We now remove the independence and depolarizing-form assumptions without changing the physical word. We use the fault-path convention of Dünnweber, Styliaris and Trivedi [15, Definition 3]. Its superset sum is the established threshold method of Aliferis, Gottesman and Preskill [42]; see also Terhal–Burkard [43] and Aharonov–Kitaev–Preskill [44]. The additional work here is a count of dense witnesses for this one-dimensional hierarchy, including overlapping tests of unbounded rank.

9.1 Noise convention and statement

Fix a finite instance and label its elementary gate locations, including waits, by a finite set U of size N. One may also specify a fixed bounded number of post-layer noise slots per site and elementary phase, of uniformly bounded arity and spatial diameter; their ideal maps are identities. All these bounds are fixed independently of the computation. A gate fault acts on the same factors as that gate, and a post-layer fault acts on its declared factors. For serially composed slots use their actual order. A shared environment starts independently of the logical input and has no circuit contacts other than at faulty slots.

More generally the implemented CPTP evolution has a linear fault-path expansion $\Phi = \sum_{S \subseteq U} \Phi_S$. A path has ideal circuit operations outside S, and arbitrary local circuit–environment superoperators on the declared factors at S, or lies in the linear span of such networks. Environment-only evolution is allowed between contacts. The path maps need not be positive or trace preserving. After inserting the fixed environment state and tracing the final environment, assume

$$\left\| \sum_{S \supseteq K} \Phi_S \right\|_\diamond \leq \eta^{|K|} \qquad (K \subseteq U). \tag{9.1}$$

This is a condition on sums with specified faults and all optional faults already summed, not just on individual paths or reduced one-time noise marginals. Composition with our noiseless endpoints preserves the bound. For simultaneous contacts within one layer, the support convention is the same: only the declared local factors couple to the bath, and the path lies in the corresponding local fault span. No serial factorization of the actual layer is required if (9.1) holds.

Theorem D (General local noise)**.** *For the unchanged circuit of Theorems A and B, and any fixed slot bounds above, there is $\eta_0 > 0$, independent of n, D, C and ε, such that every CPTP implementation satisfying (9.1) with $\eta \leq \eta_0$ retains their preparation, decoder, rank choice, stopping time, polynomial resource bound and requested error at most $2^{-h} \leq \varepsilon$. In the quantum-input assertion keep the same noiseless recursive endpoints; the estimate holds uniformly with a reference. In particular each ideal elementary CPTP gate G_u, including waits, resets and dephasings, may be replaced by an arbitrary CPTP map E_u on its factors with $\|E_u - G_u\|_\diamond \leq \eta_0$. Post-layer wire, whole-site or bounded-neighborhood CPTP perturbations within η_0 of the identity in diamond norm are allowed as well. This includes coherent errors without added depolarization. The threshold is positive but astronomically small.*

The product case is the specified all-zero-input computation of Theorem A; the quantum case is its reference-entangled input/output extension. We prove a full trace-norm estimate in the former and a diamond-norm estimate for the recovered logical channel in the latter, stronger than the stated distance conventions.

9.2 Exact recovery on a certified support

Lemma 9.1 (Replacement span)**.** *Let F be a deterministic finite set of post-layer site keys for which the deterministic good-top conclusion holds in the complete-replacement model. After the fixed recursive recovery the specified product-input output is the ideal pure state, and the quantum-input channel is the ideal unitary channel. The same holds for arbitrary trace-preserving fault networks whose faulty gates have every output factor covered by F at that layer, and for arbitrary errors on the flagged factors. A shared environment with contacts only there is allowed. More generally every linear fault term on this support has recovered output in the one-dimensional span of the specified pure state, or recovered channel in the span of the ideal unitary channel, respectively.*

Proof. Complete replacement on a d-dimensional factor has Kraus operators $|a\rangle\langle b|/\sqrt{d}$, spanning its full matrix algebra. For a certified pattern, each recovered Kraus amplitude on the specified pure input lies on the ideal output ray: its positive rank-one contribution is part of a sum supported on that ray. For quantum inputs apply the same argument to a normalized maximally entangled logical input and its untouched reference. Purity of the recovered Choi state implies that every recovered Kraus operator is a scalar multiple of the intended logical unitary, on the entire logical input space. Endpoint Kraus choices include tracing all unused outputs. These are the exact conditional statements of Lemmas 3.2, 5.4 and 7.1, not an inference from an approximately correct classical distribution.

This span also covers replacing an ideal gate, including a nonunitary one. If G_i are its Kraus operators and W is any desired operator on its input/output factors, trace preservation gives

$$W = \sum_i (WG_i^\dagger)G_i.$$

Expand each $WG_i^\dagger$ in the replacement basis on all output factors, and choose identities on extra flagged factors. Apply this to simultaneous tensor products of affected gates. Thus every substituted recovered amplitude belongs to the same ray. Expand joint circuit–environment operators in circuit matrix units with environment coefficients, and contract those coefficients only afterwards. Multilinearity gives the same conclusion for arbitrary correlated contacts on this finite network. Cross terms between any two amplitudes also lie on the same recovered ray; consequently the conclusion holds for arbitrary local linear superoperators and their linear combinations. The logical coefficient is independent of the logical input, by the Choi argument. For an actual CPTP network its coefficient is one by trace preservation. For a linear fault term it need not be one. □

9.3 Dense witnesses from the deterministic hierarchy

Use max distance in physical site and period date, with bounded elementary phases and key types absorbed in fixed constants. Put $\rho = 2^{\log_T P}$, so $1 < \lambda \le \rho \le 2$ by Section 6.

Lemma 9.2 (Density at every member)**.** *There are fixed $C \ge 1$ and $c > 0$ such that a named rank-i query has a nonempty witness V of actually flagged primitive keys satisfying*

$$|V \cap B_\infty(u, CT^j)| \ge c\rho^j \quad (u \in V,\ 0 \le j \le i). \tag{9.2}$$

No independence is assumed. On the finite ring, any fault support that can spoil the recovered answer has such a witness through the chosen rank k in its covered site keys, with circular spatial distance.

Proof. First consider a failed rank-a protected structural test, $a \ge 1$. Its rectangle has width $< w_R P^a$ and height $< w_R T^a$ and fails $(a-1)$-sparsity. Iterate the deterministic disjoint-child implication from Section 2.2, retaining intersection with every ancestor and the initial test, as in [12, Lemma 12.1]. The resulting binary tree has depth $a-1$. At a node of remaining depth ℓ, its 2^ℓ nonempty leaf regions are pairwise disjoint and contained in its doubled rank-ℓ box. Choose a flagged native point in each leaf and an actual primitive flag in that point's implementing stencil. Distinct native points may choose the same primitive key, but a key belongs to at most a fixed m_* stencils, for example $(2r_x+1)(2r_t+1)$ times the fixed type multiplicity. This subtree supplies at least $2^\ell/m_*$ distinct primitive keys. For each selected key choose a leaf that selected it; the keys below its remaining-depth-ℓ ancestor lie within $C_{\mathrm{str}}T^\ell$ of it, accounting for both stencil collars and $P \le T$. Hence the union of all selected keys, with duplicates removed, has density at least

$$c_{\mathrm{str}} 2^{\min(j,a-1)} \quad \text{in radius } C_{\mathrm{str}}T^j \quad \text{at every member, for every } j \ge 0.$$

The keys remain in the full primitive support of this failed test. Restricted leaf regions, rather than possibly overlapping doubled leaf boxes, provide disjointness. A failed direct test has a flagged key and obeys this statement with $a = 1$ and $c_{\mathrm{str}} \le 1$.

Now induct on the quantum naming rank i. Maintain also that each chosen key belongs to the full support of an actually failed base test at a descendant base query. At rank zero choose a failed certifying test and its witness. This proves the density assertion at $j = 0$ and the additional invariant.

For $i \ge 1$, examine all failed tests at descendant base queries in the rank-i footprint of Lemma 5.2. If any has index a with $C_* T^a > P^{i-1}$, choose its structural or direct witness. Set $\alpha = \log_T P \le 1$ and $K_* = 1 + \alpha + \log_T C_*$. Then

$$a - 1 \ge \alpha i - K_*, \qquad \min(j, a-1) \ge \alpha j - K_* \quad (0 \le j \le i).$$

The preceding density gives (9.2) with $c \le c_{\mathrm{str}} 2^{-K_*}$. This witness may extend beyond the query footprint, but stays in the full support of its failed test, preserving the invariant.

Otherwise every failed test at such a descendant query has its full support within P^{i-1} of that query. The named parent has two named rank-$(i-1)$ children with complete descendant footprints separated by at least $8P^{i-1}$, by Lemma 5.2. Inductively chosen keys lie within P^{i-1} of their respective footprints, by the invariant and the present case assumption. Their two witness sets are therefore disjoint. Their union has diameter at most $C_Q T^i$ for a fixed C_Q, using the parent's footprint and the two collars. For $j < i$ the child containing a chosen member already supplies its density. At $j = i$ its ball contains both sets and hence at least $2c\rho^{i-1} \ge c\rho^i$ members. Taking $C = \max(C_{\mathrm{str}}, C_Q, 1)$ completes the induction and the invariant.

For finite-ring use, take its actual injective chart from Lemma 7.1. Lift just the prescribed flags in the ordinary replacement-model backward cone to the line; set every other positive line flag, and every negative flag, to zero. If no prescribed top query is named, the deterministic reductions give exactly the recovered ideal answer. The ordinary replacement-model output on the line equals the ring output since its entire circuit cone and initialized row agree in that chart. Only finitely many reductions are needed at this horizon; recursive structural clock queries terminate at negative dates as specified in Section 2.2. This uses a pointwise sparse-history implication, not conditioning on a zero-probability infinite clean extension. Lemma 9.1 then certifies arbitrary faults on the original ring support, including covered whole-gate faults. Contrapositively an uncertified support names a rank-k top query and supplies the witness just proved. Its lifted keys are distinct ring keys; circular distance can only shorten their spatial distances. □

Associate each faulty slot with all its post-layer output sites; a post-layer slot marks every site it acts on. A slot targets at most a fixed g_* site keys, within a fixed spatial distance r_* of its slot coordinate. From a site-key witness choose, for each of its keys, a slot in the support targeting it, and let W be the image. For any slot $s \in W$ choose a preimage u. The neighbors of u in (9.2) choose slots within radius $CT^j + 2r_*$ of s, and a slot accounts for at most g_* of those keys. Thus W has the same density form with fixed enlarged C and reduced c. The elementary phases have bounded type multiplicity. Consequently a support with no dense slot subset through rank k is certified by Lemma 9.1.

9.4 Counting all supports in the indicator polynomial

Lemma 9.3 (Dense-set count and Möbius coefficients)**.** *Let U consist of $N \ge 1$ labeled locations on a ring times integer dates, with boundedly many types at each coordinate. Fix an integer $A_* \ge 2$, $1 < B_* \le 2$, and constants $C, c > 0$. Call a nonempty W dense through rank k if every $u \in W$ has at least cB_*^j members of W within max distance CA_*^j, for $0 \le j \le k$.*

Write $f(S) = 1$ when S contains such a W, and zero otherwise. There are fixed $C_1, C_2 > 0$ such that

$$a_K = \sum_{S \subseteq K} (-1)^{|K|-|S|} f(S), \qquad f(S) = \sum_{K \subseteq S} a_K, \tag{9.3}$$

$$|a_K| \le 2^{|K|}, \qquad \#\{K \subseteq U : K \text{ dense}, \ |K| = m\} \le \left[C_1 (2N)^{C_2/B_*^k}\right]^m. \tag{9.4}$$

Here $a_\varnothing = 0$; $a_K \neq 0$ implies that K itself is dense and $|K| \ge cB_*^k$.

Proof. The inversion follows by cancellation of the interior alternating sum unless its subset equals S. The coefficient bound is immediate. If some $u \in K$ belongs to no dense subset of K, then $f(S \cup \{u\}) = f(S)$ for every $S \subseteq K \setminus \{u\}$, so the terms for a_K cancel in pairs. Otherwise every member has a dense witness within K, and hence K itself satisfies all the density inequalities. At rank k these force $|K| \ge cB_*^k$.

Cut the ring at a fixed origin and use ordinary coordinates $0, \dots, L-1$. Partition space and date into nested grid squares of side A_*^j, $0 \le j \le k$, clipping to the coordinate domain if necessary. For each occupied square at level j choose one representative in K. Its circular ball contains at least cB_*^j keys of K. Conversely any particular key belongs to balls of representatives from at most a fixed d_C squares: its circular spatial neighborhood is at most two intervals of length $2CA_*^j + 1$, or the whole shorter ring, and its temporal neighborhood is one such interval. These meet only boundedly many grid squares. Double counting gives at most $d_C m/(cB_*^j)$ occupied squares at level j.

There are at most N possible occupied top squares, since each must contain a location of U. At most $q = d_C m/(cB_*^k) \ge 1$ are selected, so summing binomial bounds gives at most $(2N)^{2q}$ choices. Each occupied square at level $j \ge 1$ has at most A_*^2 child slots; choosing the occupied slots costs at most $2^{A_*^2}$. The total number of such parent decisions is at most

$$\frac{d_C}{c} m \sum_{j=1}^{k} B_*^{-j} \le \frac{d_C}{c(B_* - 1)} m.$$

At level zero there are boundedly many labels per coordinate, so choosing labels costs another fixed constant to the power m. This nested encoding specifies every K and proves (9.4), with constants independent of N, k and the ring length. □

9.5 Superset summation and the factor of two

Proof of Theorem D. Use Lemma 9.3 for slots with $A_* = T$, $B_* = \rho$ and the constants obtained from the site-to-slot witness. Compose each Φ_S with the prescribed preparation and recovery, obtaining M_S. In the product case regard M_S as the resulting output operator; in the quantum case it is a logical linear map. Whenever $f(S) = 0$, Lemma 9.1 puts M_S in the span of the ideal output, or the ideal channel $\mathcal{U}_C$, respectively. This includes the empty support. Such terms need not vanish and need not be normalized.

To handle their normalization explicitly, in the product case put $P(X) = \rho_{\text{ideal}} \operatorname{Tr} X$. In the channel case choose any fixed logical density matrix τ and put

$$P(\Gamma) = \mathcal{U}_C \operatorname{Tr}[\Gamma(\tau)].$$

Both projections have norm at most one, in trace norm or diamond norm as appropriate, since $|\operatorname{Tr} \Gamma(\tau)| \le \|\Gamma\|_\diamond$. Hence $\|I - P\| \le 2$. They fix every certified linear term. The actual recovered output has trace one, or is a trace-preserving channel, so its projection is the ideal answer. Therefore its deviation $\Delta_{\log}$ obeys

$$\Delta_{\log} = (I - P) \sum_S f(S) M_S, \qquad \|\Delta_{\log}\| \le 2 \left\| \sum_S f(S) M_S \right\|. \tag{9.5}$$

Faulty terms without a dense witness have the correct logical ray but may carry a nonunit trace coefficient. The projection accounts for their normalization at a cost of at most two, without assuming termwise trace preservation or that nonempty good paths vanish.

Insert (9.3) and sum all optional faults first:

$$\sum_S f(S) M_S = \sum_{K \subseteq U} a_K \sum_{S \supseteq K} M_S.$$

No independence is used. Contractivity of the endpoints and (9.1) give $\| \sum_{S \supseteq K} M_S \| \le \eta^{|K|}$. Using (9.4) yields

$$\|\Delta_{\log}\| \le 2 \sum_{m \ge \lceil c\rho^k \rceil} \left[2C_1 \eta (2N)^{C_2/\rho^k} \right]^m. \tag{9.6}$$

Taking absolute values of individual paths before the superset sum would introduce an exponential circuit-volume factor and does not prove this bound.

There is a fixed c_0 with $N \le c_0 L t$. The unchanged formulas of Lemma 7.1 give

$$t = VT^k, \quad L \le (2n + 4V + 23)P^k + 2Rt + 2(h+1), \quad h + \lceil \log_2 W \rceil + 1 \le \lambda^k,$$

where $W = (V+1)(n+2V+9)$ and $P \le T$. Thus fixed constants satisfy

$$\log(2N) \le c_1 [1 + \log W + \log(1+h) + k] \le c_N \rho^k.$$

For the last inequality use $n, V \le W$, $\log(1+h) \le h$, $\lambda \le \rho$, and boundedness of k/ρ^k. Natural logarithms are used here. Set

$$\eta_0 = \frac{\min\{1/2,\ 2^{-3/c}\}}{2C_1 \exp(C_2 c_N)} > 0. \tag{9.7}$$

The series base in (9.6) is then at most $\theta = \min\{1/2, 2^{-3/c}\}$. Since $\rho^k \ge \lambda^k \ge h+1$, its sum is at most

$$4\theta^{c\rho^k} \le 4\, 2^{-3\rho^k} \le 2^{-h}.$$

This includes the factor of two in (9.5). Measuring the recovered product output gives the original classical law by the full-space effect identity of Lemma 3.3. Measurement and the half-trace-distance convention only decrease the asserted error. All resources and input/output maps are unchanged. The fixed constants in the stencil, density and grid bounds make (9.7) astronomically small; no practical threshold estimate is claimed. □

9.6 Local CPTP errors and simultaneous bath interactions

For serial local CPTP replacements write $\Delta_u = E_u - G_u$ and expand the circuit multilinearly. The sum with every location in K faulty is the contracted circuit with Δ_u on K and actual CPTP maps elsewhere. Submultiplicativity and CPTP diamond norm one give (9.1). This proves the claimed arbitrary local perturbation case, including coherent overrotations and errors in reset. It also covers a shared bath if, in a specified serial order, the actual joint map at u is within η of $G_u \otimes B_u$ in joint diamond norm for an environment-only CPTP map B_u. Merely bounding reduced one-time marginals is insufficient.

The proof above uses only the local support span and (9.1), so it also covers simultaneous Hamiltonian–bath contacts within a layer. One sufficient strength condition can be seen directly. In a dilation of the ideal disjoint gates, including private environments for nonunitary gates, let the additional Hamiltonian be $\sum_u H_u(s)$, with each term coupling only the factors

of location u and the shared bath during its layer. Arbitrary bath-only evolution is included in the ideal dynamics. Suppose

$$2\int_{\text{layer of } u} \|H_u(s)\|\, ds \le \eta \quad \text{for every } u.$$

Group the time-ordered expansion by the set of interaction labels that occur. These are local fault-span paths; disjoint ideal gate supports in a layer allow each affected support to be covered at that layer's output. To bound their superset sum, multiply H_u by $z_u \in [0,1]$. The mixed finite difference in variables z_u, $u \in K$, with all other variables set to one, is exactly the sum over paths containing K. Express it as the integral of the mixed derivative over $[0,1]^K$. Duhamel's formula inserts each superoperator $-i[H_u(s),\cdot]$ once, summing all insertion orders. The intervening propagators remain CPTP for every interpolated Hamiltonian, and the inserted superoperator has diamond norm at most $2\|H_u(s)\|$. The ordered integration regions partition the product of the insertion-time intervals. Consequently the sum has norm at most

$$\prod_{u\in K} 2\int \|H_u(s)\|\, ds \le \eta^{|K|}.$$

The same argument applies across layers and survives the final environment trace. No ordering of the simultaneous physical bath contacts is imposed. This supplies (9.1) in a familiar Hamiltonian case; the theorem's hypothesis is the superset bound itself.

10 One fixed discrete update and its storage horizon

This section is an elementary register-bank consequence of Theorem A and its endpoint and site-noise extensions. It retains the noise after every original elementary layer. It does not remove a noiseless timing structure.

Corollary 10.1 (One fixed channel)**.** *There is one fixed finite-range CPTP update $\mathcal{K}$, translation-invariant by one site, on a fixed finite register at each site, applied identically at every discrete step, that gives the computation and polynomial resource conclusions of Theorem A. Independent wire replacement, or independent replacement of an entire enlarged site, acts after every application of $\mathcal{K}$ at any rate up to the common p_0. The quantum-input conclusion of Theorem B holds with its noiseless endpoints.*

For some fixed $a_{\text{mem}} > 0$ and every sufficiently large ring size L, the same update stores a qubit entangled with an untouched reference through every integer time $0 \le m \le \lfloor a_{\text{mem}} L \rfloor$. A one-time noiseless encoder and a time-dependent noiseless CPTP readout, both independent of the noise rate, give recovered Bell fidelity at least $15/16$ throughout that interval. They also preserve an arbitrary input qubit and reference to trace distance at most $1/16$ on this interval. Nothing is claimed after the horizon.

The onsite bank shift in this construction is one ideal gate on $D_{\text{layer}}Q_{\text{site}}$ qubits at once. Noise inside a decomposition of that gate is not covered by this corollary. Like the original noiseless schedule, the shift is a noiseless timing structure.

Proof. Write the fixed elementary word as $G_0, \dots, G_{d-1}$, where $d = D_{\text{layer}}$ and each bank contains Q_{site} qubits per site. At every site install d named banks indexed by r modulo d. In parallel apply layer G_r across bank r, for every r, then perform the same onsite tensor-factor permutation $r \mapsto r+1$. Their composition is $\mathcal{K}$. Its range and the number of stages specifying it are fixed; it commutes with translation by one site. No initialized phase counter or step-dependent choice of channel is used.

Put the original preparation in bank zero and fill the remaining banks with zeros. After m steps the designated stream is in bank m mod d. Its joint marginal with the reference is exactly

that of the original first m noisy elementary layers. Indeed the parallel channels on traced-out banks preserve the identity in the adjoint, so only $G_{m \bmod d}$ acts on the designated marginal, and the permutation merely renames its factors. Independent wire noise restricts to exactly the original wire noise. For enlarged-site replacement, condition on the independent site-hit set at the present step. Tracing unused banks at a hit leaves the original site's maximally mixed register with its input discarded, tensor the exterior marginal. This is exactly the old whole-site replacement. The hit sets are independent across steps, so induction applies even when previous site hits correlated banks. Leave the reference untouched throughout.

At time dt, bank zero therefore has the original output. Apply the original decoder for computation, or the recursive quantum recovery for quantum inputs. The number of sites is unchanged, the local register has the fixed factor d, and the time in updates is d times the number of periods, preserving the polynomial resource conclusion. The extra banks contain no copies of an unknown input. This proves the fixed-channel computation assertion directly from Theorems A–C.

For storage choose $n = 1$, target depth zero and the identity circuit in Theorem B. Then $V = 30$ and $W = 31 \cdot 70$ are fixed. For requests $h \geq 1$, let $k(h), t(h), L_h$ be precisely (7.1)–(7.2). Since $P \leq T$ and $1 < \lambda < T$, there is a fixed C_{mem} with

$$L_h \leq C_{\mathrm{mem}} t(h), \qquad L_h \geq 2Rt(h), \qquad L_h \longrightarrow \infty.$$

Here $P^{k(h)} \leq t(h)/30$ and $h \leq \lambda^{k(h)} \leq T^{k(h)}$ prove the upper bound; the cone term proves the lower bound. For all sufficiently large h, consecutive powers of λ differ by more than one, so increasing h by one raises k by at most one. Hence $t(h+1) \leq Tt(h)$ eventually.

For any sufficiently large L, take h immediately before the first index h' with $L_{h'} > L$. This first crossing exists and gives

$$L_h \leq L < L_{h+1} \leq C_{\mathrm{mem}} t(h+1) \leq C_{\mathrm{mem}} T t(h).$$

No monotonicity of L_h is needed. Also $h \to \infty$ as $L \to \infty$. Keep this h's initialized chart on the larger ring and put filler outside it. The full replacement-model backward cone remains in the injective chart of Lemma 7.1, so its original conditional good-support conclusion and probability bound still hold. Enlarging the ring changes no designated marginal. The bank construction thus has a declared endpoint at $dt(h) \geq dL/(C_{\mathrm{mem}} T)$ updates.

At any integer $m \leq dt(h)$, read out by applying the remaining *ideal* $\mathcal{K}$ updates to that endpoint and then the original recovery on bank zero. This is one CPTP readout depending only on L, h, m and the known hardware; it does not act during storage or depend on the unknown input or actual noise rate. Keep the actual flags before m and set all later flags to zero. Coupling them to full-horizon independent flags gives a subset of that full support. The naming and bad-top events are increasing in flags, so every certified full pattern also certifies this thinned one. Lemma 9.1, or the original deterministic conditional recovery, makes its recovered identity output exact. Bad patterns have probability at most 2^{-h}. Thus the recovered joint state is the ideal input/reference state with weight at least $1 - 2^{-h}$ plus an arbitrary state of the remaining weight. For a Bell input its overlap with the Bell state is at least $1 - 2^{-h}$; for every input/reference state its trace distance from that state is at most 2^{-h}.

Choose, for example, $a_{\mathrm{mem}} = d/(2C_{\mathrm{mem}} T)$ and take L sufficiently large that the chosen $h \geq 4$. This gives the claimed 15/16 fidelity and 1/16 trace-distance bounds at every allowed read time, under either original noise convention. It proves a finite interval, with no assertion at later times. Finally the whole bank permutation was treated as one noiseless onsite gate before the prescribed noise; the proof includes no faults within a finer decomposition of it. □

11 Scope, limitations and counting restrictions

The contribution is a fixed translation-invariant, time-periodic noisy rule on a line with a product preparation and a gate-independent classical output decoder. The initialized hierarchy and synchronous schedule remain part of the model. The polynomial bound is an existence bound; code sizes, local registers, ranges, periods and the inverse threshold can be extremely large. Near-linear space and time of the form $(n + D)\,\mathrm{polylog}((n + D)/\varepsilon)$ remain open here. The finite-chart argument pays for the entire physical evolution. It yields no storage guarantee at times superlinear in ring length. These are distinct open improvements; neither follows by iterating a one-tick storage estimate.

Classical program placement in a quantum automaton has noiseless precedents in Arrighi, Fargetton and Wang [3], Shepherd, Franz and Werner [33], and Vollbrecht and Cirac [35, 36]. Nagaj and Wocjan [30] give computation by a fixed one-dimensional Hamiltonian from programmed product input, and Kohler, Piddock, Bausch and Cubitt [26] establish related translation-invariant Hamiltonian universality. These works do not provide the continuing-noise theorem proved here. Fitzsimons and Twamley [16] and Kay [28] establish one-dimensional global-control thresholds with an externally scheduled pulse sequence, and Kay [27] treats errors in the control unit of such schemes. Kay [28] and Pérez Delgado [31] also discuss fault-tolerant quantum cellular automata as an open direction. Harrington's colony decoder [25, Chapter 5] was studied for phenomenological data and measurement faults with reliable classical state. Breuckmann et al. [40] revisit the 2D decoder and 4D Toom-like rules numerically, and Lake [41] proves a threshold for a 2D message-passing memory using noiseless classical buffers.

The two-dimensional construction of Dünnweber, Styliaris and Trivedi [15] already has the principal fixed-rule computation features. Our one-dimensional construction follows their proposed route. We retain a prescribed hierarchical classical boundary, so no self-organization result is obtained. The open directions of Balasubramanian, Davydova and Lake [5] include constant rate and continuous-time dynamics as well as translation invariance and universal computation; this paper gives universal computation and spatial translation invariance with a fixed discrete-time period. The original elementary-layer actions need not be identical. Corollary 10.1 realizes identical discrete updates on a larger fixed register; its ideal onsite bank shift remains a noiseless timing structure and is not protected against noise within a decomposition. It proves neither constant rate nor a time-independent continuous-time implementation. This distinction also follows the taxonomy of Drechsler, Oreg and Aharonov [14], whose 4D storage results include both synchronous channels and periodically driven Lindbladians. Dünnweber, Styliaris and Trivedi [15, Section VII] give a time-independent local Lindbladian construction and outline a threshold extension for weak local unitary jumps, leaving a full proof of that extension open. The stronger efficiency target raised by Gidney and Bergamaschi [17] remains open in this model.

Trivedi and Cirac [38] show a high-noise classical simulation regime and conditional low-noise hardness for families of continuous-time local systems. Their low-noise examples embed compiled circuits in two or more dimensions. Shtanko et al. [39] derive rate-dependent bounds for continuous-time autonomous memory, including a global toric-code recovery example.

The elementary gate set deliberately contains non-Clifford Toffoli and explicit reset. The rapid forgetting theorem for noisy Clifford circuits of Nelson, Rajakumar and Gullans [29] therefore does not cover this rule. Resets are nonunital operations, as needed to escape the limitations of Aharonov, Ben-Or, Impagliazzo and Nisan [1] for noisy reversible circuits whose inputs are all present at time zero. The thermal mixing results of Bergamaschi and Chen [4], Chen and Ding [10], and Bardet and collaborators [8] concern detailed-balance dynamics; the present driven circuit has no such hypothesis. The local stabilizer Hamiltonian and code restrictions of Bravyi and Terhal [9] and Bravyi, Poulin and Terhal [7] likewise do not give a restriction on this active circuit with nonlocal encoded stabilizers. Theorem D

extends the threshold to general local fault-path noise satisfying (9.1), including coherent local perturbations; the model distinctions above do not extend that hypothesis further.

Proposition 11.1 (Measurement-alphabet bound)**.** *If a deterministic decoder of M_{out} measured bits approximates n independent unbiased bits within total variation distance $\varepsilon < 1$, then*

$$M_{\mathrm{out}} \geq n + \log_2(1-\varepsilon). \tag{11.1}$$

In particular $Q_{\mathrm{site}}L$ must satisfy this bound for the depth-one circuit of n Hadamards.

Proof. The decoder's image contains at most $2^{M_{\mathrm{out}}}$ outputs. The uniform law on $\{0,1\}^n$ assigns that image mass at most $2^{M_{\mathrm{out}}-n}$, whereas the decoded law assigns it mass one. Their total variation distance is at least $1 - 2^{M_{\mathrm{out}}-n}$. Rearranging gives (11.1). The initial noise or probabilities of the measured strings cannot enlarge the deterministic image. The additive resource argument in Theorem A is always at least n at positive width, even as ε tends to one, and is compatible with this restriction. □

Proposition 11.2 (Finite precision descriptions and empty width)**.** *The numerical guarantee of Theorem A cannot be combined with losslessly recoverable descriptions of every rational tolerance in the same bounded classical basis preparation. If width and depth both zero are admitted with a nonempty ring, replace the resource argument by* $\max(1, n+D) + \log_2(1/\varepsilon)$.

Proof. Fix $n = D = 1$ and a target circuit. For all $\varepsilon \in (1/2, 1)$ the bound is at most $A3^A$ sites. A finite classical basis alphabet on any of those finitely many ring lengths gives only finitely many preparations, even if length itself is side information. Infinitely many distinct dyadic rational tolerances already contradict lossless recovery. Storing a sufficient integer request instead gives the full numerical guarantee for every tolerance, as Section 2.1 specifies; it does not encode that real number exactly.

If $n = D = 0$, for any fixed A the unfloored bound $A(\log_2(1/\varepsilon))^A$ becomes smaller than one as ε tends to one, although a nonempty ring has at least one site. With the floored argument use one site and the empty-output decoder, whose error is zero. If a physical request field is required, $h+1$ sites record it in unary; since $h + 1 \leq X + 2 \leq 3X$ for the floored $X = 1 + \log_2(1/\varepsilon)$, the constant A of (7.3) already covers this case. The positive-width theorem and its displayed formula are unchanged. Depth zero at positive width is already included by $B = (7n+8)(nD+1)$. □

The counting and representability restrictions are basic obstructions, not additional efficiency claims. The established extensions are preservation of arbitrary reference-entangled inputs with noiseless endpoints, the common-threshold whole-site model, the general local-noise theorem, and the elementary fixed-channel corollary with its ideal bank shift and linear storage horizon. Stronger overhead, lifetime and initialization results require further arguments.

A Versioned local inputs

In this table C denotes the classical source [13], version 2.1.0, doi:10.5281/zenodo.24408026; M denotes the quantum-memory source [12], version 3.1.0, doi:10.5281/zenodo.24431439. The listed statements and indicated proof passages are present in these versions with the hypotheses and conclusions used here. The classical statements are applied to the fixed-rate homogeneous never-halting family, not the searching single-rule noise diagram. The scalar-layout change is justified by M, Lemma 13.6. Expanded compiler lists and the additional raw-read count are proved in this paper; they are not asserted to be verbatim conclusions about the older hardware.

Source and locator	Required hypothesis	Matching use here
C, Section 3, Lemma 3.1	Aligned occupied input; true addressed operands.	Complete clean row, addresses, age, packet and structural/scratch values; Sections 2.2, 5, 6.
C, Sections 3.2, 3.3, 3.5; Appendix A	Large w; typed packets; true immutable family literal.	Period, packet format, literal promotion and terminating search; Sections 2.2, 5.3, 7.
C, Section 4.1, Definition 4.1 and Lemma 4.2	Restricted sets zero outside; deterministic mesh.	Sparsity and disjoint-child implication, with pair count $8(P+2)^2(T+2)^2$; Section 2.2 and site-witness proof.
C, Section 5, Definition 5.1 and following bound	Same-origin nominal/served queries; service date below T.	Lookahead $T^j - 1$ and negative terminal queries; Sections 2.2, 6.
C, Section 5, Proposition 5.3	Protected sparse history; ordinary unhalted lower ranks; true operands.	Classification, service, native issue, empty exception in a wholly $(j-1)$-sparse test; Sections 2.2, 6.
C, Section 5.5, Lemma 5.18 and full-home specialization	Proper live start; prescribed live holds; protected lower calls.	Endpoint/full-home service including live ballots; structural descent, Section 2.2.
C, Section 5.6, Lemma 5.24	Six served live inputs, common tribe, two live central transitions; full W_0; $(j-1)$-sparsity.	Two-period service excluding $2T$; full row in proof; actual raw-test dimensions, Section 6.
C, Section 6.4, Lemma 6.4 and proof	Homogeneous never-halting fixed-rate family; aligned row.	Clean negative tower, deterministic boundary, endpoint substitution; Sections 2.2, 6, 7.
M, Section 2.2	Independent wire/layer or site/layer replacement.	Both noise conventions, Section 2.1; no native iid assumption.
M, Section 5.1, Lemma 5.1 and proof	Particular punctured self-dual Reed–Muller CSS family.	One total recovery for the weight-q union span; exact real tables/logical words; Sections 2.3, 3.3.
M, Section 5.2, Lemma 5.2; preparation and final operator proof paragraphs	Native faults with arbitrary joint coefficients; $8(f+j) \le q-8$.	Verified V_q: at most k deviations for $k \le q$; normalization; 8j residual errors and coefficientwise action; Sections 2.3, 3, 5.4.
M, Section 12, Lemma 12.1	Independent primitive keys; finite stencils; zero negative flags; test dimensions.	Dependent-key witness constants, Sections 2.2, 6; site proof supplied in Theorem C.
M, Section 12, Lemma 12.4	Total parser, final period, one-way dephased banks, exact row, scalar layout.	Four interface clauses; expanded-word check, Lemma 6.1. Extra raw count is proved here.
M, Section 12, Lemma 12.5 and decoded-symbol proof	Never-halting homogeneous family on line; exact row; protected two-period tests.	Full-symbol projection, W_*, N_{clk}, radii and recursion; Section 2.2. Raw budget proved in Section 6.
M, Section 12, Lemma 12.6	Injective initialized interval with entire actual past cone; one line coupling.	Effective boundary and exact finite-ring use; Section 7.
M, Section 13, Lemma 13.1	Particular code; $q - 8 \ge 160b + 64$; b+1 fresh ECs, deletion, common cuts.	First complete operation, 9b cut type, arbitrary-input normalization; longer-list joint proof given in Section 5.4.

Source and locator	Required hypothesis	Matching use here
M, Section 13, Lemma 13.2	Fresh private decisions, dummies, paired shuttles, closed tape ends, terminal exchange.	Deletion pattern and K; expanded-list/spectator proof in Lemma 5.1.
M, Section 13, Lemma 13.3	Clean tower, scalar layout, base raw reads, packet held to ballot.	Low-bit linkage and clean-oracle invariant; Sections 2.2, 5.4.
M, Section 13, Lemma 13.4	Before-E naming; full three-cover union; $T \geq P \geq 4$.	Budgets, radii, separated pair; exact expanded geometry reproved in Lemma 5.2.
M, Section 13, Lemmas 13.5 and 13.7	Finite lists; private tagged tape; end-cell tile; constants before w.	Exponents 32/35, syntactic coefficient, timetable; expanded-list bounds proved in Lemma 5.3.
M, Section 13, Lemma 13.6	Low-word-bit scalar layout; typed halves; endpoint padding; same total parser.	Classical serialization checks and rank-independent positions; Section 2.2.

References

[1] D. Aharonov, M. Ben-Or, R. Impagliazzo and N. Nisan, Limitations of noisy reversible computation, preprint, 1996. arXiv:quant-ph/9611028.

[2] D. Aharonov and M. Ben-Or, Fault-tolerant quantum computation with constant error rate, *SIAM J. Comput.* **38** (2008), 1207–1282. doi:10.1137/S0097539799359385. Preprint: arXiv:quant-ph/9906129.

[3] P. Arrighi, R. Fargetton and Z. Wang, Intrinsically universal one-dimensional quantum cellular automata in two flavours, *Fundam. Inform.* **91** (2009), 197–230. doi:10.3233/FI-2009-0041. Preprint: arXiv:0704.3961.

[4] T. Bergamaschi and C.-F. Chen, Fast mixing of quantum spin chains at all temperatures, preprint, 2025, revised 2026. arXiv:2510.08533v2.

[5] S. Balasubramanian, M. Davydova and E. Lake, A local automaton for the 2D toric code, *Quantum* **10** (2026), 2125. doi:10.22331/q-2026-06-02-2125. Preprint: arXiv:2412.19803.

[6] M. Ben-Or, D. Gottesman and A. Hassidim, Quantum refrigerator, preprint, 2013. arXiv:1301.1995.

[7] S. Bravyi, D. Poulin and B. M. Terhal, Tradeoffs for reliable quantum information storage in 2D systems, *Phys. Rev. Lett.* **104** (2010), 050503. doi:10.1103/PhysRevLett.104.050503. Preprint: arXiv:0909.5200.

[8] I. Bardet, Á. Capel, L. Gao, A. Lucia, D. Pérez-García and C. Rouzé, Rapid thermalization of spin chain commuting Hamiltonians, *Phys. Rev. Lett.* **130** (2023), 060401. doi:10.1103/PhysRevLett.130.060401. Preprint: arXiv:2112.00593.

[9] S. Bravyi and B. M. Terhal, A no-go theorem for a two-dimensional self-correcting quantum memory based on stabilizer codes, *New J. Phys.* **11** (2009), 043029. doi:10.1088/1367-2630/11/4/043029.

[10] H. Chen and Z. Ding, Spectral analysis beyond the spectral gap: rapid mixing of quantum Gibbs samplers in one dimension, preprint, 2026. arXiv:2610.05848v2.

[11] B. S. Cirel'son, Reliable storage of information in a system of unreliable components with local interactions, in R. L. Dobrushin, V. I. Kryukov and A. L. Toom (eds.), *Locally*

Interacting Systems and Their Application in Biology, Lecture Notes in Mathematics **653**, Springer, Berlin, Heidelberg, 1978, pp. 15–30. doi:10.1007/BFb0070081.

[12] S. Douglas and N. Mghirbi, Undecidable quantum storage: no algorithm can tell from the noise rate whether a fixed circuit remembers, preprint, version 3.1.0, November 2026. doi:10.5281/zenodo.24431439.

[13] S. Douglas and N. Mghirbi, An uncomputable noise diagram on the line and a finite-energy Gibbs diagram on the square lattice, preprint, version 2.1.0, November 2026. doi:10.5281/zenodo.24408026.

[14] N. Drechsler, Y. Oreg and D. Aharonov, Autonomous quantum memories from classical stochastic processes, preprint, 2026. arXiv:2610.05589v1.

[15] G. Dünnweber, G. Styliaris and R. Trivedi, Quantum memory and autonomous computation in two dimensions, preprint, version 3, 25 September 2026. arXiv:2601.20818v3.

[16] J. Fitzsimons and J. Twamley, Globally controlled fault tolerant quantum computation, preprint, 2007. arXiv:0707.1119.

[17] C. Gidney and T. Bergamaschi, A constant rate quantum computer on a line, preprint, 2025. arXiv:2502.16132.

[18] P. Gács, Reliable computation with cellular automata, *J. Comput. Syst. Sci.* **32** (1986), 15–78. doi:10.1016/0022-0000(86)90002-4.

[19] P. Gács, Self-correcting two-dimensional arrays, in S. Micali (ed.), *Randomness in Computation*, Advances in Computing Research **5**, JAI Press, Greenwich, CT, 1989, pp. 223–326. Author's copy.

[20] P. Gács, Reliable cellular automata with self-organization, *J. Stat. Phys.* **103** (2001), 45–267. doi:10.1023/A:1004823720305. Corrected and strengthened manuscript: arXiv:math/0003117.

[21] P. Gács and J. H. Reif, A simple three-dimensional real-time reliable cellular array, *J. Comput. Syst. Sci.* **36** (1988), 125–147. doi:10.1016/0022-0000(88)90024-4.

[22] D. Gottesman, Fault-tolerant quantum computation with local gates, *J. Mod. Opt.* **47** (2000), 333–345. doi:10.1080/09500340008244046.

[23] D. Gottesman, *Surviving as a Quantum Computer in a Classical World*, textbook manuscript, 7 May 2024. Textbook manuscript.

[24] L. F. Gray, A reader's guide to Gács's "positive rates" paper, *J. Stat. Phys.* **103** (2001), 1–44. doi:10.1023/A:1004824203467.

[25] J. W. Harrington, Analysis of quantum error-correcting codes: symplectic lattice codes and toric codes, Ph.D. thesis, California Institute of Technology, 2004. doi:10.7907/AHMQ-EG82.

[26] T. Kohler, S. Piddock, J. Bausch and T. Cubitt, Translationally-invariant universal quantum Hamiltonians in 1D, *Ann. Henri Poincaré* **23** (2022), 223–254. arXiv:2003.13753.

[27] A. Kay, Error correcting the control unit in global control schemes, preprint, 2005. arXiv:quant-ph/0504197.

[28] A. Kay, Deriving a fault-tolerant threshold for a global control scheme, preprint, 2007. arXiv:quant-ph/0702239v2.

[29] J. Nelson, J. Rajakumar and M. J. Gullans, Non-Clifford gates are required for long-term memory, in *41st Computational Complexity Conference (CCC 2026)*, Leibniz International Proceedings in Informatics **383** (2026), 38:1–38:13. doi:10.4230/LIPIcs.CCC.2026.38. Preprint: arXiv:2510.08451.

[30] D. Nagaj and P. Wocjan, Hamiltonian quantum cellular automata in one dimension, *Phys. Rev. A* **78** (2008), 032311. doi:10.1103/PhysRevA.78.032311.

[31] C. A. Pérez Delgado, *Quantum Cellular Automata: Theory and Applications*, Ph.D. thesis, University of Waterloo, 2007. hdl:10012/3316.

[32] N. Selub, A. Bhardwaj and E. Lake, Local decoders for fault-tolerant quantum computation and translation-invariant stabilizer codes, preprint, 2026. arXiv:2609.11457v2.

[33] D. J. Shepherd, T. Franz and R. F. Werner, Universally programmable quantum cellular automaton, *Phys. Rev. Lett.* **97** (2006), 020502. doi:10.1103/PhysRevLett.97.020502.

[34] O. Shtanko and K. Sharma, Complexity of local quantum circuits under nonunital noise, *PRX Quantum* **6** (2025), 030347. doi:10.1103/xwz2-wfr4. Preprint: arXiv:2411.04819.

[35] K. G. H. Vollbrecht and J. I. Cirac, Reversible universal quantum computation within translation-invariant systems, *Phys. Rev. A* **73** (2006), 012324. doi:10.1103/PhysRevA.73.012324.

[36] K. G. H. Vollbrecht and J. I. Cirac, Quantum simulators, continuous-time automata, and translationally invariant systems, *Phys. Rev. Lett.* **100** (2008), 010501. doi:10.1103/PhysRevLett.100.010501.

[37] C. A. Pattison, J. Haah and J. Preskill, Fault-tolerant quantum processor with minimal external control, preprint, submitted 18 November 2026. arXiv:2611.23919v1. Especially Theorems 4.5 and 5.7.

[38] R. Trivedi and J. I. Cirac, Simulability transitions in continuous-time dynamics of local open quantum systems, *Phys. Rev. Lett.* **129** (2022), 260405. doi:10.1103/PhysRevLett.129.260405. Preprint: arXiv:2110.10638.

[39] O. Shtanko, Y.-J. Liu, S. Lieu, A. V. Gorshkov and V. V. Albert, Bounds on autonomous quantum error correction, *Quantum* **9** (2025), 1804. doi:10.22331/q-2025-07-22-1804.

[40] N. P. Breuckmann, K. Duivenvoorden, D. Michels and B. M. Terhal, Local decoders for the 2D and 4D toric code, *Quantum Inf. Comput.* **17** (2017), 181–208. arXiv:1609.00510.

[41] E. Lake, Local active error correction from simulated confinement, preprint, 2025. arXiv:2510.08056.

[42] P. Aliferis, D. Gottesman and J. Preskill, Quantum accuracy threshold for concatenated distance-3 codes, *Quantum Inf. Comput.* **6** (2006), 97–165. arXiv:quant-ph/0504218.

[43] B. M. Terhal and G. Burkard, Fault-tolerant quantum computation for local non-Markovian noise, *Phys. Rev. A* **71** (2005), 012336. doi:10.1103/PhysRevA.71.012336. Preprint: arXiv:quant-ph/0402104.

[44] D. Aharonov, A. Kitaev and J. Preskill, Fault-tolerant quantum computation with long-range correlated noise, *Phys. Rev. Lett.* **96** (2006), 050504. doi:10.1103/PhysRevLett.96.050504. Preprint: arXiv:quant-ph/0510231.