The Lovász number is not alone:
a continuum of Rényi points in the entanglement-assisted spectrum of graphs

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Abstract

Li and Zuiddam's entanglement-assisted asymptotic spectrum of graphs consists of the normalized graph parameters that are additive under disjoint unions, multiplicative under strong products and monotone under entanglement-assisted cohomomorphisms; the entanglement-assisted Shannon capacity is their pointwise minimum. To our knowledge, the Lovász number ϑ was its only known element. We find a continuum of others.

Three quantities of Duan and Winter, $\log_2 \vartheta$, the minimal Holevo capacity $C_{\min}$ and the logarithm of the no-signalling simulation cost Σ, are the smallest information radii of classical–quantum channels compatible with a graph, measured with Petz–Rényi divergences of orders 0, 1 and 2. At order zero this is due to Schrijver (as credited by Dalai) and to Duan and Winter; at order one it is the min–max formula for the Holevo capacity. Interpolating in the order gives a family ϑ_α, $0 \le \alpha \le 2$, and we prove that every member lies in the spectrum. The main step, multiplicativity under strong products for channels that need not factor, follows from an exact nesting identity for compressions. Sandwiched Rényi radii of order $\beta \in [1/2, \infty]$ give the same family, at $\alpha = 2 - 1/\beta$.

On the five-cycle the family takes every value in $[\sqrt{5}, 5/2]$. A Hilbert–Schmidt inequality for the pentagon gives $C_{\min}(C_5) = \log_2(5/2)$, proving a conjecture of Duan and Winter. We also settle the two open links of their inequality chain: the graph-level no-signalling simulation rate always equals its single-letter value $\log_2 \Sigma(G)$, and a 32-vertex graph from the Mermin–Peres magic square has $C_{\min} < \log_2 \Sigma$, certified in exact rational arithmetic. Finally, Σ never exceeds the projective rank of the complement; a rank inequality for the same square makes this strict on a 1440-vertex graph.

1 Introduction

Shannon's zero-error capacity $\Theta(G)$ of a graph is notoriously hard to compute [Sha56]. Zuiddam found a clean way to organize what we know about it [Zui19]. Consider every graph parameter that adds under disjoint union, multiplies under the strong product, equals 1 on a single vertex and respects the cohomomorphism preorder. These parameters form the *asymptotic spectrum of graphs*, and Strassen's duality theorem [Str88] says that Θ is their pointwise minimum. So every spectral point is an upper bound on the Shannon capacity, and many of the classical upper bounds turn out to be spectral points [Zui19]: the Lovász number [Lov79], the fractional Haemers bounds [Hae79, Bla13, BC19], the complement of the projective rank [CMR+14, MR16], and the fractional clique cover number. The ordinary spectrum is by now known to be uncountable [Vra21, Vra22].

Li and Zuiddam carried the theory over to quantum and entanglement-assisted communication [LZ21]. With shared entanglement the preorder changes. Now $G \le_* H$ means that, with the help of an entangled state, one use of a channel with confusability graph H can do the job of one use of a channel with confusability graph G. The duality survives: the entanglement-assisted Shannon

capacity Θ_* is the minimum over a new spectrum $X(\mathcal{G}, \leq_*)$, where $\mathcal{G}$ denotes the class of finite simple graphs. This spectrum has proved much harder to populate. Beigi and, independently, Duan, Severini and Winter had shown that $\Theta_* \leq \vartheta$ [Bei10, DSW13], and Li and Zuiddam proved that ϑ itself lies in $X(\mathcal{G}, \leq_*)$, using the entanglement-assisted monotonicity of ϑ due to Cubitt et al. [CMR+14]. They found no other element. They also showed that the fractional clique cover number is not in the spectrum $X(\mathcal{G}, \leq_q)$ of the quantum preorder [LZ21, Theorem 22] (Theorem 21 in arXiv:1810.00744v2). This point of the ordinary spectrum equals, by linear programming duality, the fractional packing number α^* used below, and $X(\mathcal{G}, \leq_q)$ contains $X(\mathcal{G}, \leq_*)$, so it is not in $X(\mathcal{G}, \leq_*)$ either. And they suggested that the conjecture $\Theta_* = \vartheta$ might be proved by showing that ϑ is the least element of the spectrum, "or even the only point" [LZ21].

It is not the only point. This paper constructs a continuum of others.

1.1 Three quantities, one family

We start from a paper of Duan and Winter [DW16] on zero-error communication assisted by no-signalling correlations. For a confusability graph G they compare several numbers attached to the classical–quantum channels $x \mapsto \rho_x$ whose confusability graph is G. There is the Lovász number $\vartheta(G)$; Duan and Winter show that $\log_2 \vartheta(G)$ is the smallest no-signalling-assisted zero-error capacity of a channel compatible with G [DW16, Theorem 5]. There is $C_{\min}(G)$, the infimum of the Holevo capacities of such channels, and $\Sigma(G)$, the infimum over such channels of the semidefinite-programming value that governs their exact simulation with no-signalling assistance. They prove the chain

$$\log_2 \vartheta(G) \;\leq\; C_{\min}(G) \;\leq\; S_{0,\mathrm{NS}}(G) \;\leq\; \log_2 \Sigma(G) \;\leq\; \log_2 \alpha^*(G), \tag{1}$$

where $S_{0,\mathrm{NS}}$ is an asymptotic simulation cost and α^* is Shannon's fractional packing number [Sha56] (Section 2.2 has the definitions). By results of Cubitt, Leung, Matthews and Winter [CLMW11], among classical channels with confusability graph G the smallest asymptotic no-signalling-assisted zero-error capacity and the smallest asymptotic no-signalling-assisted simulation cost both equal $\log_2 \alpha^*(G)$. This is how α^* enters the chain [DW16, Section V].

Three of the quantities in (1), $\log_2 \vartheta$, $C_{\min}$ and $\log_2 \Sigma$, are really one quantity in disguise. Fix a channel compatible with G and a state σ. Measure the distance from each output ρ_x to σ with the Petz–Rényi divergence D_α and take the worst case. This worst case, $\max_x D_\alpha(\rho_x \| \sigma)$, is an order-$\alpha$ *information radius*. Minimizing it over channels and centers gives a graph parameter:

$$\log_2 \vartheta_\alpha(G) = \inf_{(\rho_x),\,\sigma} \max_{x \in V(G)} D_\alpha(\rho_x \| \sigma), \qquad 0 \leq \alpha \leq 2. \tag{2}$$

Then

$$\vartheta_0 = \vartheta, \qquad \vartheta_1 = 2^{C_{\min}}, \qquad \vartheta_2 = \Sigma. \tag{3}$$

The first is a theorem of Dalai, who credits it to Schrijver [Dal13b, Theorem 8]. Duan and Winter prove it in the proof of their Theorem 5, by purifying the center [DW16, Section V], and the case of a pure center with general projections is due to Cabello, Severini and Winter [CSW10]; Dalai and Winter note the observations of Schrijver and of Duan and Winter [DW17]. The middle identity is the familiar min–max formula for the Holevo capacity [OPW97, SW01]; the third is elementary once the right formula is in place (Proposition 3.3). So the three Duan–Winter quantities are samples, at orders 0, 1 and 2, of one monotone family $\alpha \mapsto \vartheta_\alpha(G)$, which is nondecreasing on $[0, 2]$ and continuous on $[0, 1)$ (Propositions 3.3 and 3.4). This also explains why they come in the order they do, since D_α is nondecreasing in α.

What happens in between?

1.2 Results

Theorem A (every order up to two is a spectral point)**.** *For every $\alpha \in [0,2]$, the graph parameter ϑ_α belongs to the entanglement-assisted asymptotic spectrum $X(\mathcal{G}, \leq_*)$.*

For $\alpha = 0$, where $\vartheta_0 = \vartheta$, membership is due to Li and Zuiddam [LZ21]. The cases $\alpha > 0$ are new.

Normalization and additivity are routine. Monotonicity under $\leq_*$ is a data-processing argument, and it is why our argument stops at order two: above order two, Petz divergences no longer satisfy data processing (Remark 4.8). The real work is multiplicativity, $\vartheta_\alpha(G \boxtimes H) = \vartheta_\alpha(G)\vartheta_\alpha(H)$. One inequality follows by tensoring witnesses. The other has to control representations of $G \boxtimes H$ that are *not* tensor products, and there is no reason to expect a near-optimal representation of a product to factor. Section 4 handles this with an exact nesting identity for the compressions that appear in the projection formula (11).

Are the new points actually new? The pentagon answers this.

Theorem B (the pentagon)**.** *For $\frac{1}{2} \leq \alpha \leq 2$ we have $\vartheta_\alpha(C_5) = \frac{5}{2}$. For $0 \leq \alpha \leq \frac{1}{2}$, writing $p = 1/(1-\alpha) \in [1,2]$,*

$$\sqrt{5}\left(\frac{\sqrt{5}}{2}\right)^{p-1} \;\leq\; \vartheta_\alpha(C_5) \;\leq\; \min\left\{\frac{5}{2},\; \sqrt{5}\cdot 5^{(p-1)/2}\right\}. \tag{4}$$

The map $\alpha \mapsto \vartheta_\alpha(C_5)$ is continuous and carries $[0, \frac{1}{2}]$ onto $[\sqrt{5}, \frac{5}{2}]$. In particular $X(\mathcal{G}, \leq_*)$ contains continuum many distinct points of the form ϑ_α.

Figure 1 shows what we know about the pentagon curve. Since $\vartheta_1(C_5) = \vartheta_2(C_5) = 5/2$, Theorem B gives the values of $C_{\min}$ and Σ on the pentagon. That is precisely what Duan and Winter conjectured after their Hadamard-graph example, two paragraphs after the chain (1), where they note that a rigorous proof "has so far eluded" them [DW16, Eq. (54), which is (53) in arXiv:1409.3426v3, and the discussion after it].

Corollary (the Duan–Winter pentagon conjecture). $\log_2 \sqrt{5} < C_{\min}(C_5) = S_{0,\mathrm{NS}}(C_5) = \log_2 \Sigma(C_5) = \log_2 \alpha^*(C_5) = \log_2 \frac{5}{2}$.

The values $\vartheta(C_5) = \sqrt{5}$ [Lov79] and $\alpha^*(C_5) = \frac{5}{2}$ [Sha56] are classical, and the upper bounds on $C_{\min}(C_5)$, $S_{0,\mathrm{NS}}(C_5)$ and $\log_2 \Sigma(C_5)$ are Duan and Winter's chain (1). What is new is the lower bound $C_{\min}(C_5) \geq \log_2 \frac{5}{2}$.

The pentagon cannot tell order one from order two. A different graph can.

Theorem C (order one is not order two)**.** *There is a graph G_{32} on 32 vertices, built from the Mermin–Peres magic square, such that*

$$2^{C_{\min}(G_{32})} \;\leq\; 6.186983\ldots \;<\; 6.194232\ldots \;\leq\; \Sigma(G_{32}).$$

Both outer bounds are certified by exact rational data. In particular $\vartheta_1 \neq \vartheta_2$.

In exact form, the two certified bounds are

$$2^{C_{\min}(G_{32})} \;\leq\; \frac{32329248967670751}{5225365592240894} \;<\; \frac{93824992214030}{15147154828811} \;\leq\; \Sigma(G_{32}),$$

the reciprocals of the rationals in (14). For this graph $\vartheta(G_{32}) = 6$ and $\alpha^*(G_{32}) = \frac{32}{5}$, so both bounds lie well inside the interval $[\vartheta, \alpha^*]$ of the Duan–Winter chain (Section 6).

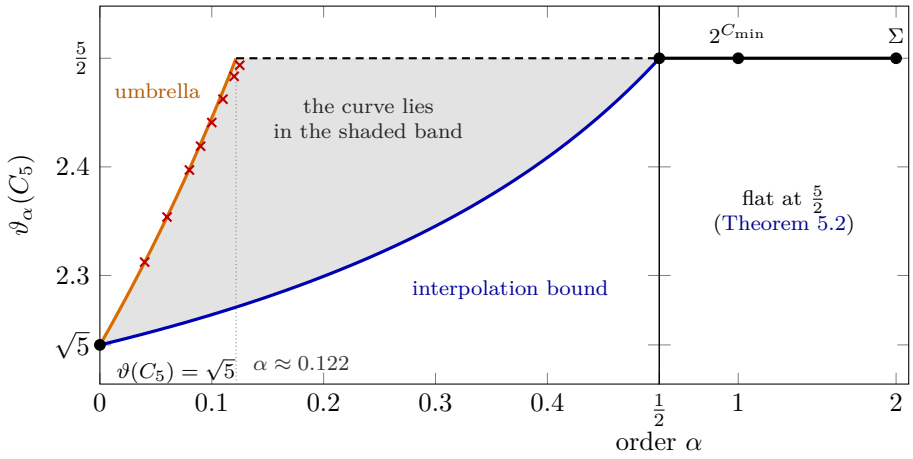


Figure 1: Bounds on $\alpha \mapsto \vartheta_\alpha(C_5)$. The horizontal scale changes at $\alpha = \frac{1}{2}$. On $[\frac{1}{2}, 2]$ the curve is exactly $\frac{5}{2}$, which gives $C_{\min}(C_5) = \log_2 \frac{5}{2}$ and $\Sigma(C_5) = \frac{5}{2}$. On $[0, \frac{1}{2}]$ it is continuous and nondecreasing and lies between the interpolation lower bound (blue) and the minimum of $\frac{5}{2}$ and the Lovász-umbrella bound obtained from the limiting handle center (orange); where this minimum is $\frac{5}{2}$ it is drawn dashed, since there it is only an upper bound. The umbrella bound is better than $\frac{5}{2}$ only for $\alpha < 1 - \ln 5/(2 \ln \frac{5}{2}) \approx 0.122$; the crossing of these upper bounds does not by itself identify a transition in the exact graph value. Crosses: numerical upper bounds from explicit witnesses, the umbrella vectors of Fig. 4(b) with the faithful center $(1-x)|c\rangle\langle c| + \frac{x}{2}(I - |c\rangle\langle c|)$ and x optimized at each order (verification script `pentagon_numerical_bounds.py`; see Data and code availability). They improve on the limiting handle center only slightly, for example $2.44084 < 2.44521$ at $\alpha = 0.1$, and they stay below $\frac{5}{2}$ until $\alpha \approx 0.128$. Continuity and the endpoint values $\sqrt{5}$ and $\frac{5}{2}$ force the curve through every value in between.

Order two also has a natural upper bound of its own. Every d/r-representation of the complementary graph $\overline{G}$, in the sense of Mančinska and Roberson [MR16], is a flat witness for G, so $\Sigma(G) \le \xi_f(\overline{G})$, where ξ_f is the projective rank (Section 7). The graph parameter $G \mapsto \xi_f(\overline{G})$ lies in Li and Zuiddam's spectrum for the quantum preorder, and whether it lies in $X(\mathcal{G}, \le_*)$ is open [LZ21]. The two sides agree on the pentagon and on Duan and Winter's Hadamard example, and if they agreed on every graph, Theorem A would settle that question. They do not.

Theorem D (order two is not the projective rank)**.** *There is a graph G_{1440} on 1440 vertices, built from the same magic square, such that*

$$\Sigma(G_{1440}) \;\le\; \frac{215940}{899} = 240.2002\ldots \;<\; 240.2033\ldots = \frac{14172}{59} \;\le\; \xi_f(\overline{G_{1440}}).$$

The upper bound is an explicit order-two witness: the magic-square representation with a center tilted slightly toward one of its vectors. The lower bound is a rank inequality for the magic square that holds in every dimension (Lemma 7.3); both bounds are proved by hand.

1.3 A taste of the method: the pentagon

Given the projection formula of Section 3, the pentagon conjecture has a short proof. Label C_5 by $\mathbb{Z}_5$ with $i \sim i \pm 1$. In a representation of the pentagon, non-neighbours get orthogonal projections:

$P_iP_{i+2} = 0$ for all i. At order $\alpha = \frac{1}{2}$, the formula of Section 3 for ϑ_α becomes

$$\frac{1}{\vartheta_{1/2}(G)} = \sup_{(P_x),\,\sigma} \min_x \|P_x\sigma^{1/2}P_x\|_2^2,$$

the supremum running over representations (P_x) of G and faithful states σ. Now the one inequality we need: for every representation of C_5 and every matrix X,

$$\sum_{i\in\mathbb{Z}_5} \|P_iXP_i\|_2^2 \;\le\; 2\,\|X\|_2^2. \tag{5}$$

The reason is local. The two neighbours $i-1$ and $i+1$ of a vertex are not adjacent to each other, so $P_{i-1}+P_{i+1}$ is again a projection. Seen from inside P_i, the sum of all five projections is therefore at most twice the identity (Lemma 5.1). Put $X = \sigma^{1/2}$. The right side of (5) is $2\,\mathrm{Tr}\,\sigma = 2$, so some vertex has $\|P_i\sigma^{1/2}P_i\|_2^2 \le 2/5$, and $\vartheta_{1/2}(C_5) \ge 5/2$. Since ϑ_α is nondecreasing in α (Proposition 3.3(ii)), the same bound holds for $\vartheta_1 = 2^{C_{\min}}$ and for $\vartheta_2 = \Sigma$.

The matching upper bound is the obvious classical channel: input i produces a uniformly random element of $\{i, i+1\}$. Its Holevo capacity is $\log_2 5 - 1 = \log_2 \frac{5}{2}$, and $T = \frac{1}{2}I$ dominates every output with $\mathrm{Tr}\,T = \frac{5}{2}$. Duan and Winter's chain (1) squeezes $S_{0,\mathrm{NS}}$ in between, and $\alpha^*(C_5) = \frac{5}{2}$ is classical [Sha56]. Besides Lemma 5.1, the argument uses the projection formula, the identities $\vartheta_1 = 2^{C_{\min}}$ and $\vartheta_2 = \Sigma$ of Proposition 3.3(i) (the first is the radius formula for the Holevo capacity) and monotonicity in α.

1.4 What is new

New points in the entanglement-assisted spectrum. To our knowledge, the parameters ϑ_α with $\alpha > 0$ are the first elements of $X(\mathcal{G}, \le_*)$ other than ϑ. Of the two routes to $\Theta_* = \vartheta$ that Li and Zuiddam suggested [LZ21], Theorems A and B rule out the stronger one, which was to show that ϑ is the only point. They say nothing about whether ϑ is the *least* point. Every ϑ_α dominates ϑ, so the bound $\Theta_* \le \vartheta$ is not improved and the conjecture $\Theta_* = \vartheta$ is untouched. A continuum with the same values on C_5 was already available in the *ordinary* asymptotic spectrum. Vrana's log-convex interpolation [Vra21] between ϑ and a fractional Haemers bound, whose value on C_5 is $\frac{5}{2}$ [BC19, Proposition 4], evaluates the vertex-transitive graph C_5 log-affinely [Vra22, Section 5], so its values on C_5 fill $[\sqrt{5}, \frac{5}{2}]$. What is new here is that the points ϑ_α lie in the entanglement-assisted spectrum $X(\mathcal{G}, \le_*)$.

Multiplicativity for joint representations. Duan and Winter proved that Σ is multiplicative for two *fixed* cq-graphs [DW16, Proposition 18]. Applied to representations of $G \boxtimes H$ that are tensor products, their result gives the order-two identity of Theorem 4.1 for the infimum restricted to such representations. Theorem 4.1 proves the graph-level identity without that restriction, at every order in $[0, 2]$; it does not recover their statement for fixed cq-graphs. At the two lower endpoints the graph-level identity is known: order zero is Lovász's multiplicativity of ϑ [Lov79], and order one is Duan and Winter's additivity of $C_{\min}$ under the strong product [DW16, Lemma 23]. Every order in $(0, 1) \cup (1, 2]$ is new, and order two is the one with an operational consequence. There it gives a single-letter formula,

$$S_{0,\mathrm{NS}}(G) = \log_2 \Sigma(G),$$

for graph-level asymptotic no-signalling simulation: at each block length n one may use any channel compatible with $G^{\boxtimes n}$, and such a channel may correlate the n uses. Duan and Winter

remarked that this identity "may be true", and that proving it requires the additivity of $\log \Sigma$ under the strong product, which they describe as unknown [DW16, Section V]. In their concluding section they list this graph-level simulation cost as open [DW16, Section VII]. For a fixed cq-graph they had already shown that the asymptotic cost is single-letter [DW16, Proposition 18 and Theorem 19]. At the level of confusability graphs, Theorem 4.1 reduces asymptotic no-signalling simulation to a one-shot problem. Their general problem, for non-commutative bipartite graphs that are not cq-graphs, is not addressed here.

The Duan–Winter chain, link by link. With (3) in hand, every link of (1) can now be decided (Fig. 2). Duan and Winter showed that the last link can be strict. They conjectured that the first is strict on the pentagon, which is the corollary above. They left open whether either of the two middle links can be strict, and remarked that the third might be an equality. It is always an equality, by multiplicativity at order two (Theorem 4.1 and Remark 4.2). The second can be strict (Theorem C). For a fixed cq-graph with two non-orthogonal pure outputs, Duan and Winter had already shown the corresponding fixed-channel inequalities to be strict [DW16, Section IV.D]; what is new is a separation of the graph-level quantities. They also wrote that "it seems that $\Sigma(G)$ and $2^{C_{\min}(G)}$ are two new interesting, multiplicative graph parameters, different from both $\vartheta(G)$ and $\alpha^*(G)$" [DW16, Section VII]. For $2^{C_{\min}}$ multiplicativity is their Lemma 23, and Theorem 4.1 gives it for Σ. Their Hadamard-graph example already separates both parameters from α^*, and the pentagon separates both from ϑ.

One family instead of three numbers. Duan and Winter already wrote $\log_2 \Sigma$ as a conditional min-entropy [DW16, Theorem 2], which is a max-relative-entropy radius. What seems to be new is that, once one minimizes over states supported in a fixed projection, this radius coincides with the *Petz* radius of order two. All three quantities then sit in a single monotone family. The sandwiched Rényi graph radii give this entire family as well: for $\frac{1}{2} \le \beta \le \infty$, $\widetilde{\vartheta}_\beta = \vartheta_{2-1/\beta}$ (Proposition 3.5). Thus the sandwiched family runs from ϑ at $\beta = \frac{1}{2}$, through $2^{C_{\min}}$ at $\beta = 1$, to Σ at $\beta = \infty$, and gives the same spectral points. The equality is between optimized graph radii, not divergences of fixed pairs of states. The picture has a useful by-product: every ϑ_α lies between ϑ and α^* (Proposition 3.3). At orders 0, 1 and 2 these bounds follow from Duan and Winter's chain; at other orders they follow from the chain together with monotonicity in α, and flat witnesses give the upper bound directly. So the family collapses to a single value on graphs with $\vartheta = \alpha^*$, perfect graphs among them, and it can only separate points on graphs with $\vartheta < \alpha^*$.

Order two is not the projective rank. Flat witnesses built from d/r-representations of the complement give $\Sigma(G) \le \xi_f(\overline{G})$, with equality whenever $\vartheta = \alpha^*$, on C_5 and on Duan and Winter's Hadamard example. To our knowledge, neither this inequality nor the question of equality appears in earlier work. Had equality held for every graph, Theorem A would have placed $\xi_f(\overline{G})$ in $X(\mathcal{G}, \le_*)$, which Li and Zuiddam leave open [LZ21, after Proposition 24]. Theorem D closes that route, and since $\vartheta_\alpha \le \Sigma$ for $\alpha \le 2$ it does so for every member of the family; it does not decide the question. The lower bound on ξ_f uses no certificate. It is one linear inequality between the normalized ranks of a representation of the 24-vertex graph built from the magic square, tight on the magic-square representation itself, and its proof is a robust form of the anticommutation relations that every operator solution of the magic square obeys [Ji13, Lemma 6]: up to errors controlled by the defects of the representation, cell observables in different rows and columns anticommute, which forces their traces to be small. Rank-counting lower bounds on the projective

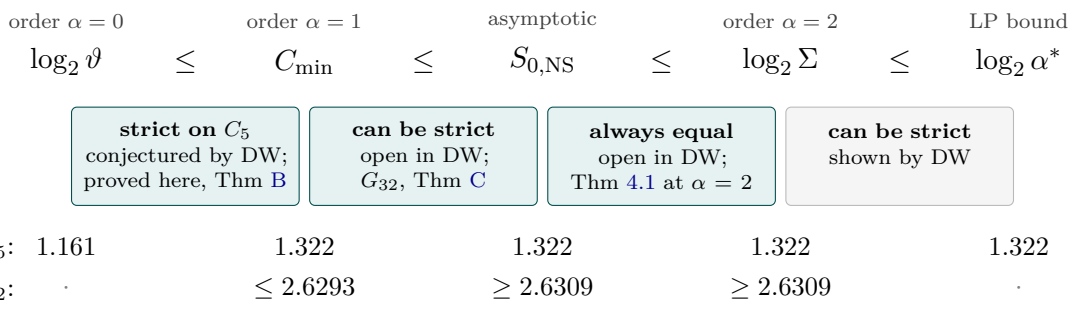


Figure 2: The Duan–Winter chain (1) after this paper. The three quantities in the first four columns that carry an order are the members $\alpha = 0, 1, 2$ of one monotone family. The last column is Shannon's fractional packing (LP) bound [Sha56]. Shaded boxes record the status of each link; teal marks what is new here. The bottom rows give values in bits (dots are values we do not need).

rank have been used before, with spectral input [WE19]; we know of no lower bound of this contextual kind.

1.5 Related work

Dalai showed that the Lovász bound and the sphere-packing bound for classical–quantum channels are both information radii of Csiszár type [Csi95] built from Rényi divergences [Dal13a, Dal13b]. At the level of graphs he considered the order-zero radius ϑ_{sp}, the minimum over projector representations (U_x) of G and density operators F of $\max_x \log(1/\operatorname{Tr} U_x F)$, and proved that it equals $\log \vartheta(G)$, a theorem he credits to Schrijver [Dal13b, Theorem 8]. This is our identity $\vartheta_0 = \vartheta$. Duan and Winter prove it as well, by purifying the center in the proof of their Theorem 5 [DW16, Section V]; the case of a pure center with general projections is due to Cabello, Severini and Winter [CSW10], and Dalai and Winter note the observations of Schrijver and of Duan and Winter [DW17]. The family ϑ_α continues this graph-level order-zero radius to all orders $0 \leq \alpha \leq 2$. For a fixed cq-channel, Duan and Winter already express their quantities as radii: the no-signalling-assisted zero-error capacity of their Theorem 4 is the order-zero radius (by the minimax formula in their proof of Theorem 5), the simulation cost of their Theorem 2 is given by a conditional min-entropy, that is, by a max-relative-entropy radius, and the Holevo capacity is the relative-entropy radius [DW16]. What is new here is the graph-level family at every order in $[0, 2]$ and the identification of its order-two member with a Petz radius.

Recent work concerns related channel quantities. Non-signalling-assisted simulation cannot increase the Rényi capacity of a classical channel, at any order. This follows from the data-processing inequality for channel divergences under non-signalling maps of Oufkir, Yao and Berta [OYB26, Lemma 24] (Lemma 25 in arXiv:2509.15200v1), and Shariat, Hoogsteder-Riera and Tomamichel give an alternative variational proof that works directly with the information-radius form of the capacity [SHRT26]. This is a channel-level classical analogue of our monotonicity step (Theorem 4.7). Concurrently, Cheng and Berta show that the Petz–Rényi channel information of quantum-input channels can be strictly superadditive for every $\alpha \in [\frac{1}{2}, 1)$ [CB26]. This does not conflict with Theorem 4.1, which concerns classical-input families and covers arbitrary, possibly entangled, joint representations of $G \boxtimes H$. Petz radii of a fixed cq-channel also have an operational meaning: Petz–Rényi capacities of Csiszár type determine the error exponent of coding assisted by activated no-signalling correlations [OTB26].

In the ordinary spectrum, continua arise in a different way. Vrana's probabilistic refinement makes the ordinary spectrum log-convex in a suitable sense [Vra21], and so interpolating between two known points produces new ones. We do not know whether an analogue holds for $\leq_*$. Section 8 gives a heuristic reason why, even if it does, ϑ_1 should not be an interpolation between ϑ and Σ. The quantum preorder $\leq_q$ relates fewer pairs of graphs than $\leq_*$ ($G \leq_q H$ implies $G \leq_* H$), so monotonicity under $\leq_q$ is the weaker requirement and $X(\mathcal{G}, \leq_q)$ contains $X(\mathcal{G}, \leq_*)$. Li and Zuiddam place the fractional real and complex Haemers bounds and the complement of the projective rank in $X(\mathcal{G}, \leq_q)$ [LZ21, Theorem 22], and Gao, Gribling and Li place a tracial Haemers bound in the asymptotic spectrum for the commuting-operator version of the quantum preorder [GGL22]. To our knowledge none of these is known to be monotone under $\leq_*$; for the Haemers bounds and the projective rank, Li and Zuiddam state this explicitly [LZ21]. Finally, Vrana studied which points of the ordinary spectrum extend to non-commutative graphs [Vra22]. We stay with classical graphs throughout.

Organization. Section 2 fixes conventions. Section 3 rewrites ϑ_α in terms of projections and proves the basic properties of the family. Section 4 proves Theorem A. Section 5 treats the pentagon and the Duan–Winter chain, Section 6 separates orders one and two, and Section 7 separates order two from the projective rank. Open problems are collected in Section 8. The appendices contain the certificate format and a few routine proofs.

2 Setup

Conventions

- Graphs are finite and simple, and they are read as *confusability graphs*: $x \sim y$ means that inputs x and y may be confused. We write $x \simeq y$ for "$x = y$ or $x \sim y$".
- Distinct non-adjacent vertices must be told apart, so they are represented by *orthogonal* objects. This is Lovász's original orientation [Lov79]. Li and Zuiddam express the same relations through graph complements.
- $G \sqcup H$ is the disjoint union and $G \boxtimes H$ the strong product, where $(x, y) \simeq (x', y')$ if and only if $x \simeq x'$ and $y \simeq y'$. K_0 is the null graph, with no vertices, K_1 is a single vertex, $\overline{K_n}$ is the edgeless graph on n vertices (a noiseless channel with n inputs), and C_5 is the pentagon.
- Logarithms are to base two unless written ln. Hilbert spaces are finite-dimensional, of arbitrary dimension. A state is *faithful* if it is invertible. Π_ρ is the support projection of ρ.

2.1 Representations and Rényi radii

Definition 2.1. A *representation* of G is a family $(P_x)_{x \in V(G)}$ of nonzero orthogonal projections on a Hilbert space such that $P_x P_y = 0$ whenever $x \neq y$ are non-adjacent. A family of states (ρ_x) is *compatible with* G if its supports form a representation.

A compatible family is the same thing as a classical–quantum channel $x \mapsto \rho_x$ whose confusability graph is contained in G: two inputs can only be confused if they are adjacent. Extra orthogonality between adjacent vertices is allowed. Duan and Winter optimize over the same class of channels. Readers who prefer channels whose confusability graph is exactly G lose nothing, because each of the infima (6)–(8) below is the same for both classes (Lemma B.1).

For a state ρ and a faithful state σ, the Petz–Rényi divergence of order $\alpha \in (0,1) \cup (1,2]$ is

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \operatorname{Tr} \rho^\alpha \sigma^{1-\alpha}.$$

At $\alpha = 1$ it is the relative entropy $D(\rho\|\sigma)$, and at $\alpha = 0$ it is $D_0(\rho\|\sigma) = -\log \operatorname{Tr} \Pi_\rho \sigma$. If σ is a state that is not faithful but $\operatorname{supp} \rho \subseteq \operatorname{supp} \sigma$, the same formulas are used with all powers and logarithms of σ taken on its support; Lemma 4.6 uses this convention. We use four standard facts [Pet86, Tom16]. The divergence is nonnegative and vanishes at $\rho = \sigma$. It is additive under tensor products. It is nondecreasing in α. And it satisfies data processing for every $\alpha \in [0,2]$.

Definition 2.2. For a graph G and $\alpha \in [0,2]$, define $\vartheta_\alpha(G)$ by

$$\log \vartheta_\alpha(G) = \inf\Big\{ \max_x D_\alpha(\rho_x\|\sigma) \ : \ (\rho_x) \text{ compatible with } G,\ \sigma \text{ faithful} \Big\}, \tag{6}$$

where the states live on a common Hilbert space of any finite dimension. For the null graph we set $\vartheta_\alpha(K_0) = 0$.

Nothing changes if singular centers are allowed with the usual support conventions: mix a singular center with a little of the maximally mixed state and let the weight go to zero.

2.2 The Duan–Winter quantities

Duan and Winter attach the following quantities to a graph [DW16]. The *minimal Holevo capacity* is

$$C_{\min}(G) = \inf_{(\rho_x) \text{ compatible}} \max_p \Big[S\Big(\sum_x p_x \rho_x\Big) - \sum_x p_x S(\rho_x) \Big]. \tag{7}$$

The *no-signalling simulation cost* is

$$\Sigma(G) = \inf\big\{ \operatorname{Tr} T \ : \ T \geq \rho_x \text{ for all } x,\ (\rho_x) \text{ compatible} \big\}. \tag{8}$$

Duan and Winter show that $\min\{\operatorname{Tr} T : T \geq \rho_x\ \forall x\}$ is the real SDP value governing exact simulation of the channel $x \mapsto \rho_x$ with no-signalling assistance. For a fixed channel, the literal one-shot noiseless message alphabet has size equal to the ceiling of this SDP value; the ceiling disappears after division by block length. At graph level, $\Sigma(G)$ is an infimum over compatible channels and is not known to be attained. Thus $\log \Sigma(G)$ is the infimum of the corresponding single-letter SDP costs. Tensoring witnesses shows that Σ is submultiplicative under $\boxtimes$, so the regularization

$$S_{0,\mathrm{NS}}(G) = \lim_{n \to \infty} \frac{1}{n} \log \Sigma(G^{\boxtimes n})$$

exists. This is Duan and Winter's graph-level asymptotic simulation cost. Finally, $\alpha^*(G)$ is Shannon's fractional packing number [Sha56]: the largest $\sum_x v_x$ over $v \geq 0$ with $\sum_{x \in C} v_x \leq 1$ for every clique C. By LP duality it is also the smallest $\sum_C w_C$ over $w \geq 0$ with $\sum_{C \ni x} w_C \geq 1$ for every vertex x.

2.3 The entanglement-assisted spectrum

We use the entanglement-assisted cohomomorphism preorder of Li and Zuiddam [LZ21], which comes from the entanglement-assisted homomorphisms of Cubitt et al. [CMR+14]. In confusability

language, $G \leq_* H$ means that there are a state ω and positive semidefinite operators ω_g^h ($g \in V(G)$, $h \in V(H)$) with

$$\sum_{h \in V(H)} \omega_g^h = \omega \quad \text{for every } g, \tag{9}$$

$$\omega_g^h \, \omega_{g'}^{h'} = 0 \quad \text{whenever } g \neq g' \text{ are non-adjacent in } G \text{ and } h \simeq h' \text{ in } H. \tag{10}$$

Lemma B.2 derives (9)–(10) from Li and Zuiddam's definition, which is phrased with graph complements. Here is the operational picture. Alice and Bob share a pure entangled state, and ω is Bob's half of it. To send g, Alice measures her half with a measurement that depends on g, obtains an outcome h, and sends h through a channel with confusability graph H. Her measurement steers Bob's half to ω_g^h (unnormalized), and (9) says that Bob's marginal does not depend on g. Suppose g and g' must be told apart but the channel might confuse h with h'. Then (10) lets Bob tell them apart anyway, by measuring his own half. So $G \leq_* H$ says that, with entanglement, one use of an H-channel does everything one use of a G-channel does. For example, $\overline{K_n} \leq_* G$ says that n messages can be sent without error over one use of G.

Definition 2.3. The *entanglement-assisted asymptotic spectrum* $X(\mathcal{G}, \leq_*)$ is the set of maps ϕ from finite graphs to $[0, \infty)$ such that

$$\phi(G \sqcup H) = \phi(G) + \phi(H), \quad \phi(G \boxtimes H) = \phi(G)\phi(H), \quad \phi(K_1) = 1, \quad G \leq_* H \Rightarrow \phi(G) \leq \phi(H).$$

Li and Zuiddam show that $\Theta_*(G) = \min_{\phi \in X(\mathcal{G}, \leq_*)} \phi(G)$, and that $\vartheta \in X(\mathcal{G}, \leq_*)$ [LZ21].

3 Power compressions and the shape of the family

The definition (6) optimizes over states. It is much easier to work with projections, and the tool for this is the following compression.

3.1 From states to projections

Let $P \neq 0$ be a projection, let σ be positive definite (usually a faithful state, but Lemma 3.6 also compresses unnormalized operators), and let $s \in [-1, 1]$. On $\operatorname{ran} P$ define

$$K_s(\sigma, P) = \begin{cases} (P\sigma^s P)^{1/s}, & s \neq 0, \\ \exp(P(\ln \sigma)P), & s = 0, \end{cases} \qquad Z_s(\sigma, P) = \operatorname{Tr} K_s(\sigma, P),$$

extended by zero off $\operatorname{ran} P$. We call K_s the *power compression* of σ to P and Z_s its *weight*. Four values of s recur. $Z_1 = \operatorname{Tr} P\sigma$ is the plain weight. $Z_{1/2} = \|P\sigma^{1/2}P\|_2^2$ is a Hilbert–Schmidt weight. Z_0 is a Gibbs weight. And $Z_{-1} = \operatorname{Tr}(P\sigma^{-1}P)^{-1}$ is the trace of the *shorted operator* $\sigma_{/P}$, the largest positive operator below σ with range in $\operatorname{ran} P$ [AT75]. All the weights are homogeneous of degree one: $K_s(c\sigma, P) = c\, K_s(\sigma, P)$ for $c > 0$.

Proposition 3.1 (supported minimization)**.** *Let $0 \leq \alpha \leq 2$ and $s = 1 - \alpha$. Then*

$$\min_{\rho:\ \operatorname{supp}\rho \subseteq \operatorname{ran} P} D_\alpha(\rho \| \sigma) = -\log Z_s(\sigma, P),$$

and the minimum is attained at $\rho = K_s(\sigma, P)/Z_s(\sigma, P)$.

Proof. For ρ supported in P we have $\operatorname{Tr}\rho^\alpha\sigma^s = \operatorname{Tr}\rho^\alpha A$ with $A = P\sigma^s P$, and $Z_s = \operatorname{Tr}A^{1/s}$ when $s \neq 0$.

Case $0 < \alpha < 1$. Here $s \in (0,1)$. Hölder's inequality for Schatten norms gives $\operatorname{Tr}\rho^\alpha A \leq \|\rho^\alpha\|_{1/\alpha}\|A\|_{1/s} = (\operatorname{Tr}A^{1/s})^s$. Since $\alpha - 1 < 0$, this becomes $D_\alpha(\rho\|\sigma) \geq \frac{s}{\alpha-1}\log Z_s = -\log Z_s$, with equality at $\rho \propto A^{1/s}$.

Case $1 < \alpha \leq 2$. Now $s \in [-1,0)$ and A is positive definite on $\operatorname{ran}P$. Write $A = \sum_k a_k|e_k\rangle\langle e_k|$ and $q_k = \langle e_k|\rho|e_k\rangle$, a probability vector. Convexity of $t \mapsto t^\alpha$ gives $\langle e_k|\rho^\alpha|e_k\rangle \geq q_k^\alpha$, so $\operatorname{Tr}\rho^\alpha A \geq \sum_k a_k q_k^\alpha$. By Hölder,

$$1 = \sum_k (a_k^{1/\alpha} q_k) a_k^{-1/\alpha} \leq \Big(\sum_k a_k q_k^\alpha\Big)^{1/\alpha} \Big(\sum_k a_k^{1/s}\Big)^{(\alpha-1)/\alpha},$$

using $-1/(\alpha-1) = 1/s$. Hence $\operatorname{Tr}\rho^\alpha A \geq (\sum_k a_k^{1/s})^s = Z_s^s$, and dividing the logarithm by $\alpha - 1 > 0$ gives $D_\alpha \geq -\log Z_s$. Equality holds when ρ is diagonal in (e_k) with $q_k \propto a_k^{1/s}$, that is, at $\rho \propto A^{1/s}$.

Case $\alpha = 1$. On $\operatorname{ran}P$ we have $\ln K_0 = P(\ln\sigma)P$, so $\operatorname{Tr}\rho\ln\sigma = \operatorname{Tr}\rho\ln(K_0/Z_0) + \ln Z_0$ and $D(\rho\|\sigma) = D(\rho\|K_0/Z_0) - \log Z_0 \geq -\log Z_0$.

Case $\alpha = 0$. $\operatorname{Tr}\Pi_\rho\sigma \leq \operatorname{Tr}P\sigma = Z_1$, with equality whenever ρ has full support in $\operatorname{ran}P$. □

The minimizers for different vertices have the supports we prescribe, so they form a compatible family whenever the projections form a representation. Conversely, the supports of any compatible family form a representation. This proves the projection formula:

$$\frac{1}{\vartheta_\alpha(G)} = \sup_{(P_x),\,\sigma} \min_{x \in V(G)} Z_{1-\alpha}(\sigma, P_x), \tag{11}$$

with the supremum over representations of G and faithful states σ on the same space.

3.2 Endpoints, flat witnesses and a sandwich

Some witnesses have the same radius at every order. Call a pair (ρ,σ) *flat* if σ commutes with a projection Π and $\rho = \Pi\sigma/\operatorname{Tr}\Pi\sigma$.

Lemma 3.2 (flat witnesses)**.** *If (ρ,σ) is flat, then $D_\alpha(\rho\|\sigma) = -\log\operatorname{Tr}\Pi\sigma$ for every $\alpha \in [0,2]$.*

Proof. With $c = \operatorname{Tr}\Pi\sigma$ we get $\operatorname{Tr}\rho^\alpha\sigma^{1-\alpha} = c^{-\alpha}\operatorname{Tr}\Pi\sigma = c^{1-\alpha}$, which gives the claim for $\alpha \neq 0,1$. The two remaining orders are direct. □

Proposition 3.3. *For every graph G:*

(i) $\vartheta_0(G) = \vartheta(G)$ [Dal13b, Theorem 8], [DW16, proof of Theorem 5], $\vartheta_1(G) = 2^{C_{\min}(G)}$ [OPW97, SW01] *and* $\vartheta_2(G) = \Sigma(G)$*;*

(ii) $\alpha \mapsto \vartheta_\alpha(G)$ *is nondecreasing on* $[0,2]$ (immediate from the monotonicity of D_α in α [Tom16])*;*

(iii) $\vartheta(G) \leq \vartheta_\alpha(G) \leq \alpha^*(G)$ *for every* $\alpha \in [0,2]$*; at* $\alpha \in \{0,1,2\}$ *these bounds follow from Duan and Winter's chain* (1) [DW16].

Proof. (i) *Order zero.* This is Dalai's Theorem 8, which he credits to Schrijver [Dal13b]. Duan and Winter give the same purification argument in the proof of their Theorem 5 [DW16, Section V], and for a pure center the passage from projections to vectors is the argument of Cabello, Severini and Winter [CSW10, proof of Eq. (6)]. We include the short argument in our notation.

By (11), $1/\vartheta_0(G) = \sup \min_x \operatorname{Tr} P_x\sigma$. Lovász's definition of ϑ through the value of an orthonormal representation reads $1/\vartheta(G) = \sup \min_x |\langle c, u_x\rangle|^2$ over orthonormal representations (u_x) of G and unit vectors c [Lov79, Section II]. Lovász uses real vectors, but complex ones give the same value: rephase each u_x so that $\langle c, u_x\rangle \geq 0$, and then realify, $v \mapsto (\operatorname{Re} v, \operatorname{Im} v)$, which preserves unit length, orthogonality and the values $\langle c, u_x\rangle$. A vector representation is a projection representation with $P_x = |u_x\rangle\langle u_x|$, and a faithful approximation of $|c\rangle\langle c|$ gives $\vartheta_0 \leq \vartheta$. Conversely, purify σ to a unit vector $c \in \mathcal{H} \otimes \mathcal{H}'$ and put $u_x \propto (P_x \otimes I)c$. Non-adjacent vertices get orthogonal vectors, and $|\langle c, u_x\rangle|^2 = \operatorname{Tr} P_x\sigma$, so $\vartheta \leq \vartheta_0$.

Order one. For each compatible family, the Holevo capacity equals the relative-entropy radius $\min_\sigma \max_x D(\rho_x\|\sigma)$ [OPW97, SW01]. Taking the infimum over families gives $\vartheta_1 = 2^{C_{\min}}$.

Order two. By (11), $1/\vartheta_2 = \sup \min_x \operatorname{Tr} \sigma_{/P_x}$. Given a compatible family and $T \geq \rho_x$ with $t = \operatorname{Tr} T$, put $\sigma = T/t$, regularized if needed. Each ρ_x/t lies below σ with range in $\operatorname{ran} P_x$, so $\operatorname{Tr} \sigma_{/P_x} \geq 1/t$. Hence $\vartheta_2 \leq \Sigma$. Conversely, if $\operatorname{Tr} \sigma_{/P_x} \geq c$ for all x, the states $\rho_x = \sigma_{/P_x}/\operatorname{Tr} \sigma_{/P_x}$ are compatible and all lie below $T = \sigma/c$, whose trace is $1/c$. Hence $\Sigma \leq \vartheta_2$.

(ii) D_α is nondecreasing in α for each pair of states, and infima preserve this.

(iii) The lower bound follows from (i) and (ii). The upper bound also follows from (i), (ii) and (1), since $\vartheta_\alpha \leq \vartheta_2 = \Sigma \leq \alpha^*$; flat witnesses give it directly. Take a fractional clique cover w with $\sum_C w_C = \alpha^*(G)$, and work on the space spanned by orthonormal vectors $|C\rangle$, one for each clique with $w_C > 0$. Put $\sigma = \sum_C w_C|C\rangle\langle C|/\alpha^*(G)$ and let Π_x project onto the cliques containing x. Non-adjacent vertices share no clique, so the flat states $\rho_x = \Pi_x\sigma/\operatorname{Tr} \Pi_x\sigma$ are compatible. By Lemma 3.2 their radius at every order is $\max_x \log(\alpha^*(G)/\sum_{C\ni x} w_C) \leq \log \alpha^*(G)$. □

Two consequences are worth stating. First, if $\vartheta(G) = \alpha^*(G)$, the whole family collapses on G. This happens for every perfect graph. Second, if some flat witness is optimal for $\vartheta_1(G)$, then $\vartheta_1(G) = \vartheta_2(G)$, because the flat radius does not depend on the order. On the pentagon, this is exactly what happens (Section 5).

3.3 Continuity

Proposition 3.4. *For every graph G, the map $\alpha \mapsto \vartheta_\alpha(G)$ is continuous on $[0, 1)$.*

Right continuity comes from fixing a near-optimal witness. Left continuity comes from a comparison $D_\alpha \leq \frac{\alpha(1-\beta)}{\beta(1-\alpha)} D_\beta$ for $\beta < \alpha < 1$, whose constant does not depend on the states or the dimension. The details are in Appendix B.1. The right-continuity argument works at every order in $[0, 2)$, so the map is right-continuous on $[0, 2)$. Left continuity at orders in $[1, 2]$ is not proved here, except on the pentagon, where the map is constant on $[\frac{1}{2}, 2]$ (Theorem 5.2).

3.4 Sandwiched radii give the same family

The Petz divergence is not the only quantum Rényi divergence. For $\beta \in [\frac{1}{2}, 1) \cup (1, \infty)$ the *sandwiched* Rényi divergence [MLDS+13, WWY14] is

$$\widetilde{D}_\beta(\rho\|\sigma) = \frac{1}{\beta - 1} \log \operatorname{Tr}\Big(\sigma^{\frac{1-\beta}{2\beta}}\, \rho\, \sigma^{\frac{1-\beta}{2\beta}}\Big)^\beta,$$

with the relative entropy $\widetilde{D}_1 = D$ at order one and the max-relative entropy $\widetilde{D}_\infty(\rho\|\sigma) = \log \min\{\lambda : \rho \leq \lambda\sigma\}$ at order infinity. This family satisfies data processing throughout $[\frac{1}{2}, \infty]$ [FL13, Bei13]. Define $\widetilde{\vartheta}_\beta(G)$ by (6) with $\widetilde{D}_\beta$ in place of D_α, using the same compatible families and faithful centers, and set $\widetilde{\vartheta}_\beta(K_0) = 0$. On this full range it gives exactly the Petz family, with a change of order.

Proposition 3.5 (sandwiched radii)**.** *For every graph G and every $\beta \in [\frac{1}{2}, \infty]$,*

$$\widetilde{\vartheta}_\beta(G) = \vartheta_{2-1/\beta}(G),$$

with $2 - 1/\infty = 2$. In particular,

$$\widetilde{\vartheta}_{1/2} = \vartheta, \qquad \widetilde{\vartheta}_{2/3} = \vartheta_{1/2}, \qquad \widetilde{\vartheta}_1 = 2^{C_{\min}}, \qquad \widetilde{\vartheta}_\infty = \Sigma.$$

Proof. Put $s = (1-\beta)/\beta \in [-1, 1]$, with $s = -1$ at $\beta = \infty$, so the corresponding Petz order is $\alpha = 1 - s = 2 - 1/\beta$. As in the proof of (11), it suffices to show that for a projection $P \neq 0$ and a faithful state σ,

$$\min_{\rho:\ \operatorname{supp}\rho \subseteq \operatorname{ran} P} \widetilde{D}_\beta(\rho\|\sigma) = -\log Z_s(\sigma, P), \tag{12}$$

with a minimizer faithful on $\operatorname{ran} P$. All inverses and compressed powers below are taken on $\operatorname{ran} P$.

Case $\frac{1}{2} \leq \beta < 1$. Here $s \in (0, 1]$. Put $A = P\sigma^s P > 0$ on $\operatorname{ran} P$ and $X = A^{1/2}\rho A^{1/2}$. The constraint on ρ becomes $X \geq 0$, $\operatorname{Tr} A^{-1}X = 1$. The map $V = \sigma^{s/2}PA^{-1/2} : \operatorname{ran} P \to \mathcal{H}$ is an isometry, and

$$\sigma^{s/2}\rho\sigma^{s/2} = VXV^*.$$

Consequently the sandwiched moment is $\operatorname{Tr} X^\beta$. Pinch X in an orthonormal eigenbasis (e_i) of A, obtaining Y. Pinching preserves $\operatorname{Tr} A^{-1}X$ and cannot decrease $\operatorname{Tr} X^\beta$: it is an average of unitary conjugations, and $X \mapsto \operatorname{Tr} X^\beta$ is concave for $0 < \beta < 1$. Writing $Ae_i = a_i e_i$ and $\langle e_i, Y e_i\rangle = a_i q_i$, we have $q_i \geq 0$ and $\sum_i q_i = 1$. Scalar Hölder with conjugate exponents $1/\beta$ and $1/(1-\beta)$ gives

$$\operatorname{Tr} X^\beta \leq \operatorname{Tr} Y^\beta = \sum_i a_i^\beta q_i^\beta \leq \Big(\sum_i a_i^{\beta/(1-\beta)}\Big)^{1-\beta} = Z_s(\sigma, P)^{1-\beta}.$$

Equality holds at $\rho = A^{1/s}/Z_s = K_s/Z_s$: then $q_i = a_i^{1/s}/Z_s$ and X is already diagonal in (e_i). Since $\beta - 1 < 0$, maximizing the moment minimizes the divergence, giving (12). At $\beta = \frac{1}{2}$ this directly reads $-\log Z_1(\sigma, P) = -\log \operatorname{Tr} P\sigma$; no limiting argument is needed.

Case $\beta = 1$. This is Proposition 3.1 at order one, with minimizer K_0/Z_0.

Case $1 < \beta < \infty$. Put $\theta = (\beta - 1)/\beta \in (0, 1)$, so $s = -\theta$, and let $A = P\sigma^{-\theta}P$, which is positive definite on $\operatorname{ran} P$, and $X = A^{1/2}\rho A^{1/2}$. The operator $\sigma^{-\theta/2}\rho\,\sigma^{-\theta/2}$ has the same nonzero eigenvalues as $\rho^{1/2}\sigma^{-\theta}\rho^{1/2} = \rho^{1/2}A\rho^{1/2}$, and hence as X, so the sandwiched moment is $\operatorname{Tr} X^\beta$. Since $\operatorname{Tr} A^{-1}X = \operatorname{Tr}\rho = 1$, Hölder's inequality with the conjugate exponents β and $\beta/(\beta-1) = 1/\theta$ gives

$$1 \leq \|X\|_\beta \, \|A^{-1}\|_{1/\theta} = (\operatorname{Tr} X^\beta)^{1/\beta} (\operatorname{Tr} A^{-1/\theta})^\theta.$$

Here $A^{-1/\theta} = (P\sigma^{-\theta}P)^{1/(-\theta)} = K_{-\theta}(\sigma, P)$, so $\operatorname{Tr} X^\beta \geq Z_{-\theta}(\sigma, P)^{-(\beta-1)}$, which is the lower bound in (12). At $\rho = K_{-\theta}/Z_{-\theta}$ we get $X \propto A^{1-1/\theta}$, so $X^\beta \propto A^{-1/\theta} = (A^{-1})^{1/\theta}$, which is the equality case of Hölder's inequality.

Case $\beta = \infty$. The condition $\rho \leq \lambda\sigma$ for a state supported in $\operatorname{ran} P$ says that ρ/λ lies below σ with range in $\operatorname{ran} P$, so $1/\lambda \leq \operatorname{Tr}\sigma_{/P} = Z_{-1}(\sigma, P)$, with equality at $\rho = \sigma_{/P}/\operatorname{Tr}\sigma_{/P} = K_{-1}/Z_{-1}$.

In every case K_s/Z_s is faithful on $\operatorname{ran} P$. Thus the minimizing states retain the prescribed supports, and for a graph with at least one vertex the same argument as in (11) yields

$$\frac{1}{\widetilde{\vartheta}_\beta(G)} = \sup_{(P_x),\,\sigma} \min_x Z_s(\sigma, P_x) = \frac{1}{\vartheta_{1-s}(G)}.$$

The null graph follows from the conventions. These identities hold in each fixed dimension before optimizing over representations and centers; no exchange of an order limit with an unbounded-dimension infimum is used. □

Thus the sandwiched radii on $[\frac{1}{2}, \infty]$ give exactly the Petz family on $[0, 2]$, from the Lovász number to the no-signalling simulation cost. By Theorem A, every $\widetilde{\vartheta}_\beta$ in this range is a point of $X(\mathcal{G}, \leq_*)$, and the two families give the same set of spectral points. This is an identity of optimized graph radii after reparameterization, not an identity between the two divergences for arbitrary fixed states. For a pure state $\psi = |\phi\rangle\langle\phi|$ and a faithful state σ the two divergences do agree: $\widetilde{D}_\beta(\psi\|\sigma) = D_{2-1/\beta}(\psi\|\sigma)$ for every $\beta \in [\frac{1}{2}, \infty]$. Indeed, for $\beta \neq 1, \infty$ the operator $\sigma^{(1-\beta)/2\beta}\psi\,\sigma^{(1-\beta)/2\beta}$ has rank one, so both sides equal $\frac{\beta}{\beta-1}\log\langle\phi|\sigma^{(1-\beta)/\beta}|\phi\rangle$; at $\beta = 1$ both are the relative entropy, and at $\beta = \infty$ both equal $\log\langle\phi|\sigma^{-1}|\phi\rangle$. Proposition 3.5 is the corresponding statement for optimized radii, where the optimal states need not be pure. At $\beta = \infty$ the identity $\widetilde{\vartheta}_\infty = \Sigma$ is Duan and Winter's conditional min-entropy formula [DW16, Theorem 2], written as a max-relative-entropy radius and optimized over compatible channels.

3.5 The tower identity

The key algebraic fact is that compressing in two steps is the same as compressing in one.

Lemma 3.6 (tower identity)**.** *Let $P \leq Q$ be projections and $s \in [-1, 1]$. Compute the outer compression below inside* $\operatorname{ran} Q$*, where $K_s(\sigma, Q)$ is faithful. Then*

$$K_s(K_s(\sigma, Q), P) = K_s(\sigma, P).$$

Consequently, if $z = Z_s(\sigma, Q)$, then $Z_s(K_s(\sigma, Q)/z,\, P) = Z_s(\sigma, P)/z$.

Proof. For $s \neq 0$, the s-th power of $K_s(\sigma, Q)$ inside $\operatorname{ran} Q$ is $Q\sigma^s Q$, and compressing further to $P \leq Q$ gives $P\sigma^s P$. For $s = 0$ the same computation works with logarithms. The second statement follows from homogeneity. □

4 Every order up to two is a spectral point

We need four properties: normalization, additivity under $\sqcup$, multiplicativity under $\boxtimes$, and monotonicity under $\leq_*$. Normalization is immediate. On K_1 the divergence is nonnegative and vanishes at $\rho = \sigma$, so $\vartheta_\alpha(K_1) = 1$. The product is the heart of the matter, so we start there.

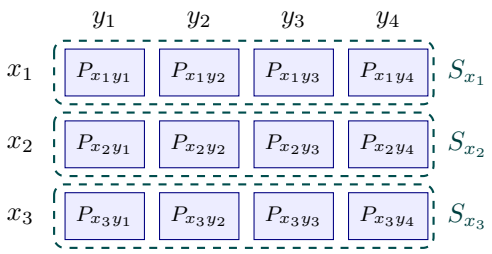
4.1 Products

Here is the idea (Fig. 3). Take any representation (P_{xy}) of $G \boxtimes H$ and group its projections by the first coordinate. The ranges in row x span a subspace, with projection S_x. If x and x' are distinguishable in G, then every cell of row x is distinguishable from every cell of row x', so $S_x \perp S_{x'}$ and the rows represent G. Inside one row, the cells represent H. All that is missing is a way to split the weight of a cell into a row part and a within-row part. The tower identity does exactly this.

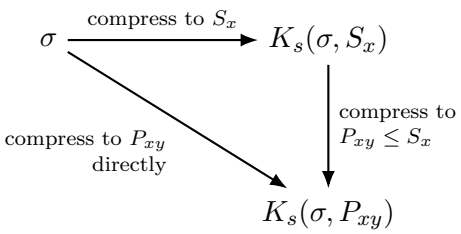
Theorem 4.1 (products)**.** *For $0 \leq \alpha \leq 2$ and all graphs G, H,*

$$\vartheta_\alpha(G \boxtimes H) = \vartheta_\alpha(G)\,\vartheta_\alpha(H).$$

Proof. If either graph is the null graph both sides vanish, so assume that both have vertices and put $s = 1 - \alpha$.



Distinguishable rows are orthogonal: (S_x) represents G.
Within one row, $(P_{xy})_y$ represents H.



Both routes agree (Lemma 3.6). With $\sigma_x = K_s(\sigma, S_x)/z_x$ this reads $Z_s(\sigma_x, P_{xy}) = Z_s(\sigma, P_{xy})/z_x$.

Figure 3: The proof of Theorem 4.1. Left: a representation of $G \boxtimes H$ that need not be a tensor product, organized by rows. Right: the tower identity lets each row be analysed with its own reduced center, so the weight of a cell factors into a G-part z_x and an H-part.

For "$\le$", take witnesses $(\rho_x), \sigma$ for G and $(\tau_y), \omega$ for H. If (x, y) and (x', y') are distinct and non-adjacent in $G \boxtimes H$, then x, x' are distinct and non-adjacent in G or y, y' are in H. In either case $(\rho_x \otimes \tau_y)(\rho_{x'} \otimes \tau_{y'}) = 0$. Since D_α is additive under tensor products, radii add.

For "$\ge$", let (P_{xy}) represent $G \boxtimes H$, let σ be faithful, and suppose $Z_s(\sigma, P_{xy}) \ge c > 0$ for every pair (x, y). Let S_x be the projection onto $\sum_y \operatorname{ran} P_{xy}$, and put $z_x = Z_s(\sigma, S_x)$.

The rows represent G. If $x \ne x'$ are non-adjacent in G, every pair $(x, y), (x', y')$ is distinct and non-adjacent in $G \boxtimes H$, so $P_{xy}P_{x'y'} = 0$ and hence $S_x S_{x'} = 0$.

Each row represents H. Fix x. If $y \ne y'$ are non-adjacent in H, then (x, y) and (x, y') are non-adjacent, so $(P_{xy})_y$ represents H on $\operatorname{ran} S_x$. Use the faithful state $\sigma_x = K_s(\sigma, S_x)/z_x$ on $\operatorname{ran} S_x$ as center. By Lemma 3.6, $Z_s(\sigma_x, P_{xy}) = Z_s(\sigma, P_{xy})/z_x \ge c/z_x$ for every y. The projection formula (11) for H gives $c/z_x \le 1/\vartheta_\alpha(H)$, that is, $z_x \ge c\,\vartheta_\alpha(H)$.

Combine. Apply (11) for G to (S_x) and σ: $c\,\vartheta_\alpha(H) \le \min_x z_x \le 1/\vartheta_\alpha(G)$. So $c \le 1/(\vartheta_\alpha(G)\vartheta_\alpha(H))$ for every admissible c, and taking the supremum gives

$$\frac{1}{\vartheta_\alpha(G \boxtimes H)} \le \frac{1}{\vartheta_\alpha(G)\,\vartheta_\alpha(H)}. \qquad \square$$

Remark 4.2. At $\alpha = 0$ and $\alpha = 1$, Theorem 4.1 recovers the multiplicativity of ϑ [Lov79] and Duan and Winter's additivity of $C_{\min}$ [DW16, Lemma 23]. At $\alpha = 2$ it says $\Sigma(G \boxtimes H) = \Sigma(G)\Sigma(H)$. Hence $S_{0,\mathrm{NS}}(G) = \log \Sigma(G)$: regularizing the graph-level value $\log \Sigma(G)$, an infimum of one-shot SDP costs, does not change it. Their fixed-cq-graph multiplicativity [DW16, Proposition 18] gives this identity only for the infimum restricted to product representations $P_{xy} = P_x \otimes Q_y$. Theorem 4.1 removes that restriction, but it does not recover their statement for fixed cq-graphs.

4.2 Disjoint unions

Two small lemmas do the work. The first rewrites a divergence after compressing the center. The second says that the weights of orthogonal blocks never add up to more than one.

Lemma 4.3 (compression identity)**.** *Let $Q \ne 0$ be a projection, $s = 1 - \alpha \in [-1, 1]$, and $\sigma_Q = K_s(\sigma, Q)/Z_s(\sigma, Q)$. Every state ρ supported in $\operatorname{ran} Q$ satisfies*

$$D_\alpha(\rho\|\sigma) = D_\alpha(\rho\|\sigma_Q) - \log Z_s(\sigma, Q).$$

Proof. For $\alpha \notin \{0,1\}$, substitute $Q\sigma^s Q = Z_s^s \sigma_Q^s$ into the Petz moment. For $\alpha = 1$ use $Q(\ln\sigma)Q = \ln\sigma_Q + (\ln Z_0)Q$ on $\operatorname{ran} Q$, and for $\alpha = 0$ use $Q\sigma Q = Z_1\sigma_Q$. □

Lemma 4.4 (orthogonal blocks)**.** *If $Q_1, \dots, Q_m$ are mutually orthogonal nonzero projections, σ is a faithful state and $s \in [-1,1]$, then $\sum_j Z_s(\sigma, Q_j) \le 1$.*

The proof is a one-line application of Jensen's inequality (Appendix B.2).

Theorem 4.5 (disjoint unions)**.** *For $0 \le \alpha \le 2$, $\vartheta_\alpha(G \sqcup H) = \vartheta_\alpha(G) + \vartheta_\alpha(H)$.*

Proof. Assume that neither graph is the null graph. For "$\le$", put witnesses for G and H with radii at most $\log a$ and $\log b$ on orthogonal summands, and use the center $(a\sigma_G \oplus b\sigma_H)/(a+b)$. Compressing this center to the first summand gives weight $a/(a+b)$ and reduced center σ_G, so by Lemma 4.3 every G-state has radius at most $\log a + \log \frac{a+b}{a} = \log(a+b)$. The same holds for H.

For "$\ge$", take a compatible family of $G \sqcup H$ with center σ and radius at most R. Let Q_G and Q_H project onto the spans of the supports of the two subfamilies. Cross pairs are non-adjacent, so $Q_G Q_H = 0$. By Lemma 4.3, the G-states have radius at most $R + \log Z_s(\sigma, Q_G)$ around σ_{Q_G}, so $\vartheta_\alpha(G) \le 2^R Z_s(\sigma, Q_G)$. Likewise $\vartheta_\alpha(H) \le 2^R Z_s(\sigma, Q_H)$. Add the two bounds and apply Lemma 4.4. □

In particular $\vartheta_\alpha(\overline{K_n}) = n$.

4.3 Entanglement-assisted monotonicity

The proof follows the protocol described after (10). Bob ends up holding his share of the entanglement together with the channel output. If τ_h is the output for h, his joint state given g is $\sum_h \omega_g^h \otimes \tau_h$. These joint states form a new channel for G, and the only question is whether its radius can be larger than the old one. It cannot. The joint state is a flagged mixture with the flag discarded, and discarding the flag is data processing.

Lemma 4.6 (flagged mixtures)**.** *Let $\omega_h \ge 0$ with $\sum_h \omega_h = \omega$ and $\operatorname{Tr}\omega = 1$, let τ_h be states and let σ be faithful. If $D_\alpha(\tau_h\|\sigma) \le R$ whenever $\omega_h \ne 0$, then for $0 \le \alpha \le 2$*

$$D_\alpha\Big(\sum_h \omega_h \otimes \tau_h \;\Big\|\; \omega \otimes \sigma\Big) \le R.$$

Proof. Add a classical flag and compare $\Omega = \sum_h |h\rangle\langle h| \otimes \omega_h \otimes \tau_h$ with $\Omega' = \sum_h |h\rangle\langle h| \otimes \omega_h \otimes \sigma$; the support of Ω lies in that of Ω'. The Petz moment splits over the flag, and the ω_h factors cancel:

$$\operatorname{Tr}\Omega^\alpha \Omega'^{1-\alpha} = \sum_h w_h \operatorname{Tr}\tau_h^\alpha \sigma^{1-\alpha}, \qquad w_h = \operatorname{Tr}\omega_h, \quad \textstyle\sum_h w_h = 1.$$

For $0 < \alpha < 1$ each term satisfies $\operatorname{Tr}\tau_h^\alpha\sigma^{1-\alpha} \ge 2^{(\alpha-1)R}$, so the average does too and $D_\alpha(\Omega\|\Omega') \le R$. For $1 < \alpha \le 2$ all inequalities reverse and the conclusion is the same. At $\alpha = 1$ the divergence is the w-average of the $D(\tau_h\|\sigma)$, and at $\alpha = 0$ we get $\operatorname{Tr}\Pi_\Omega\Omega' = \sum_h w_h \operatorname{Tr}\Pi_{\tau_h}\sigma \ge 2^{-R}$. Tracing out the flag maps $\Omega \mapsto \sum_h \omega_h \otimes \tau_h$ and $\Omega' \mapsto \omega \otimes \sigma$, and Petz divergences of order $\alpha \in [0,2]$ do not increase under partial traces [Pet86, Tom16]. □

Theorem 4.7 (entanglement-assisted monotonicity)**.** *If $G \le_* H$, then $\vartheta_\alpha(G) \le \vartheta_\alpha(H)$ for every $\alpha \in [0,2]$.*

Proof. If G is the null graph there is nothing to prove, and otherwise (9) forces H to have a vertex. Take ω, ω_g^h as in (9)–(10), restricted to $\operatorname{supp}\omega$, and a compatible family (τ_h) for H with faithful center σ and radius at most R. Define

$$\tau'_g = \sum_h \omega_g^h \otimes \tau_h, \qquad \sigma' = \omega \otimes \sigma.$$

Each τ'_g has trace $\operatorname{Tr}\omega = 1$. If $g \neq g'$ are non-adjacent, every term of $\tau'_g\tau'_{g'} = \sum_{h,h'} \omega_g^h \omega_{g'}^{h'} \otimes \tau_h\tau_{h'}$ vanishes. When $h \simeq h'$ this is (10). Otherwise h, h' are distinct and non-adjacent in H, and then $\tau_h\tau_{h'} = 0$. So (τ'_g) is compatible with G, and Lemma 4.6 bounds its radius around σ' by R. □

Proof of Theorem A. Normalization was noted at the start of the section. Additivity is Theorem 4.5, multiplicativity is Theorem 4.1, and monotonicity is Theorem 4.7. At $\alpha = 0$ these are properties of ϑ, and membership of ϑ in $X(\mathcal{G}, \leq_*)$ is Li and Zuiddam's theorem [LZ21], whose monotonicity step is due to Cubitt et al. [CMR+14]. □

Remark 4.8 (orders above two)**.** The only place the restriction $\alpha \leq 2$ enters is data processing in Lemma 4.6. Proposition 3.1, Lemma 3.6 and Lemmas 4.3 and 4.4 hold verbatim for $s < -1$, that is, for $\alpha > 2$, so data processing is the only obstruction in this argument. Petz data processing fails above order two, and the inequality of Lemma 4.6 itself fails too: the accompanying verification scripts (see Data and code availability) exhibit an explicit finite-dimensional instance at order three in which it fails by more than half a bit. So this argument says nothing about $\alpha > 2$.

5 The pentagon

Throughout this section C_5 is labelled by $\mathbb{Z}_5$ with $i \sim i \pm 1$, and a representation satisfies $P_iP_{i+2} = 0$ for every i. We begin at order one half, where everything is exact, and then work down to order zero.

5.1 Order one half and above

Lemma 5.1 (Hilbert–Schmidt inequality)**.** *For every representation of C_5 and every operator X,*

$$\sum_i \|P_iXP_i\|_2^2 \leq 2\|X\|_2^2.$$

Proof. Put $T = \sum_i P_i$. The terms $P_iP_{i\pm2}P_i$ vanish, and P_{i-1} and P_{i+1} are orthogonal to each other, so

$$P_iTP_i = P_i + P_i(P_{i-1} + P_{i+1})P_i \leq 2P_i. \tag{13}$$

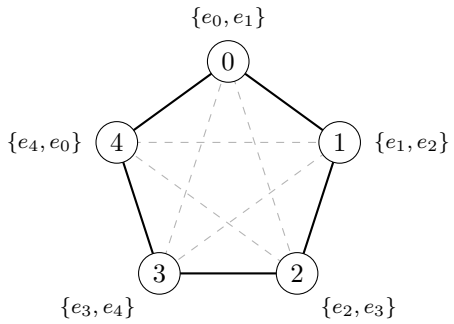
Consider the map $\mathcal{A} : (X_i)_i \mapsto \sum_i X_i$ from $\bigoplus_i P_i\mathcal{B}_2P_i$ to the Hilbert–Schmidt class $\mathcal{B}_2$. For $X_i = P_iX_iP_i$, cyclicity gives $\operatorname{Tr} X_i^*X_j = \operatorname{Tr}((P_jX_i)^*(X_jP_i))$, so

$$|\operatorname{Tr} X_i^*X_j| \leq \tfrac{1}{2}(\|P_jX_i\|_2^2 + \|X_jP_i\|_2^2).$$

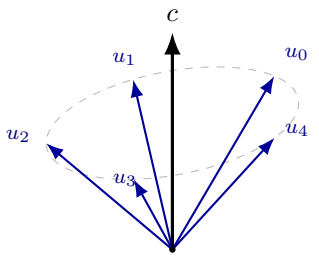
Summing over i and j and using (13),

$$\Big\|\sum_i X_i\Big\|_2^2 \leq \tfrac{1}{2}\sum_i[\operatorname{Tr}(X_i^*TX_i) + \operatorname{Tr}(X_iTX_i^*)] \leq 2\sum_i\|X_i\|_2^2.$$

So $\|\mathcal{A}\|^2 \leq 2$. The adjoint of $\mathcal{A}$ is $X \mapsto (P_iXP_i)_i$, and it has the same norm. □



(a) flat witness, $\frac{1}{2} \le \alpha \le 2$



(b) Lovász's umbrella, $\alpha = 0$

Figure 4: Two pentagon representations attaining the known values in the indicated regimes. Solid lines are edges and dashed chords are the non-edges, which must be orthogonal. (a) Vertex i gets the uniform state on $\{e_i, e_{i+1}\}$ and the center is $I/5$. Non-adjacent supports are disjoint, and the radius is $\log \frac{5}{2}$ at every order; this witness is optimal for $\frac{1}{2} \le \alpha \le 2$. (b) Unit vectors u_i on a cone around the handle c with $|\langle c, u_i\rangle|^2 = 1/\sqrt{5}$ and $u_i \perp u_{i+2}$. With the singular center $|c\rangle\langle c|$, a limit of faithful centers, this attains $\vartheta(C_5) = \sqrt{5}$ at order zero; for these vectors no faithful center attains it. For $0 \le \alpha \le \frac{1}{2}$, faithful centers tending to $|c\rangle\langle c|$ make the exponentiated radius tend to $5^{1/(2-2\alpha)}$, giving an upper bound on $\vartheta_\alpha(C_5)$. For $0 < \alpha < \frac{1}{2}$, the exact graph value remains undetermined.

Theorem 5.2. *$\vartheta_\alpha(C_5) = \frac{5}{2}$ for every $\alpha \in [\frac{1}{2}, 2]$.*

Proof. At $s = \frac{1}{2}$ the weight is $Z_{1/2}(\sigma, P) = \|P\sigma^{1/2}P\|_2^2$. Lemma 5.1 with $X = \sigma^{1/2}$ gives $\sum_i Z_{1/2}(\sigma, P_i) \le 2 \operatorname{Tr} \sigma = 2$. So the minimum over i is at most $\frac{2}{5}$, and (11) gives $\vartheta_{1/2}(C_5) \ge \frac{5}{2}$. By monotonicity in α the same lower bound holds on $[\frac{1}{2}, 2]$. For the upper bound, $\alpha^*(C_5) = \frac{5}{2}$ and Proposition 3.3(iii) applies. Explicitly, the flat witness is $\rho_i = \frac{1}{2}(|i\rangle\langle i| + |i+1\rangle\langle i+1|)$ on $\mathbb{C}^5$ with $\sigma = I/5$, and it has radius $\log \frac{5}{2}$ at every order. □

5.2 Below order one half

Order zero is governed by a trace-norm version of Lemma 5.1, with the constant $\sqrt{5}$ in place of 2. Interpolating between the two gives a lower bound at every order in between.

Lemma 5.3 (trace-norm inequality)**.** *For every representation of C_5 and every operator X, $\sum_i \|P_i X P_i\|_1 \le \sqrt{5}\, \|X\|_1$.*

Proof. If (P_i) represents G and v is a vector, pick unit vectors $u_i \in \operatorname{ran} P_i$ parallel to $P_i v$ (arbitrary if $P_i v = 0$). They form an orthonormal representation of G, and $\sum_i \|P_i v\|^2 = \sum_i |\langle u_i, v\rangle|^2 \le \vartheta(\overline{G})\|v\|^2$ by Lovász's Theorem 5 applied to $\overline{G}$ [Lov79]. That theorem is stated for real vectors; to apply it, rephase each u_i so that $\langle u_i, v\rangle \ge 0$ and realify, $w \mapsto (\operatorname{Re} w, \operatorname{Im} w)$, which preserves unit length, orthogonality and the values $\langle u_i, v\rangle$. The pentagon is self-complementary, so the constant is $\sqrt{5}$. For $X = ab^*$ we have $\|P_i a b^* P_i\|_1 = \|P_i a\|\|P_i b\|$, and Cauchy–Schwarz proves the claim for rank one. A singular value decomposition and the triangle inequality do the rest. □

Lemma 5.4 (interpolation)**.** *For $1 \le p \le 2$, every representation of C_5 and every operator X,*

$$\sum_i \|P_i X P_i\|_p^p \le B(p)\, \|X\|_p^p, \qquad B(p) = 5^{(2-p)/2}\, 2^{p-1}.$$

Proof. Consider the map $T : X \mapsto \bigoplus_i P_i X P_i$ into block-diagonal operators on $\mathbb{C}^5 \otimes \mathcal{H}$. For block-diagonal operators $\|\bigoplus_i Y_i\|_p^p = \sum_i \|Y_i\|_p^p$, so the claim says that T has norm at most $B(p)^{1/p}$ from $S_p(\mathcal{H})$ to $S_p(\mathbb{C}^5 \otimes \mathcal{H})$. By Lemma 5.3 the norm is at most $\sqrt{5}$ for $p = 1$, and by Lemma 5.1 it is at most $\sqrt{2}$ for $p = 2$. The Schatten classes form a complex interpolation scale with equal norms [PX03, Eq. (2.1)], and the complex method is exact of exponent θ [BL76], here $\theta = 2 - 2/p$. So the norm from S_p to S_p is at most $\sqrt{5}^{1-\theta}\sqrt{2}^{\theta} = 5^{(2-p)/(2p)}2^{(p-1)/p}$. Raise this to the power p. □

Theorem 5.5. *For $0 \le \alpha \le \frac{1}{2}$ and $p = 1/(1-\alpha)$, the bounds (4) hold.*

Proof. For the lower bound put $s = 1/p$ and $X = \sigma^s$ in Lemma 5.4. Then $\|X\|_p^p = \operatorname{Tr}\sigma = 1$ and $\|P_i\sigma^s P_i\|_p^p = Z_s(\sigma, P_i)$, so $\sum_i Z_s(\sigma, P_i) \le B(p)$. The minimum is at most the average, so $\vartheta_\alpha(C_5) \ge 5/B(p) = \sqrt{5}(\sqrt{5}/2)^{p-1}$.

The step from the minimum to the average loses nothing. Given a representation (P_i) of C_5 with center σ, the projections $P_i' = \bigoplus_{k\in\mathbb{Z}_5} P_{i+k}$ on $\mathbb{C}^5 \otimes \mathcal{H}$ again represent C_5, and with the center $\sigma' = \bigoplus_k \sigma/5$ homogeneity and additivity over orthogonal blocks give $Z_s(\sigma', P_i') = \frac{1}{5}\sum_k Z_s(\sigma, P_k)$ for every i. So the minimum for (P_i') and σ' is the average for (P_i) and σ, and the supremum of the minimum equals the supremum of the average.

The bound $\frac{5}{2}$ is Theorem 5.2 combined with monotonicity. For the other upper bound, take the umbrella of Fig. 4(b): unit vectors $u_i = (\cos\gamma, \sin\gamma\cos\frac{2\pi i}{5}, \sin\gamma\sin\frac{2\pi i}{5})$ with $\cos^2\gamma = 1/\sqrt{5}$, which satisfy $u_i \perp u_{i+2}$. For a pure state, $\operatorname{Tr}|u\rangle\langle u|^\alpha\sigma^{1-\alpha} = \langle u|\sigma^{1-\alpha}|u\rangle$. As faithful centers approach $|c\rangle\langle c|$ with $c = (1,0,0)$, this tends to $|\langle c, u_i\rangle|^2 = 5^{-1/2}$ because $1 - \alpha > 0$. The radius therefore tends to $\frac{1}{1-\alpha}\log\sqrt{5} = \log 5^{p/2}$. □

5.3 A continuum of spectral points

Proof of Theorem B. The values on $[\frac{1}{2}, 2]$ are Theorem 5.2, and the bounds on $[0, \frac{1}{2}]$ are Theorem 5.5. By Propositions 3.3 and 3.4, $\alpha \mapsto \vartheta_\alpha(C_5)$ is continuous and nondecreasing on $[0, \frac{1}{2}]$. It runs from $\vartheta(C_5) = \sqrt{5}$ to $\frac{5}{2}$, so by the intermediate value theorem its image is $[\sqrt{5}, \frac{5}{2}]$. Parameters with different values on C_5 are different functions, and each of them lies in $X(\mathcal{G}, \le_*)$ by Theorem A. □

The lower bound in (4) exceeds $\sqrt{5}$ as soon as $\alpha > 0$. So no ϑ_α with $\alpha > 0$ equals ϑ. Points of the *ordinary* spectrum whose values on C_5 fill $[\sqrt{5}, \frac{5}{2}]$ were already available through Vrana's interpolation (Section 1.4); the new content of Theorem B is that the points ϑ_α lie in $X(\mathcal{G}, \le_*)$.

Remark 5.6 (an explicit ladder)**.** Readers who want explicit separations without the intermediate value theorem can use $\alpha_k = 1/(8^{k+1}+1)$, $k \ge 0$. Then $p_k - 1 = 8^{-(k+1)}$, and the bounds (4) give $\vartheta_{\alpha_{k+1}}(C_5) \le \sqrt{5}\cdot 5^{(p_k-1)/16} < \sqrt{5}\,(\sqrt{5}/2)^{p_k-1} \le \vartheta_{\alpha_k}(C_5)$. The strict inequality is $5^{1/16} < \sqrt{5}/2$, that is, $2^{16} = 65536 < 78125 = 5^7$. Together with $\vartheta_{\alpha_0}(C_5) \le 5^{9/16} < \frac{5}{2}$, this shows that $\vartheta, \vartheta_{\alpha_0}, \vartheta_{\alpha_1}, \ldots, \vartheta_{1/2}$ are pairwise distinct.

5.4 The Duan–Winter chain

Duan and Winter prove the chain (1) [DW16, Eq. (54)]. They show that its last link can be strict, and they conjecture that on the pentagon every term except the first equals $\log\frac{5}{2}$. They also ask whether the two middle links can ever be strict. We can now answer all of this.

Since the corollaries below answer their questions, we quote the relevant passages. The quotations, and the section, theorem, proposition and lemma numbers we cite from [DW16], are from arXiv:1409.3426v3, where the chain is numbered (53) rather than (54). There a graph G comes with $S_G = \operatorname{span}\{|i\rangle\langle j| : i \sim j\}$, where "$i \sim j$ means confusability: $i = j$ or $\{i, j\} \in E$

is an edge of the graph". For a cq-graph K the condition $K^\dagger K < S_G$ says that the channel is compatible with G in the sense of Definition 2.1, and $G \times H$ denotes their strong product, our $G \boxtimes H$. Section V defines

> $\Sigma(G) := \inf\{\Sigma(K) : K \text{ cq-graph with } K^\dagger K < S_G\}$, and the asymptotic simulation cost (regularization) $S_{0,\mathrm{NS}}(G) := \lim_{n\to\infty} \frac{1}{n} \log \Sigma(G^n)$, where $G^n = G \times \cdots \times G$ denotes the n-fold strong graph product. The latter limit exists and equals the infimum because evidently $\log \Sigma(G \times H) \leq \log \Sigma(G) + \log \Sigma(H)$.

Their Lemma 23 reads: "For any two graphs G and H, $C_{\min}(G \times H) = C_{\min}(G) + C_{\min}(H)$." They derive the chain from it as a corollary and continue:

> Note that it may be true that $S_{0,\mathrm{NS}}(G) = \log \Sigma(G)$, but to prove this we would need to show the additivity relation $\log \Sigma(G \times H) = \log \Sigma(G) + \log \Sigma(H)$, which remains unknown.

After an example in which the last link is strict, they write:

> We think that also the leftmost inequality in (53) can be strict. But although the pentagon $G = C_5$ seems to be a candidate, for which we conjecture (based on some ad hoc calculations) that $C_{\min}(G) = S_{0,\mathrm{NS}}(G) = \log \Sigma(G) = \log \alpha^*(G) = \log \frac{5}{2}$, whereas $\log \vartheta(G) = \frac{1}{2} \log 5$, a rigorous proof of this has so far eluded us. Whether the other two inequalities can be strict remains an open question.

Their Proposition 18 does not settle the middle link. It states that $\Sigma(K_1 \otimes K_2) = \Sigma(K_1)\Sigma(K_2)$ for cq-graphs K_1, K_2, which concerns two fixed cq-graphs, that is, two fixed families of output supports. The graph quantity $\Sigma(G \boxtimes H)$ is an infimum over all channels compatible with $G \boxtimes H$, and most of them are not of the form $K \otimes L$. So Proposition 18 gives only $\Sigma(G \boxtimes H) \leq \Sigma(G)\Sigma(H)$, the inequality quoted above. The reverse inequality needs a bound for non-product channels. That is the additivity they list as unknown, and Theorem 4.1 at order two supplies it (Remark 4.2). Operationally, $S_{0,\mathrm{NS}}(G)$ is the cost rate of exact no-signalling-assisted simulation when, at each block length n, one may choose any channel compatible with $G^{\boxtimes n}$. Such a channel may correlate the n uses. Corollary 5.8 says that this freedom does not lower the rate below the single-letter value $\log \Sigma(G)$.

Corollary 5.7 (the pentagon conjecture)**.** $\log \sqrt{5} < C_{\min}(C_5) = S_{0,\mathrm{NS}}(C_5) = \log \Sigma(C_5) = \log \alpha^*(C_5) = \log \frac{5}{2}$.

Proof. Theorem 5.2 and Proposition 3.3 give $C_{\min}(C_5) = \log \Sigma(C_5) = \log \frac{5}{2}$, and (1) squeezes $S_{0,\mathrm{NS}}(C_5)$ between them. The fractional packing number of C_5 is $\frac{5}{2}$ [Sha56] and $\vartheta(C_5) = \sqrt{5}$ [Lov79]. □

The values $\vartheta(C_5) = \sqrt{5}$ and $\alpha^*(C_5) = \frac{5}{2}$ are classical, and the upper bounds on $C_{\min}(C_5)$, $S_{0,\mathrm{NS}}(C_5)$ and $\log \Sigma(C_5)$ are Duan and Winter's chain. The new ingredient is the lower bound $C_{\min}(C_5) \geq \log \frac{5}{2}$. This proof uses only Lemma 5.1, the projection formula, the radius formula for the Holevo capacity and monotonicity in α. The product theorem is not needed for the pentagon, but it is needed for the general statement.

Corollary 5.8 (the middle links)**.** *For every graph G, $S_{0,\mathrm{NS}}(G) = \log \Sigma(G)$. There is a graph with $C_{\min}(G) < S_{0,\mathrm{NS}}(G)$.*

Proof. The first statement is Remark 4.2. The second is Theorem C, since $C_{\min}(G_{32}) < \log \Sigma(G_{32}) = S_{0,\mathrm{NS}}(G_{32})$. □

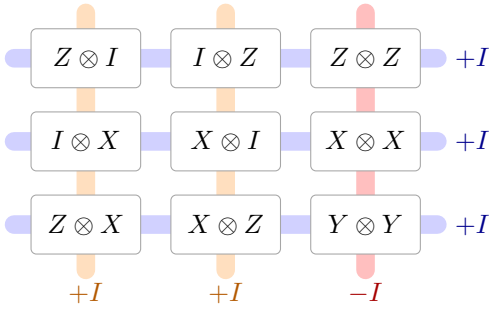


Figure 5: The Mermin–Peres magic square. Each row and each column is a *context*: its three observables commute, and their product is the operator shown at its end. A vertex of the seed graph F is a context together with a joint eigenvalue pattern (three signs with the prescribed product). That gives four vertices per context and 24 in all.

For a fixed cq-graph, strict inequalities of this kind were known: for the cq-graph K of two non-orthogonal pure states, Duan and Winter show $C_{0,\mathrm{NS}}(K) < C_{\min E}(K) < S_{0,\mathrm{NS}}(K)$ [DW16, Section IV.D and Fig. 4]. Corollary 5.8 separates the graph-level quantities, each of which is an infimum over all channels compatible with G.

For non-commutative bipartite graphs K, Duan and Winter also asked whether $\Sigma(K)$ can be strictly submultiplicative [DW16, Section VII]. Wang and Duan answered this with explicit qubit–qutrit examples, for which $\Sigma(K \otimes K) < \Sigma(K)^2$ [WD16]. For the graph-level quantity $\Sigma(G)$, with the optimization over compatible channels built in, Theorem 4.1 shows that no such gap occurs.

6 Order one versus order two

6.1 Where to look

Lemma 3.2 tells us where *not* to look. If some flat witness is optimal at order one, then orders one and two agree. This is what happens on the pentagon, and on every graph whose best order-one channel is flat. To separate the endpoints we need a graph whose good channels are genuinely quantum. Contextuality scenarios are a natural source, since they come with rigid noncommutative representations. We take the most familiar one, the Mermin–Peres magic square [Mer90, Per90], and then add weights by cloning vertices. This section uses Section 3 but not Theorem 4.1.

6.2 The graph

Index the nine cells of a 3×3 array by $(i, j) \in \{0, 1, 2\}^2$, and fill them with the two-qubit observables of Fig. 5. The six contexts are the three rows and the three columns. A vertex of the *seed graph* F is a context together with signs for its three cells: signs with product $+1$ on rows and on the first two columns, and with product -1 on the last column. Two distinct vertices are non-adjacent if they belong to the same context, or if they give opposite signs to a shared cell. That makes 24 vertices, each non-adjacent to exactly nine others (216 ordered non-adjacent pairs).

For a context whose first two observables are A and B, the vertex with first two signs (s, t) is represented by the joint eigenprojection

$$Q_{A,B;s,t} = \tfrac{1}{4}(I + sA)(I + tB),$$

a rank-one projection on $\mathbb{C}^2 \otimes \mathbb{C}^2$. These projections represent F. In fact $Q_v Q_w = 0$ holds exactly for the non-adjacent pairs, with no accidental orthogonality. The ranges of these 24 projections are the 24 vectors, in six bases of $\mathbb{C}^4$, that Cubitt, Leung, Matthews and Winter took from Peres's proof of the Kochen–Specker theorem to show that entanglement can increase the one-shot zero-error capacity of a classical channel [CLMW10, Fig. 2]. The six contexts give their six bases, and the confusability graph of their channel is the orthogonality graph of the 24 vectors, the complement of F. Let vertex 0 be the first-row vertex with signs $(+,+,+)$, so $Q_0 = |00\rangle\langle 00|$, and sort the vertices by overlap $j_v = 4 \operatorname{Tr} Q_v Q_0 \in \{0,1,2,4\}$. The four classes have sizes $9, 8, 6, 1$. Class 0 consists of the nine non-neighbours of vertex 0, and class 4 is vertex 0 itself.

Weights enter through cloning. For a profile $k = (k_0, k_1, k_2, k_4)$ of nonnegative integers, the *clone graph* $F[k]$ replaces every class-j vertex by k_j pairwise non-adjacent copies (deleting it if $k_j = 0$). Each copy inherits its parent's adjacencies to all other vertices. Our main example is

$$G_{32} = F[(2,1,1,0)], \qquad |V(G_{32})| = 2 \cdot 9 + 8 + 6 = 32.$$

6.3 Two weights and a bridge

Two of the weights of Section 3 matter here. The Gibbs weight $w_\sigma(P) = Z_0(\sigma, P)$ belongs to order one, and the harmonic weight $h_\sigma(P) = Z_{-1}(\sigma, P) = \operatorname{Tr} \sigma_{/P}$ belongs to order two. For a profile k and a seed vertex v, write $k_v = k_{j_v}$ for the number of copies of v in $F[k]$. Let $t_W(F[k])$ be the supremum of $\min_{v:\, k_v > 0} w_\sigma(P_v)/k_v$ over representations (P_v) of the seed F and faithful states σ. Define $t_H(F[k])$ in the same way with h_σ. Deleted vertices, those with $k_v = 0$, may be represented by $P_v = 0$.

Proposition 6.1 (bridge)**.** *For every profile k,*

$$2^{C_{\min}(F[k])} \le \frac{1}{t_W(F[k])} \qquad \text{and} \qquad \Sigma(F[k]) \ge \frac{1}{t_H(F[k])}.$$

Proof. Upper bound on $C_{\min}$. Start from a representation (P_v) of F and a state σ. Let $A = \prod_{v:k_v>0} \{1, \dots, k_v\}$, work on $\bigoplus_{f \in A} \mathcal{H}$, and put

$$\sigma' = \frac{1}{|A|} \bigoplus_{f \in A} \sigma, \qquad P'_{v,a} = \bigoplus_{f \in A:\, f(v) = a} P_v$$

(zero in the other summands). Copies of one vertex live in disjoint sets of summands, so they are orthogonal, and non-adjacent seed vertices stay orthogonal. So $(P'_{v,a})$ represents $F[k]$. Functional calculus on the direct sum gives $w_{\sigma'}(P'_{v,a}) = w_\sigma(P_v)/k_v$. Now apply the projection formula (11) at $\alpha = 1$ together with $\vartheta_1 = 2^{C_{\min}}$.

Lower bound on Σ. Take a compatible family $(\rho_{v,a})$ for $F[k]$ and $T \ge \rho_{v,a}$, put $t = \operatorname{Tr} T$ and $\sigma = T/t$ (regularized if needed), and let $P_{v,a}$ be the supports. As in Proposition 3.3, $h_\sigma(P_{v,a}) \ge 1/t$. Copies of one vertex are non-adjacent, so $P_v = \sum_a P_{v,a}$ is a projection, and (P_v) (with $P_v = 0$ for deleted v) represents F. The harmonic weight is superadditive over orthogonal blocks: for a positive definite block matrix M, each diagonal block of M^{-1} dominates the inverse of the corresponding block of M. Applied to $M = P_v \sigma^{-1} P_v$ this gives $h_\sigma(P_v) \ge \sum_a h_\sigma(P_{v,a}) \ge k_v/t$. Hence $t_H(F[k]) \ge 1/t$. □

The upper bound on $C_{\min}$ needs one good witness. The lower bound on Σ needs an upper bound on t_H valid for *every* representation, in every dimension. That is the job of the next subsection.

graph	profile (k_0, k_1, k_2, k_4)	vertices	$2^{C_{\min}} \le$	$\Sigma \ge$	certified gap $\ge$ (bits)
G_{32}	$(2, 1, 1, 0)$	32	$6.186983\ldots$	$6.194232\ldots$	0.001689
G_{44}	$(1, 2, 3, 1)$	44	$8.949075\ldots$	$8.962394\ldots$	0.002145
G_{72}	$(4, 3, 2, 0)$	72	$105/8$	$13.191800\ldots$	0.007324

Table 1: Certified bounds for three clone graphs. Upper bounds on $2^{C_{\min}}$ come from explicit rational Gibbs witnesses: a direct sum of the magic-square representation, with center proportional to $\mathrm{diag}(A^4, B^4, B^4, B^4)$ and $(A, B) = (41, 56)$, $(56, 51)$, $(3, 4)$, and one-dimensional clique representations (see the text). Lower bounds on Σ come from rational dual certificates for the moment relaxation. The gap is $\log_2$ of the ratio of the two bounds, rounded down.

6.4 Harmonic weights as moments

Let (P_v) represent F on $\mathcal{H}$ and let σ be faithful. Fix an orthonormal basis (e_i) of $\mathcal{H}$, write $\mathrm{vec}(M) = \sum_{i,j} \langle e_i, M e_j \rangle\, e_i \otimes e_j$, and take all transposes in this basis. Put $\Omega = \mathrm{vec}(\sqrt{\sigma}) \in \mathcal{H} \otimes \mathcal{H}$, let F_v be the orthogonal projection onto $\sigma^{-1/2} \operatorname{ran} P_v$, and let $E_v = F_v^{\mathsf{T}}$.

Lemma 6.2 (harmonic weight as a moment)**.** *For every v,*

$$(P_v \otimes E_v)\Omega = (I \otimes E_v)\Omega \qquad \text{and} \qquad \langle \Omega, (I \otimes E_v)\Omega \rangle = h_\sigma(P_v).$$

Proof. With $(A \otimes B)\,\mathrm{vec}(M) = \mathrm{vec}(AMB^{\mathsf{T}})$, the first identity says $P_v \sqrt{\sigma} F_v = \sqrt{\sigma} F_v$. This holds because $\sqrt{\sigma} F_v$ has range $\operatorname{ran} P_v$. For the second, $\sqrt{\sigma} F_v \sqrt{\sigma}$ lies below σ and has range in $\operatorname{ran} P_v$. If $0 \le A \le \sigma$ has range in $\operatorname{ran} P_v$, then $\sigma^{-1/2} A \sigma^{-1/2}$ is a contraction supported in $\operatorname{ran} F_v$, hence at most F_v. So $\sqrt{\sigma} F_v \sqrt{\sigma} = \sigma_{/P_v}$, and its trace is the moment. □

So harmonic weights are expectations in the vector Ω of two commuting families of projections, $P_v \otimes I$ and $I \otimes E_w$. The first family obeys the orthogonality relations of F. The second obeys no relations of its own, but it is tied to the first by the identity of Lemma 6.2. This is exactly the setting of noncommutative moment relaxations [NPA08, Mgh26]. We form the Gram matrix of the 409 vectors

$$\Omega, \quad (P_a \otimes I)\Omega, \quad (I \otimes E_b)\Omega, \quad (P_a \otimes E_b)\Omega \;\; (a = b \text{ or } a \sim b).$$

The pairs with $a \not\sim b$ are left out because those vectors vanish: $(P_a \otimes E_b)\Omega = (P_a P_b \otimes E_b)\Omega = 0$. The Gram matrix is positive semidefinite, its corner entry is $\operatorname{Tr} \sigma = 1$, and the entry $\langle \Omega, (I \otimes E_v)\Omega \rangle$ is $h_\sigma(P_v)$. Many entries coincide or vanish in every model. We record these coincidences in a table of 1009 moment keys, using four rewriting rules together with symmetry (Appendix A). Maximizing t subject to positivity, the corner normalization and $h_v \ge t k_v$ gives an upper bound on $t_H(F[k])$ in every dimension. A rational dual solution then certifies a rational upper bound.

6.5 Certified values

Table 1 lists the certified bounds. For G_{32} the certified rational bounds, not the exact values of t_W and t_H, are

$$t_W(G_{32}) \;\ge\; \frac{5225365592240894}{32329248967670751} \;>\; \frac{15147154828811}{93824992214030} \;\ge\; t_H(G_{32}). \tag{14}$$

For G_{44} and G_{72} the certified harmonic bounds are $t_H \le 31406288103814/281475531618259$ and $t_H \le 21337116511217/281474976708840$.

Proof of Theorem C. Combine (14) with Proposition 6.1. The reciprocals of the two rationals, 6.186983… and 6.194232…, are the bounds in Theorem C. □

The Gibbs witnesses. Each upper bound on $2^{C_{\min}}$ comes from one representation of the seed on $\mathbb{C}^4 \oplus \mathbb{C}^m$. Vertex v gets $Q_v \oplus \bigoplus_{C \ni v} |C\rangle\langle C|$, where C runs over m weighted cliques of F, each with its own basis vector $|C\rangle$ ($m = 18$, 10 and 9 cliques, all of size five, for G_{32}, G_{44} and G_{72}). Non-adjacent vertices share no clique, so this is a representation. The center is

$$\sigma = \frac{w}{A^4 + 3B^4} \operatorname{diag}(A^4, B^4, B^4, B^4) \;\oplus\; \bigoplus_C w_C |C\rangle\langle C|$$

in the basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ of $\mathbb{C}^4$, with $(A, B) = (41, 56)$, $(56, 51)$ and $(3, 4)$ and positive rational weights w and w_C. If σ is diagonal in an orthonormal basis $(|i\rangle)$ with eigenvalues λ_i, the Gibbs weight of a rank-one projection Q is $Z_0(\sigma, Q) = \prod_i \lambda_i^{\langle i|Q|i\rangle}$. Since $4\langle 00|Q_v|00\rangle = j_v$, the magic-square block contributes the rational number $wA^{j_v}B^{4-j_v}/(A^4 + 3B^4)$ to $w_\sigma(P_v)$, each clique containing v contributes w_C, and Gibbs weights add over orthogonal blocks. By homogeneity, $t_W(F[k]) \geq \min_{k_v > 0} w_\sigma(P_v)/(k_v \operatorname{Tr}\sigma)$, and this minimum is the certified value, for G_{32} the left rational in (14). The stored weights have $\operatorname{Tr}\sigma = 999999999/10^9$ for G_{32} and G_{44} and $\operatorname{Tr}\sigma = 1$ for G_{72}; for G_{32} the magic-square block has weight $w = 345767129/10^9$.

Where the separations sit. In all three graphs the Lovász number equals the independence number, and the fractional packing number is larger:

$$\vartheta = 6,\ \alpha^* = \tfrac{32}{5}\ (G_{32}); \qquad \vartheta = 8,\ \alpha^* = \tfrac{37}{4}\ (G_{44}); \qquad \vartheta = 12,\ \alpha^* = \tfrac{72}{5}\ (G_{72}).$$

So the certified values of Table 1 lie strictly inside the interval $[\vartheta, \alpha^*]$ that the Duan–Winter chain allows, and the gap between orders one and two is a small part of it. The exact positivity of each dual matrix Z is proved in integers (Appendix A); in floating point its smallest eigenvalue is about 2.0×10^{-5}, 6.2×10^{-7} and 2.0×10^{-5} for G_{32}, G_{44} and G_{72}. These margins only guided the rounding. The values of ϑ and α^* are checked exactly by the verification script `theorem_c_context.py` (see Data and code availability): an independent set and a rational dual matrix for ϑ, and a rational packing and clique cover of equal value for α^*.

7 Order two versus the projective rank

7.1 The flat bound

Let ξ_f be the projective rank of Mančinska and Roberson [MR16], the infimum of d/r over d/r-representations: rank-r projections on $\mathbb{C}^d$, one for each vertex, with adjacent vertices orthogonal. A d/r-representation (P_x) of $\overline{G}$ is a representation of G in the sense of Definition 2.1, and the flat witness $\rho_x = P_x/r$, $\sigma = I/d$ has radius $\log(d/r)$ at every order (Lemma 3.2). Hence

$$\vartheta_\alpha(G) \leq \xi_f(\overline{G}) \qquad (0 \leq \alpha \leq 2), \tag{15}$$

and in particular $\Sigma(G) \leq \xi_f(\overline{G})$. This is at least as strong as the bound α^* of Proposition 3.3(iii), since $\xi_f(\overline{G}) \leq \alpha^*(G)$ [LZ21]. Equality $\Sigma(G) = \xi_f(\overline{G})$ holds whenever $\vartheta(G) = \alpha^*(G)$. It holds on C_5, where both sides equal $\frac{5}{2}$ (by Theorem 5.2, (15) and $\alpha^*(C_5) = \frac{5}{2}$), and on the complement of the Hadamard graph H_n in Duan and Winter's example, where both equal n [DW16, Section V]: there $\Sigma = n$, and the vectors of H_n give an $n/1$-representation of H_n. Theorem D shows that it can fail.

7.2 The order-two bridge is exact

For the clone graphs of Section 6, the lower bound on Σ in Proposition 6.1 is an equality.

Proposition 7.1. *Let k be a profile with at least one positive entry. If (P_v) and σ are as in the definition of $t_H(F[k])$, then*

$$\Sigma(F[k]) \le \max_{v:\, k_v>0} \frac{k_v}{h_\sigma(P_v)}.$$

Consequently $\Sigma(F[k]) = 1/t_H(F[k])$.

Proof. Let L be a common multiple of the positive k_v. On $\mathcal{H} \otimes \mathbb{C}^L$ put $\sigma' = \sigma \otimes I/L$. For each v with $k_v > 0$, split the standard basis of $\mathbb{C}^L$ into k_v consecutive blocks of L/k_v vectors, let $\Pi_{v,a}$ project onto the a-th block, and put $P'_{v,a} = P_v \otimes \Pi_{v,a}$. Copies of one vertex get orthogonal blocks, and non-adjacent seed vertices have orthogonal P_v, so $(P'_{v,a})$ represents $F[k]$. Since $P'_{v,a}\sigma'^{-1}P'_{v,a} = P_v\sigma^{-1}P_v \otimes L\,\Pi_{v,a}$, the shorted operator factorizes, $\sigma'_{/P'_{v,a}} = \sigma_{/P_v} \otimes \Pi_{v,a}/L$, and

$$h_{\sigma'}(P'_{v,a}) = h_\sigma(P_v)\,\frac{\operatorname{Tr}\Pi_{v,a}}{L} = \frac{h_\sigma(P_v)}{k_v}.$$

The projection formula (11) at $\alpha = 2$, with $\vartheta_2 = \Sigma$ (Proposition 3.3), gives the displayed bound. Taking the supremum over (P_v) and σ gives $\Sigma(F[k]) \le 1/t_H(F[k])$, and Proposition 6.1 gives the reverse inequality. □

7.3 A rank inequality for the magic square

In this subsection $\tau = \operatorname{Tr}/d$ is the normalized trace on $\mathbb{C}^d$. We use three facts about rank: it is subadditive, $\operatorname{rank}(XY) \le \min(\operatorname{rank} X, \operatorname{rank} Y)$, and multiplying by an invertible operator does not change it. Number the six contexts of the seed F as $c = 0, \dots, 5$: the rows are $0, 1, 2$ from the top and the columns are $3, 4, 5$ from the left (Fig. 5). As before, vertex 0 is the row-0 vertex with signs $(+,+,+)$.

Lemma 7.2 (anticommutators bound traces)**.** *If X and Y are Hermitian unitaries on $\mathbb{C}^d$, then $|\operatorname{Tr} X| \le \operatorname{rank}(XY + YX)$.*

Proof. The operator $X + YXY = (XY + YX)Y$ has the rank of $XY + YX$, norm at most 2 and trace $2\operatorname{Tr} X$. The trace of an operator is at most its rank times its norm in absolute value. □

Lemma 7.3 (rank inequality)**.** *Let (P_v) be projections on $\mathbb{C}^d$, one for each vertex of F, with $P_vP_w = 0$ whenever $v \ne w$ are non-adjacent; zero projections are allowed. For each context c put $S_c = \sum_{v\in c} P_v$. Then*

$$4\tau(P_0) + 7\tau(S_0) + 11\tau(S_1) + 9\tau(S_2) + 11\tau(S_3) + 11\tau(S_4) + 9\tau(S_5) \le 59, \tag{16}$$

with equality for the magic-square representation (Q_v).

The four vertices of a context are pairwise non-adjacent, so each S_c is a projection, and $\delta_c = 1 - \tau(S_c)$ is the normalized rank of its defect $I - S_c$. The coefficients of the contexts sum to 58, so (16) says

$$4\tau(P_0) \le 1 + 7\delta_0 + 11\delta_1 + 9\delta_2 + 11\delta_3 + 11\delta_4 + 9\delta_5.$$

If every context is complete, this is $\tau(P_0) \le \frac{1}{4}$: the vertex-0 outcome takes at most a quarter of the space. The lemma makes this robust, with a penalty linear in the defects. For (Q_v) every context is complete and $\tau(Q_0) = \frac{1}{4}$, which gives equality.

Proof. Completion. In each context, add the defect $I - S_c$ to one of its four projections; in row 0, add it to P_0. Each context now carries four orthogonal projections summing to I, labelled by its sign patterns. For a cell (i, j), let U_{ij} be the sum of the completed projections of row i, each weighted by the sign it gives to the cell, and define V_{ij} in the same way from column j. These are Hermitian unitaries. The three U_{ij} in a row commute and multiply to I. The three V_{ij} in a column commute and multiply to I, except in the last column, where they multiply to $-I$.

Rows and columns agree off the defects. Fix a cell (i, j). Let $U_\pm$ be the sum of the original projections of row i that give the cell the sign $\pm$, and $V_\pm$ the same for column j. Two vertices that give a shared cell opposite signs are non-adjacent, so $U_+V_- = U_-V_+ = 0$. Let z lie in the kernels of both defects, $I - S_i$ and $I - S_{3+j}$. Then $U_+z + U_-z = z = V_+z + V_-z$. Put $x = U_+z$ and $y = V_+z$. Then $\langle x, z - y\rangle = \langle z, U_+V_-z\rangle = 0$ and $\langle z - x, y\rangle = \langle z, U_-V_+z\rangle = 0$, so

$$\|x\|^2 = \langle x, z\rangle = \langle x, y\rangle = \langle z, y\rangle = \|y\|^2.$$

Hence $\|x - y\|^2 = 0$ and $U_{ij}z = 2x - z = 2y - z = V_{ij}z$. The two kernels meet in a subspace of codimension at most $d(\delta_i + \delta_{3+j})$, so

$$\eta_{ij} := \operatorname{rank}(U_{ij} - V_{ij})/d \le \delta_i + \delta_{3+j}.$$

Put $e_j = \sum_i \eta_{ij}$ and $\varepsilon = e_0 + e_1 + e_2$. Summing over the nine cells gives $\varepsilon \le 3\delta$, where $\delta = \sum_c \delta_c$.

Relations. Write A, B for U_{00}, U_{01}, write D, E for U_{10}, U_{11}, and write H, J for U_{20}, U_{21}. The U's in a row commute and multiply to I, so the last cells of the three rows carry AB, DE and HJ. In column 0, $V_{20} = V_{00}V_{10} = V_{10}V_{00}$. Replacing each V_{i0} by U_{i0} costs at most η_{i0} in normalized rank, so

$$\operatorname{rank}(H - AD)/d, \quad \operatorname{rank}(H - DA)/d \;\le\; e_0,$$

and in the same way $\operatorname{rank}(J - BE)/d$ and $\operatorname{rank}(J - EB)/d$ are at most e_1. In the last column $V_{22} = -V_{02}V_{12} = -V_{12}V_{02}$, so

$$\operatorname{rank}(HJ + AB\,DE)/d, \quad \operatorname{rank}(HJ + DE\,AB)/d \;\le\; e_2.$$

Since $HJ - AD\,BE = (H - AD)J + AD(J - BE)$, comparing HJ with $AD\,BE$ and with $-AB\,DE$ gives $\operatorname{rank}(ADBE + ABDE) \le d\varepsilon$. Now $ADBE + ABDE = A(DB + BD)E$, and A and E are unitary, so $\operatorname{rank}(BD + DB) \le d\varepsilon$. In the same way, $DAEB + DEAB = D(AE + EA)B$ gives $\operatorname{rank}(AE + EA) \le d\varepsilon$. Finally, A and D lie in column 0, where V_{00} and V_{10} commute, so $\operatorname{rank}(AD - DA) \le 2d(\eta_{00} + \eta_{10}) \le du_0$ with $u_0 = 2(\delta_0 + \delta_1 + 2\delta_3)$. Likewise $\operatorname{rank}(BE - EB) \le du_1$ with $u_1 = 2(\delta_0 + \delta_1 + 2\delta_4)$.

Traces. By Lemma 7.2, $|\tau(B)| \le \varepsilon$ and $|\tau(A)| \le \varepsilon$. Since A and B commute, AB is a Hermitian unitary. The identities $\{AB, D\} = A\{B, D\} + [D, A]B$ and $\{AB, E\} = A[B, E] + \{A, E\}B$, together with Lemma 7.2, give

$$|\tau(AB)| \le \varepsilon + \min(u_0, u_1) \le \varepsilon + 2(\delta_0 + \delta_1 + \delta_3 + \delta_4).$$

The completed row-0 projection with signs $(+, +, +)$ is $\frac{1}{4}(I + A)(I + B)$, and it equals $P_0 + (I - S_0)$. Taking normalized traces,

$$4\tau(P_0) + 4\delta_0 = 1 + \tau(A) + \tau(B) + \tau(AB) \le 1 + 3\varepsilon + 2(\delta_0 + \delta_1 + \delta_3 + \delta_4).$$

With $\varepsilon \le 3\delta$ this is $4\tau(P_0) \le 1 + 7\delta_0 + 11\delta_1 + 9\delta_2 + 11\delta_3 + 11\delta_4 + 9\delta_5$, which is (16). □

The relations used in the proof are robust forms of the anticommutation relations of the magic square: in an exact operator solution, cell observables in different rows and columns anticommute [Ji13, Lemma 6].

7.4 Proof of Theorem D

Let $G_{1440} = F[(59, 60, 61, 63)]$, so $k_0 = 59$, $k_1 = 60$, $k_2 = 61$ and $k_4 = 63$; it has $9 \cdot 59 + 8 \cdot 60 + 6 \cdot 61 + 63 = 1440$ vertices. In the seed, row 0 consists of vertex 0 and three class-0 vertices, rows 1 and 2 consist of class-1 vertices, and each column consists of two class-0 and two class-2 vertices. So every context of G_{1440} has $63 + 3 \cdot 59 = 4 \cdot 60 = 2 \cdot 61 + 2 \cdot 59 = 240$ vertices.

Lower bound. Let $(R_{v,a})$ be a d/r-representation of $\overline{G_{1440}}$, where (v, a) is the a-th copy of the seed vertex v. Distinct copies of one vertex are non-adjacent in G_{1440}, so $P_v = \sum_a R_{v,a}$ is a projection of rank $k_v r$. If $v \neq w$ are non-adjacent in F, all their copies are non-adjacent, so $P_v P_w = 0$. Lemma 7.3 applies with $\tau(P_0) = 63r/d$ and $\tau(S_c) = 240r/d$ for every context, and gives

$$\frac{r}{d}\,(4 \cdot 63 + 58 \cdot 240) = \frac{r}{d} \cdot 14172 \leq 59.$$

So $d/r \geq 14172/59$ for every d/r-representation, and $\xi_f(\overline{G_{1440}}) \geq 14172/59$.

Upper bound. Take the magic-square representation (Q_v) and, in the basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, the faithful state $\sigma = \operatorname{diag}(31, 29, 29, 29)/118$. For a rank-one projection $Q = |\psi\rangle\langle\psi|$ we have $h_\sigma(Q) = 1/\langle\psi, \sigma^{-1}\psi\rangle$. Since $|\langle 00|\psi_v\rangle|^2 = j_v/4$ for a unit vector ψ_v in $\operatorname{ran} Q_v$,

$$\frac{1}{h_\sigma(Q_v)} = 118\Big(\frac{j_v}{4 \cdot 31} + \frac{4 - j_v}{4 \cdot 29}\Big) = \frac{59\,(62 - j_v)}{899}.$$

Hence $k_v/h_\sigma(Q_v) = 59\,k_v(62 - j_v)/899$, and $k_j(62 - j)$ equals 3658, 3660, 3660 and 3654 for $j = 0, 1, 2, 4$. By Proposition 7.1,

$$\Sigma(G_{1440}) \leq \frac{59 \cdot 3660}{899} = \frac{215940}{899}.$$

Explicitly, with $L = \operatorname{lcm}(59, 60, 61, 63)$ and $\Pi_{v,a}$ as in the proof of Proposition 7.1, the states $\rho_{v,a} = (k_v/L)\,Q_v \otimes \Pi_{v,a}$ are compatible with G_{1440}, and all of them lie below $T = (215940/899)\,\sigma \otimes I/L$, whose trace is $215940/899$.

Separation. Finally, $14172/59 - 215940/899 = 168/53041 > 0$. This proves Theorem D. The script `projective_rank_separation.py` re-checks the class structure of the contexts, the harmonic weights and this arithmetic in exact rational arithmetic (see Data and code availability).

Where the profile comes from. Both bounds are linear in the profile. For every profile k with positive entries, the same argument gives

$$\xi_f(\overline{F[k]}) \geq \frac{83k_0 + 80k_1 + 62k_2 + 11k_4}{59}, \qquad \Sigma(F[k]) \leq \max_j \frac{k_j}{h_j(a)},$$

where $h_j(a)$ is the harmonic weight of a class-j projection Q_v at the center $\operatorname{diag}(a, b, b, b)$, $a + 3b = 1$. At the central state every harmonic weight is $\frac{1}{4}$, and for profiles proportional to the weights the two bounds coincide. Tilting the center toward $|00\rangle$, with $a = \frac{1}{4} + 3\epsilon$ and $b = \frac{1}{4} - \epsilon$, changes the weights of the classes $j = 0, 1, 2, 4$ by $-\epsilon$, 0, ϵ and 3ϵ to first order. For profiles proportional to the tilted weights, the lower bound then exceeds the upper bound by a relative $12\epsilon/59$ to first order. So the separation appears at first order in the tilt. The gain is small, and the clone counts must follow the weights closely. For G_{1440} the relative gap is 1.3×10^{-5}; its profile is the smallest we found.

8 Open problems

Is ϑ the least point? Every ϑ_α dominates ϑ, so this paper leaves the conjecture $\Theta_* = \vartheta$ where it was. A proof that ϑ is the minimum of $X(\mathcal{G}, \leq_*)$ would settle it. So would a single spectral point below ϑ on some graph, in the opposite direction.

Above order two. Remark 4.8 shows that our monotonicity argument breaks down above order two. Is some suitably defined ϑ_α with $\alpha > 2$ still in $X(\mathcal{G}, \leq_*)$? Flat witnesses show that any such extension is at most α^*. A natural guess for the $\alpha \to \infty$ endpoint is the complement of the projective rank. Flat representations give the corresponding upper bound, but we are missing a dimension-free argument for the other direction. By Theorem D, such an extension would have to keep growing beyond order two on G_{1440}.

The projective rank. Theorem D shows that $\Sigma(G)$ and $\xi_f(\overline{G})$ differ, so Theorem A does not place $\xi_f(\overline{G})$ in $X(\mathcal{G}, \leq_*)$. Whether it lies in $X(\mathcal{G}, \leq_*)$ remains open [LZ21, after Proposition 24]. For G_{1440}, each context is an independent set of 240 vertices, so $240 \leq \Sigma(G_{1440}) \leq \frac{215940}{899}$, while $\frac{14172}{59} \leq \xi_f(\overline{G_{1440}}) \leq \alpha^*(G_{1440})$. What are the exact values? Is Lemma 7.3 the best linear inequality of its kind for the seed, and do other contextuality scenarios give larger gaps? How small can a graph with $\Sigma(G) < \xi_f(\overline{G})$ be?

Other divergences. The proof of Theorem A uses three features of the Petz family. These are a closed form for the supported minimum with the tower property, the block structure used for unions and flagged mixtures, and data processing. For sandwiched divergences the answer is complete on the full data-processing range $[\frac{1}{2}, \infty]$: Proposition 3.5 gives the supported minimum $-\log Z_{(1-\beta)/\beta}$ and identifies the graph radii with the Petz family on $[0, 2]$. In particular, the orders $\frac{1}{2} \leq \beta < 1$ give precisely the Petz sub-one family, not additional spectral points. Do other quantum divergences with these three features yield spectral points outside this common family?

Convexity. In the ordinary spectrum, Vrana's probabilistic refinement makes the spectrum log-convex [Vra21]. If the same held for $\leq_*$, interpolating between ϑ and Σ would produce another continuum. The following is a heuristic, conditional on such a theory for $\leq_*$; it suggests that at least part of the Rényi family is not of this form. In the ordinary spectrum, vertex-transitive graphs are evaluated log-affinely along such interpolations [Vra22, Section 5]. If the same were true for $\leq_*$, an interpolant with value $\frac{5}{2}$ on C_5 would carry no weight on ϑ, so it would be Σ itself. Since $\vartheta_1(C_5) = \frac{5}{2}$ and $\vartheta_1 \neq \Sigma$ by Theorem C, ϑ_1 would then not be such an interpolant; we do not know whether the ϑ_α with $\alpha < \frac{1}{2}$ would be. Making any of this rigorous would need an entanglement-assisted theory of probabilistic refinements, which we do not have.

The pentagon curve. What is $\vartheta_\alpha(C_5)$ for $0 < \alpha < \frac{1}{2}$? The bounds in Fig. 1 do not meet. The flat witness in Fig. 4(a) is optimal on $[\frac{1}{2}, 2]$, and the umbrella in panel (b) attains the order-zero value. At intermediate orders, the gap between the displayed bounds does not by itself decide which representation or center is optimal; the exact graph value remains open.

Small separations. How small can a graph with $C_{\min} < \log \Sigma$ be? Thirty-two vertices suffice, and we have no reason to think this is minimal. Can the rational certificate behind Theorem C be replaced by a short analytic inequality, in the spirit of Lemma 5.1? And is the first level of the moment relaxation already exact for the three graphs of Table 1?

Non-commutative graphs. Duan, Severini and Winter's quantum Lovász number extends ϑ to non-commutative graphs [DSW13], and Li and Zuiddam place it in the entanglement-assisted spectrum of non-commutative graphs. There the analogue of $\Theta_* = \vartheta$ fails: Wang and Duan exhibit non-commutative graphs whose entanglement-assisted zero-error capacity is strictly below the quantum Lovász number [WD18], so that spectrum contains another point, although no explicit one is known [LZ21, Theorem 33 and the remark after it]. Does the whole family ϑ_α extend, and does the product theorem survive? A positive answer at order two would give a single-letter formula for graph-level no-signalling simulation over non-commutative graphs, in contrast with the fixed-K quantity $\Sigma(K)$, which can be strictly submultiplicative [WD16].

Data and code availability

The certificate data behind Theorem C and Table 1, the exact data for the values of ϑ and α^* quoted in Section 6, and the verification scripts named in the text are archived on Zenodo under the all-versions DOI 10.5281/zenodo.22983498, and are also available at `https://github.com/Apsiape/the-lovasz-number-is-not-alone`. They include an exact check of the arithmetic behind Theorem D. The archive documents the data format and pins the expected output of every script.

Reproducibility

The certificate data and their verification scripts accompany this paper (see Data and code availability). They are needed only for the finite rational claims of Section 6. The proof of Theorem D is complete by hand; the script `projective_rank_separation.py` re-checks its arithmetic. An additional deterministic matrix regression checks instances of the supported sandwiched minimum in Proposition 3.5, including the sub-one orders and endpoints. These finite numerical checks are not used as proof; the dimension-independent statements are proved in the text.

AI-assisted tools

Generative AI tools were used throughout the research and preparation of this manuscript, including for mathematical exploration, proof development and critique, literature organization, computational assistance, and drafting. The authors determined which results and arguments to include and take responsibility for the final manuscript. Exact computer-assisted claims are supported, where indicated, by accompanying deterministic certificates and verification scripts.

A The certificate

Words and moments. For words $L = a_1 \cdots a_m$ and $R = b_1 \cdots b_n$ in seed vertices, write $P_L = P_{a_1} \cdots P_{a_m}$, $E_R = E_{b_1} \cdots E_{b_n}$ and

$$m(L, R) = \langle \Omega, (P_L \otimes E_R)\Omega \rangle.$$

Every entry of the Gram matrix of Section 6 is such a moment. The following rules hold in every model.

(R1) Equal adjacent letters in L or in R merge, since $P_a^2 = P_a$ and $E_b^2 = E_b$.

(R2) If L contains two adjacent letters $a \neq b$ that are non-adjacent in F, then $m(L, R) = 0$, since $P_a P_b = 0$.

(R3) If R ends in y, then $m(L, R) = m(Ly, R)$, since $(I \otimes E_y)\Omega = (P_y \otimes E_y)\Omega$.

(R4) If R begins with y, then $m(L, R) = m(yL, R)$, by the adjoint of the same identity.

Reversing both words conjugates the moment, since $(P_L \otimes E_R)^* = P_{\bar{L}} \otimes E_{\bar{R}}$. Let $\mathcal{A}_0$ be the 48 automorphisms of F that fix vertex 0 (equivalently, that preserve the overlap classes). A relabelled representation is again a representation, and the constraints $h_v \geq t k_v$ are invariant because k is constant on classes. So we may average the Gram matrix over $\mathcal{A}_0$ and then take real parts. Both operations preserve positivity, and together they identify symmetry-related moments and reversed pairs. The verifier builds the key table from these rules only. So every identification in the table holds for the Gram matrix of every representation, in every dimension. The result has 1009 keys.

Deleted vertices. In G_{32} and G_{72} the class-4 vertex is deleted. It stays in the seed and is represented by $P_0 = 0$, and hence $E_0 = 0$. All four rules remain valid for zero projections, no radial constraint is imposed on deleted vertices, and the relaxation needs no change.

The relaxation and its dual. Let $\Gamma(y)$ be the real symmetric matrix with one variable y_κ per key, write 1 for the key of $m(\varnothing, \varnothing) = \operatorname{Tr}\sigma$, and let κ_v be the key of $m(\varnothing, v) = h_\sigma(P_v)$. Entries of the Gram matrix whose moment vanishes identically by the rules (R1)–(R4) are 0 in $\Gamma(y)$; they carry no key and no equation below, so the corresponding entries of Z are restricted only by $Z \succeq 0$. The primal problem is to maximize t subject to

$$\Gamma(y) \succeq 0, \qquad y_1 = 1, \qquad y_{\kappa_v} \geq t k_v \quad (k_v > 0).$$

A dual certificate consists of $Z \succeq 0$, multipliers $\mu_v \geq 0$ and a number λ such that, for every key κ,

$$\sum_{(i,j)\colon \mathrm{key}(i,j)=\kappa} Z_{ij} + \sum_{v\colon \kappa_v=\kappa} \mu_v = \lambda\,[\kappa = 1]. \tag{17}$$

Pairing with $\Gamma(y)$ gives $0 \leq \langle Z, \Gamma(y)\rangle = \lambda - \sum_v \mu_v y_{\kappa_v}$, and therefore

$$t_H(F[k]) \leq \frac{\lambda}{\sum_v \mu_v k_v}.$$

Exact positivity. The certificate file (see Data and code availability) contains, for each graph, the rational Gibbs witness, the multipliers μ, the nonzero entries of Z, one for each orbit of $\mathcal{A}_0$ together with transposition, and a shift γ. The verifier rebuilds the seed, its 216 ordered non-adjacent pairs, the two-qubit representation, the overlap classes, the 48 automorphisms, the 409 monomials and the key table. It checks every equation (17) and $\mu \geq 0$ in rational arithmetic, and it recomputes the Gibbs witness exactly. To prove $Z \succeq 0$, it clears denominators, $DZ \in \mathbb{Z}^{409\times 409}$, rounds a floating-point Cholesky factor of the shifted matrix $Z - \gamma I$ at scale 2^{40} to an integer matrix L, and checks in exact integer arithmetic that

$$R = 2^{80} DZ - DLL^{\mathsf{T}}$$

is symmetric and diagonally dominant with nonnegative diagonal. Then $R \succeq 0$ and $Z = LL^{\mathsf{T}}/2^{80} + R/(2^{80}D) \succeq 0$. The shift, $\gamma = 10^{-5}$, $2 \cdot 10^{-7}$ and 10^{-5} for G_{32}, G_{44} and G_{72}, leaves room on the diagonal of R for the rounding error. Only the integer check counts. Whether the floating-point factorization succeeded plays no role.

Negative controls. The verifier runs two tampered certificates, and both must be rejected. One breaks an affine equation. The other changes only entries attached to moments that vanish identically, so every affine equation still holds but exact positivity fails.

How the certificate was found. We followed the recovery procedure of [Mgh26]. A numerical SDP solution is symmetrized, rounded to dyadic rationals, and projected exactly onto the affine constraints by a minimum-norm correction, and positivity is then certified as above. None of this matters for correctness, which rests only on the exact checks.

B Deferred proofs

B.1 Continuity in the order

Proof of Proposition 3.4. Write $f(\alpha) = \log \vartheta_\alpha(G)$.

Right continuity at $\alpha_0 \in [0, 2)$. Pick a compatible family and a faithful center whose order-α_0 radius is at most $f(\alpha_0) + \varepsilon$. For fixed states with a faithful center, $\alpha \mapsto D_\alpha$ is continuous on $[0, 2]$, including from the right at 0, because $\rho^\alpha \to \Pi_\rho$, and at 1, where the limit is the relative entropy. So $\limsup_{\alpha \downarrow \alpha_0} f(\alpha) \le f(\alpha_0) + \varepsilon$, and monotonicity gives the reverse inequality.

Left continuity at $\alpha \in (0, 1)$. Let $0 < \beta < \alpha$ and $\psi(t) = \ln \operatorname{Tr} \rho^t \sigma^{1-t}$. By Hölder's inequality ψ is convex, and $\psi(1) = 0 \ge \psi(0)$. So $\psi(\beta) \le (1 - \frac{\beta}{\alpha})\psi(0) + \frac{\beta}{\alpha}\psi(\alpha) \le \frac{\beta}{\alpha}\psi(\alpha)$. Since both orders lie below one, this rearranges to

$$D_\alpha(\rho\|\sigma) \le \frac{\alpha(1-\beta)}{\beta(1-\alpha)} D_\beta(\rho\|\sigma).$$

The constant does not depend on the states or the dimension. Applying it to near-optimal witnesses at order β gives $f(\beta) \le f(\alpha) \le \frac{\alpha(1-\beta)}{\beta(1-\alpha)} f(\beta)$, and the constant tends to 1 as $\beta \uparrow \alpha$. □

B.2 Orthogonal blocks

Proof of Lemma 4.4. For $s \neq 0$ let $A = \sigma^s$ and $f(t) = t^{1/s}$. For $s = 0$ let $A = \ln \sigma$ and $f = \exp$. In each case f is convex on an interval containing the spectrum of A and $f(A) = \sigma$. Diagonalize $Q_j A Q_j$ on $\operatorname{ran} Q_j$ in an orthonormal basis $(e_{jk})_k$. Its eigenvalues are $\langle e_{jk}, A e_{jk}\rangle$, so Jensen's inequality for the spectral measure of A gives

$$Z_s(\sigma, Q_j) = \sum_k f(\langle e_{jk}, A e_{jk}\rangle) \le \sum_k \langle e_{jk}, \sigma e_{jk}\rangle.$$

The vectors $(e_{jk})_{j,k}$ are orthonormal, so the total is at most $\operatorname{Tr} \sigma = 1$. □

B.3 Exact confusability graphs

Lemma B.1. *In (6), (7) and (8), restricting to channels whose confusability graph is exactly G does not change the infimum.*

Proof. Restricting can only increase an infimum, so we show the reverse. Take a compatible family (ρ_x) on $\mathcal{H}$ with center σ (or dominator T). If $E(G) = \varnothing$, every compatible family already has exact confusability graph G, so there is nothing to prove. Assume henceforth that $|E(G)| > 0$.

Add an orthonormal vector f_e for each edge e of G. For a vertex x of degree $d_x > 0$, let Π_x project onto the f_e with $e \ni x$, and let Π_E project onto all of them. Put

$$\rho'_x = (1-\varepsilon)\rho_x \oplus \frac{\varepsilon}{d_x}\Pi_x, \qquad \sigma' = (1-\varepsilon)\sigma \oplus \frac{\varepsilon}{|E|}\Pi_E, \qquad T' = T \oplus \varepsilon\,\Pi_E,$$

and leave isolated vertices unchanged. Adjacent x, y now overlap on $f_{\{x,y\}}$, while non-adjacent vertices share no new vector. So the confusability graph is exactly G. Every Petz moment, relative entropy and support weight splits over the two summands, and the new summand contributes $O(\varepsilon)$. So radii converge to the original ones as $\varepsilon \to 0$. Also $T' \geq \rho'_x$ and $\operatorname{Tr} T' \to \operatorname{Tr} T$. The statement for $C_{\min}$ follows from the order-one radius identity. □

B.4 The entanglement-assisted preorder

Li and Zuiddam state the preorder for graphs through complements. Their Definition 7 [LZ21] (Definition 6 in arXiv:1810.00744v2) writes $G \xrightarrow{*} H$ if there are $d \in \mathbb{N}$ and $d \times d$ positive semidefinite matrices ρ and ρ_g^h ($g \in V(G)$, $h \in V(H)$) such that

(i) $\sum_{h \in V(H)} \rho_g^h = \rho$ for every $g \in V(G)$, and

(ii) $\rho_g^h \rho_{g'}^{h'} = 0$ whenever $\{g, g'\} \in E(G)$ and $\{h, h'\} \notin E(H)$.

The entanglement-assisted cohomomorphism preorder is then $G \leq_* H$ if and only if $\overline{G} \xrightarrow{*} \overline{H}$. Their graphs are confusability graphs, as ours are.

Lemma B.2. *For graphs G and H, the relation $G \leq_* H$ of Li and Zuiddam, that is $\overline{G} \xrightarrow{*} \overline{H}$, holds if and only if there are a state ω and positive semidefinite operators ω_g^h satisfying (9) and (10).*

Proof. Apply the definition to $\overline{G}$ and $\overline{H}$. A pair $\{g, g'\}$ is an edge of $\overline{G}$ exactly when $g \neq g'$ and g, g' are non-adjacent in G. A pair $\{h, h'\}$ is a non-edge of $\overline{H}$ exactly when $h = h'$ or $h \sim h'$ in H, that is, when $h \simeq h'$. The case $h = h'$ is included because graphs have no loops. So condition (ii) for $\overline{G}$ and $\overline{H}$ is (10), and condition (i) is (9) with $\omega = \rho$. Both conditions are homogeneous, so dividing ρ and every ρ_g^h by $\operatorname{Tr}\rho$ makes ρ a state. (A nonzero ρ is implicit in the definition, since $\rho = 0$ would relate every pair of graphs; Li and Zuiddam note that ρ may even be taken positive definite. In Cubitt et al.'s Definition 1, from which the definition is taken, ρ is the reduced state of a bipartite pure state, so it is a nonzero state [CMR+14, Definition 1].) Conversely, a state ω and operators ω_g^h satisfying (9)–(10) satisfy (i) and (ii) with $\rho = \omega$ in dimension $d = \dim \mathcal{H}$. □

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