Matter that pays the price of perfection*

Seth Douglas Nidhal Mghirbi

Version 1.0.0, October 2026

Abstract

We construct translation-invariant nearest-neighbour lattice families whose fixed-coupling thermodynamic gap stays positive while the entanglement entropy of a growing cell-anchored region has any prescribed nonnegative limit-computable logarithmic coefficient. Ordered lower and upper coefficients have the exact second-level one-sided arithmetic classes within an effective cell model. Separately, a referee history cell converts low energy into a nonlocal-game strategy with identical pure-state Schmidt coefficients and a mixed-state distillable-entanglement certificate. The extraction cost is controlled by the clock grid's corner-to-uniform resistance: linear, logarithmic, and bounded for one, two, and three clocks. We give a conditional material theorem with explicit layout and selection hypotheses, a worst-case whole-cell perturbation law, and necessary temperature, preparation-time, and work bounds. Disjoint quasi-local spin-lattice measurements yield only the closure of finite-dimensional correlations. All gap limits use factorial tori before the coupling limit; the entanglement cut grows inside a selected cell. The game-driven material construction remains conditional.

1 Main statements and conventions

A finite nonlocal game can demand unbounded entanglement as its loss tends to zero. We ask how that demand can enter a Hamiltonian. There are two distinct statements. An explicit lattice construction programs a ground-state entanglement coefficient directly. A referee cell instead certifies entanglement through a game's precision price. The first is an unconditional construction using an imported ruler; the second is an unconditional theorem about an ideal legal history space. Their composition into a particular game-driven translation-invariant material has additional, explicitly stated hypotheses.

Entropies and distillation rates are in bits. Natural logarithms are written ln. For a finite game G, $D_G(\varepsilon)$ is the least common local Hilbert-space dimension of a finite strategy of loss at most ε, allowing mixed shared states. The usable price is

$$d_G^{\rightarrow}(\varepsilon) = \inf\{E_D^{A\to B}(\rho) : \text{a finite strategy on } \rho \text{ has loss at most } \varepsilon\}. \tag{1}$$

An empty infimum is $+\infty$. Whenever an energy bound gives a loss larger than one, the resulting price statement is understood simply as the trivial bound at loss one. We use no precision asymptotic there. These conventions are those of [24, 26]; in particular, the infimum is over the original mixed strategy state.

*The authors developed this work with AI models from more than one developer and take responsibility for every claim. How the work was done, and what the machine checks cover, is described at the end of the paper.

The distinct-level gap of a nonscalar finite Hamiltonian is $\Delta(L,g) = E_1^{\mathrm{distinct}} - E_0$: the whole ground eigenspace is removed. Write

$$\Delta_\infty(g) = \lim_{n\to\infty} \Delta(n!, g)$$

when the limit exists. The volume limit always precedes $g \downarrow 0$. A classical block fixes every wall and tableau label. Our designated cell entropies are evaluated in one minimizing classical block, off the pitch ties. Degenerate quantum cell spaces require a further state choice; they never imply that all ground vectors factor over cells.

Theorem 1.1 (Programmed entanglement with a positive gap)**.** *Let $0 \le a \le b < \infty$, with $a \in$ left-Σ_2^0 and $b \in$ left-Π_2^0. There is a fixed finite-local-dimension, rational, translation-invariant onsite and nearest-neighbour family on $\mathbb{Z}^2$, affine in g, such that*

$$\Delta(L,g) \ge 1/16 \quad (g \le 0), \qquad \Delta_\infty(g) \ge 5/8 \quad (0 < g < \bar{g}).$$

The second assertion includes pitch ties. Off ties, the cell-anchored bottom-prefix region $R_c(g)$ has, in every single-classical-block ground state,

$$S(R_c) = \log_2 \mathrm{SR}(R_c : R_c^c) = k(h), \quad k(h) = b_0 |A \cap [0, \ell(h))|, \quad \ell(h) = \mathrm{bitlength}(h-1), \tag{2}$$

where b_0 is a positive integer and $h = p_*(g) - 1$. The lower and upper limits of $S(R_c)/\log_2(1/g)$ are (a,b). Consequently every nonnegative Δ_2^0 real is an existing coefficient; when it equals κ, $\mathrm{SR}(R_c : R_c^c) = g^{-\kappa + o(1)}$.

This is a bound on the fixed-g thermodynamic gap, not a positive finite-volume bound uniform in g near every level crossing. The region grows within a cell and has boundary length $O(\log(1/g))$. The entropy does not violate an area law. The pitch diverges, but it is not asserted to be a connected-correlation length. The proof is in Section 3.

For the second main theorem let $n = T + 1$, $M = n^k$, and let L_k be the unnormalized combinatorial Laplacian on $\{0, \dots, T\}^k$. Its edge quadratic form is $\sum_{\{v,w\}} \|\psi_v - \psi_w\|^2$. Assume an exact local rewinding $W = W_A \otimes W_B$ gives

$$W^* H W = L_k \otimes I + |0^k\rangle\langle 0^k| \otimes (I - P) + |T^k\rangle\langle T^k| \otimes Q. \tag{3}$$

Here $P = |\pi, 0\rangle\langle \pi, 0| \otimes I_{\mathrm{sys}}$ initializes coherent questions and local work, and Q is the final losing projection conjugated by the two honest local, question-controlled circuits. Section 4 specifies this assumption fully. Let $G_k = (L_k^+)_{0^k, 0^k}$ and let μ_1 be its first positive eigenvalue.

Theorem 1.2 (Referee extraction and resistance)**.** *For every unit legal vector Ψ of energy λ, a finite strategy on $W^*\Psi$ has loss at most $C_k M \lambda$, with*

$$C_k = 2(3 + 4\sqrt{G_k})^2 + \frac{2}{M\mu_1}. \tag{4}$$

It retains exactly the Schmidt coefficients of Ψ. Thus

$$\mathrm{SR}(\Psi) \ge D_G(C_k M \lambda), \qquad S_A(\Psi) \ge d_G^{\rightarrow}(C_k M \lambda).$$

For every legal density matrix ρ,

$$E_D^{A\to B}(\rho) \ge d_G^{\rightarrow}(C_k M \,\mathrm{tr}(\rho H)). \tag{5}$$

The resistance has the asymptotics

$$G_1 \sim T/3, \qquad G_2 = (2/\pi)\ln T + O(1), \qquad G_3 < 14.$$

For $k \geq 2$, $2/(M\mu_1) \leq 1/2$; in particular three clocks admit the absolute extraction constant $C_3 < 647$. Two clocks admit the alternative elementary constant $5(1+\sqrt{2T})^2$.

The resistance is for injection at a corner and uniform withdrawal, namely $(\delta_c - M^{-1}\mathbf{1})^* L_k^+ (\delta_c - M^{-1}\mathbf{1})$. It is not obtained by shorting all other vertices together. A third idle clock improves the extraction coefficient while increasing the referee-volume dilution M. Section 4 proves the theorem and the logarithmic obstruction. Sections 6 and 9 give, respectively, the conditional material composition and necessary physical resource bounds.

1.1 Imported results

We use the released version 2.0.0 of *Uncomputable critical exponents* [25]. Its Lemmas 2.1–2.2 and Corollary 2.3 give the five-symbol ruler, its charging estimate, and selection/separation on factorial tori. Section 5 gives the functional nearest-neighbour tableau with the bottom-prefix reader, and Section 11 anchors it inside ruler cells. Lemma 8.1 supplies a computable binary word with any admissible density endpoints, with each bit computable in time polynomial in its index's bitlength. Proposition 10.1 is the limit calculus. Theorem 1.1 of that paper is the earlier gap-exponent construction; none of its counter-spectrum argument is needed for our Bell-pair layer.

From *The uncomputable price of precision*, version 4.0.0 [26], we use Theorem 1.1 for the matched dimension and original-state usable price, Theorem 1.2 for the halting game, Theorem 7.1 for its explicit finite models, Theorem 3.1 for its fixed polynomial-to-Boolean compiler, and Theorem 12.3 for its original-state usable coefficient. In particular

$$\begin{gathered} \alpha_K = \frac{3+\theta_K}{4}, \qquad \theta_K = \sum_{e\in K} 4^{-(e+1)}, \\ \log_2 D_{G_K}(\varepsilon) = \alpha_K \log_2(1/\varepsilon) + o(\log(1/\varepsilon)), \\ d^{\rightarrow}_{G_K}(\varepsilon) = \alpha_K \log_2(1/\varepsilon) + o(\log(1/\varepsilon)). \end{gathered} \tag{6}$$

For even N, the finite model has

$$K_N = \{e < N/2 : h(e) < 2^N\}, \quad \theta_N = \sum_{e\in K_N} 4^{-(e+1)}, \quad d_N = 2^{3N+k_N}, \quad k_N = \theta_N N + O(\log(N+2)), \tag{7}$$

with defining operator defects at most $3\cdot 4^{-N}$ and $\theta_N \uparrow \theta_K$. The compiler and the modification in Section 5 give strategies of system dimension $c_0 d_N$ and loss at most $\varepsilon_N = C_1 16^{-N}$, with fixed constants. The Boolean correspondence is imported from *The price of precision*, version 1.1.0, Theorem 4.9 [24]. That paper's Theorem 11.1 already gives matching pure Schmidt-rank, pure-entropy, and mixed-state formation coefficients for every positive rational. We do not re-prove these game or ruler theorems.

1.2 Relation to earlier work

Computational lattice constructions begin here with the gap undecidability of Cubitt, Pérez-García and Wolf [22]; marker segments and undecidable entanglement behavior occur in [9], size-sensitive phase changes in [10], and continuous-parameter checkerboard/marker constructions in Bausch, Cubitt and Watson [11]. Purcell, Li and Cubitt [45] place a transition at a halting probability.

Undecidable renormalisation destinations [55] and complexity of critical locations [54] concern different asymptotic questions. Schmidhuber's algorithmic ensemble [47] has computable critical inverse temperature $\ln 2$ and a partition function equal to a halting probability there. Public bootstrap-percolation papers [7, 6] announce algorithmic exponent uncomputability across update families. We make no claim to the first uncomputable lattice exponent.

Arithmetic limit classifications [59] and density-pair realization [27] are mathematical inputs. Symbolic growth invariants and entropy classifications already encode noncomputability in local rules [40, 34]. Commensurate staircases [5, 32, 3] and wall-density transitions [44] also precede the present construction. Finite analytic spectral branches are governed by Rellich's theorem as presented by Kato [38]. Our new realization concerns a programmed logarithmic coefficient for a specified growing cut in a distinct-level-gapped family. Unusual chain entanglement laws [43, 58, 46] and standard pure-state entanglement concentration [14] are separate precedents; pair counting itself is elementary.

Feynman–Kitaev computation [31, 39], local histories [2, 1, 33], multiple clocks [16], modified and graph clocks [8], idling and endpoint analysis [17], and nonunitary histories [52] are established. Classical network resistance [21, 28] and resistance-based quantum witnesses [12] are also established. The contribution here is their quantitative use in fresh-question extraction with unchanged Schmidt data and an explicit game-price certificate. Precision-dependent dimension witnesses already occur in Slofstra–Vidick [49]; the matched prices used here are the imports just stated.

The quasi-local approximation argument below is standard in the spirit of Scholz–Werner [48]; tensor/commuting separation is supplied by [37]. Thermal information [56], entangling rates [15, 53], squashed entanglement [20], and correlation work costs [36] are prior inequalities. Our resource bounds substitute the imported price into these inequalities. Certified equilibrium algorithms [30] do not imply computability of an arbitrary limiting coupling coefficient.

Instability already has direct precedents in Castilla-Castellano–Lucia [18] and the October version of Xu et al. [57]; their gap conventions must be kept distinct from ours. Rotationally symmetric gap undecidability [19] is another separate result. Strict-boundary stability [41, 42], robust computation in tilings [29], and low-temperature quasicrystalline Gibbs states [50] do not supply the selection and interface rules required by our conditional composition. We assert neither an open interaction-ball stability theorem nor a general local-noise law.

2 The ruler and cell transfer

We state precisely the imported geometric interface. With $r = 1/16$, a perfect cell of pitch $p \geq 2$ has interior side $h = p - 1$, and the classical density is

$$f_g(p) = -\frac{g}{p} + \frac{r}{p^2}, \qquad g_p = r\left(\frac{1}{p} + \frac{1}{p+1}\right), \qquad f_g(p) - f_g(p+1) = \frac{g_p - g}{p(p+1)}. \tag{8}$$

The five symbols are W, V, A, S, D. For orientation, the allowed horizontal pairs are

$$WW, AA, SA, DS, DD, AV, SV, VS, VD;$$

the allowed top-to-bottom pairs are

$$WS, WA, WV, SW, DW, VW, AA, AS, SD, DD, VV.$$

Every other pair costs one. An onsite W bonus is $-g$; a vertical VW bond costs r. We use the forcing and charging theorems of [25, Lemmas 2.1–2.2], rather than a new rule-level proof.

For small $g > 0$, $p_*(g) = p$ on (g_p, g_{p-1}), with both $p, p+1$ minimizing at g_p. Each stripe can choose its own horizontal phase. The source's Corollary 2.3 gives separation from every nonminimizing classical block by at least $\min(5/8, L\delta(g))$, after also making the vacuum excess exceed that bound, where

$$\delta(g) = \min_{p \notin \mathcal{P}_*(g)} p(f_g(p) - \min_q f_g(q)) > 0.$$

At a tie $\mathcal{P}_*(g)$ contains both minimizers. For fixed rational g, a factorial-volume cutoff is computable. Every fixed positive g also has eventual commensurability and separation. In particular

$$p_*(g) = 2r/g + O(1), \qquad \log_2 h_*(g) = \log_2(1/g) + O(1). \tag{9}$$

For $g \leq 0$ the two wall-free vacua minimize, with classical margin at least r.

The flags in [25, Sections 5 and 11] tie the top, bottom, left and right of a tableau to its enclosing walls. In a correct cell the classical content is the unique functional tableau of an $h \times h$ box. All extra tests are nonnegative unit penalties. All quantum terms below are positive, conditioned on local classical labels, and act either onsite or on a bond entirely inside a cell. Exterior quantum registers have one-dimensional frozen kernels and a specified positive excitation bound.

Lemma 2.1 (Reparametrization)**.** *Suppose $\phi(g)$ samples every sufficiently large integer, only large integers as $g \downarrow 0$, and $\ln \phi(g) = \ln(1/g) + O(1)$ uniformly. If $Q(h) > 0$ and $|\ln Q(h)| = O(\ln h)$, the lower and upper power exponents of $Q(\phi(g))$ are those of $Q(h)$. If $Q(h) = O(\ln h)$, the same is true of the logarithmic coefficients.*

Proof. On a plateau with $\phi(g) = h$, the denominators differ by $O(1)$. The bounded normalized numerators make the difference of ratios $o(1)$. Every tail integer occurs, and small g excludes any finite prefix. Consequently the two sets of tail limit points coincide. The argument does not require a computable convergence rate. □

Proposition 2.2 (Selected-block transfer)**.** *Suppose a selection theorem gives minimizing perfect blocks of pitch p, with all other classical blocks at energy at least $E_0 + \gamma$. Let the frozen excitation bound be $\gamma_f > 0$, and include all tableau, flag, and exterior margins in $\gamma_{\text{eff}} = \min(\gamma, \gamma_f, \gamma_{\text{other}})$. In each minimizing block assume exactly*

$$H = E_0 I + \sum_c (K_h - e(h)I)_c + \sum_w F_w, \qquad e(h) = \min \operatorname{spec} K_h, \tag{10}$$

on disjoint factors, with $\ker F_w = \mathbb{C}|0\rangle_w$. Below $E_0 + \gamma_{\text{eff}}$, the full spectrum is the sum spectrum of the cells, with multiplicities summed over the minimizing blocks. The full gap equals $\Delta^+(K_h)$ if that number is less than γ_{eff}; otherwise the full gap is at least γ_{eff}. Here $\Delta^+(K_h) = +\infty$ if the cell has no positive excitation above its ground space. The ground space is

$$\bigoplus_{\sigma \ mminimizing} |\sigma\rangle \otimes \left(\bigotimes_c \ker(K_h - e(h)I) \right) \otimes |\text{frozen}\rangle. \tag{11}$$

Effective specified cell quantities with finite normalized endpoints transfer by Lemma 2.1. Uniformly computable scalar cell data have lower, upper, and existing finite limits in left-Σ^0_2, left-Π^0_2, Δ^0_2, respectively.

Proof. All classical labels are conserved. Nonminimizing blocks and excited frozen factors cannot contribute below the stated margin. Diagonalize the disjoint factors in (10); eigenvalues add and ground spaces tensor. This proves the spectral statement and (11). If $d_h = \dim\ker(K_h - e(h)I)$, the degeneracy is $N_{\min} d_h^{L^2/p^2}$. The stripe phase count has $\log N_{\min} = O((L/p+1)\log p)$ at fixed pitch, so its entropy density vanishes; the quantum residual entropy density is $(\log d_h)/p^2$.

Reparametrization proves the transfer assertion for the specified quantity. For arithmetic, uniformly approximate each normalized cell datum by a rational within 2^{-h}. For a rational sequence a_h,

$$q < \liminf_h a_h \quad \Longleftrightarrow \quad \exists r \in \mathbb{Q}\ \exists H\ [r > q \text{ and } \forall h \geq H :\ a_h \geq r].$$

This is a second-level existential left cut. The upper endpoint is dual; an existing finite limit is computable in the halting set. For an irrational endpoint complement the appropriate strict cut; a rational endpoint is itself computable. This is membership of an individual real in a cut class, not a uniform exact-comparison algorithm for arbitrary Cauchy names. It agrees with [25, Proposition 10.1] and the prior calculus [59]. □

If every cell ground state is unique, a single-block ground vector is a product over cells. More generally, that product is an available choice. For such states, restrictions to any region tensor over its cell intersections, so all Rényi entropies add over cut cells; whole-cell unions have zero entropy and different cells' observables factor. These conclusions are not statements about every vector in (11). A free logical qubit in each cell permits a Bell state across two cells in a single classical block, giving a whole cell entropy one. Repeated finite Bell dimers even give a pure clustering example. Arbitrary cell marginals must therefore be used when cell kernels are degenerate.

Lemma 2.3 (Small-energy perfect-stripe staircase)**.** *Let $0 \leq e(h) \leq ch^{-3}$ for $h \geq 1$ and set*

$$F_g(p) = -g/p + (r + e(p-1))/p^2, \qquad p \geq 2.$$

Define $d_p = ((p+1)/p)e(p-1) - (p/(p+1))e(p)$ and $g'_p = g_p + d_p$. Then $F_g(p) - F_g(p+1) = (g'_p - g)/(p(p+1))$. The ties g'_p strictly decrease for $p \geq p_0 = \lceil 4c/r \rceil + 2$, and all sufficiently large pitches are selected in order among perfect stripes, with $p_* g \to 2r$.

Proof. Subtract the two densities. Nonnegativity gives $|d_p| \leq 2c/(p-1)^3$, so

$$g'_p - g'_{p+1} \geq \frac{2r}{p(p+2)} - \frac{4c}{(p-1)^3}.$$

For $p \geq p_0$ use $4c \leq r(p-2)$ and $2(p-1)^3 - p(p+2)(p-2) = p^3 - 6p^2 + 10p - 2 > 0$ for integers $p \geq 2$. The last polynomial is positive at $2, 3, 4$ and increasing thereafter; the case $c = 0$ is immediate. At sufficiently small g, the finitely many smaller pitches have positive density, whereas a pitch near $2r/g$ has negative density. Adjacent differences then change sign exactly once on the tail. Finally $g'_p = 2r/p + O(p^{-2})$. □

This lemma compares perfect stripes. Selection against irregular configurations with positive cell energy requires the local charging hypothesis in Section 6. Zero cell energy needs only the imported ruler theorem.

3 A gapped family with programmed entanglement

Fix $b_0 \geq 1$, put $d = 2^{b_0}$, and take a fast word A from [25, Lemma 8.1]. The imported bottom reader selects precisely the first $\ell(h) = \mathrm{bitlength}(h-1)$ interior columns at the final tableau row. For all sufficiently large h, the output at its column j is $A(j)$. Indeed the first ℓ bits are written by time $2^{\ell-1} - 2 < h - 1$ in the imported timing convention. A computable small-g cutoff makes both rows of every pair interior.

Each site carries $\mathbb{C}^d$. Add a classical partner bit, functionally determined by whether the lower neighbour is an active bottom-prefix site. A two-site equality test determines it uniquely and costs two if violated. On each active vertical pair put

$$H_{\mathrm{pair}} = I - |\Phi_d\rangle\langle\Phi_d|, \qquad |\Phi_d\rangle = d^{-1/2} \sum_{j=0}^{d-1} |j, j\rangle. \tag{12}$$

Every other quantum register, including wall registers, receives $I - |0\rangle\langle 0|$. In a correct tableau these supports are disjoint. All matrix entries of the Bell projection are zero or $1/d$; the whole construction therefore has rational coefficient matrices, fixed local dimension, and nearest-neighbour interactions. A finite alphabet enlargement suffices for all added classical flags. For this construction give every forbidden wall pair and every tableau/flag violation cost two. This leaves the perfect densities (8) unchanged. The imported wall estimate strengthens by exactly the number of wall violations, so a defective wall block has margin at least 13/8. The larger penalties will also make the thermodynamic gap an actual eventually constant limit.

Proof of Theorem 1.1. In a perfect correct block the quantum Hamiltonian has a unique ground state, a tensor product of Bell pairs and frozen sites. Its ground energy is zero and its gap is one. In every other classical block all additional quantum and tableau terms remain nonnegative. Thus no such block can undercut the imported classical lower bounds. For $g \leq 0$ the wall-free vacua freeze every register and the margin is at least $1/16$. At fixed small $g > 0$, defective wall blocks have margin at least 13/8, and incorrect tableau/flag blocks cost at least two. Choose factorial volume large enough that all wrong clean stripe pitches and the vacua have excess greater than one. This is possible by the imported positive separation $\delta(g)$. Inside every minimizing block the quantum spectrum is integral and has a unit excitation. Thus $\Delta(L, g) = 1$ for all sufficiently large factorial volumes. This proves existence of the thermodynamic limit and the stated lower bound, also at ties by excluding both minimizing pitches when computing the separation.

Off ties, all cells have side $h = p_*(g) - 1$. A single-block vector is a product because the cell kernels are one-dimensional. Exactly $m(h) = |A \cap [0, \ell(h))|$ Bell pairs cross the designated bottom-prefix cut. Its reduced density matrix has $d^{m(h)}$ equal nonzero eigenvalues, each $d^{-m(h)}$. Hence every Rényi entropy in bits equals $b_0 m(h) = k(h)$, and its rank is $2^{k(h)}$. Writing $\rho_j(A) = |A \cap [0, j)|/j$, we obtain

$$\frac{k(h_*(g))}{\log_2(1/g)} = b_0 \rho_{\ell(h_*(g))}(A) + o(1). \tag{13}$$

Every sufficiently large pitch, and hence every sufficiently large prefix length, occurs. Lemma 2.1 gives the two density endpoints. Choose an integer $b_0 \geq \max(1, b)$ and the imported fast word with endpoints $a/b_0, b/b_0$. This proves attainment. Equal endpoints are precisely the nonnegative Δ_2^0 targets by the imported density theorem. Zero is included, for example by an eventually empty word. Local dimension and the finite program may depend on the target. □

The union of all designated bottom prefixes has entropy $k(h)$ per cell, or $k(h)/p^2$ per lattice site in bits. Whole-cell unions have zero entropy in these states. A label-free variant uses the maximum

entropy of a horizontal segment of length $s(g)$ satisfying $\ell(h) \le s(g) \le p/2$. Once $s(g) + \ell(h) < p$, such a segment can meet at most one active prefix, and a segment covering that prefix attains exactly $k(h)$. Thus the maximum equals the cell entropy; a factor-of-two comparison would not suffice to identify this logarithmic coefficient.

At a fixed finite commensurate volume, approaching a classical pitch crossing from one side produces an arbitrarily small positive competing block splitting. This explains why the theorem uses a fixed-g thermodynamic distinct-level gap. It supplies no identification with every infinite-volume GNS gap and no conventional continuous critical point. In the quantum pair layer, connected correlations vanish unless the supports touch a common pair; a diverging structural pitch is compatible with this observation.

Corollary 3.1 (Exact arithmetic classes in an effective cell class)**.** *For uniquely grounded selected cells with uniformly effective algebraic finite matrices and a computably specified cell-anchored region, finite lower, upper, and existing entropy or log-Schmidt-rank coefficients belong to left-*Σ_2^0*, left-*Π_2^0*,* Δ_2^0*, respectively. Together with Theorem 1.1, these classes are exact in this class.*

Proof. Exact real algebra computes the simple ground eigenvector and its reduced algebraic matrix. Its exact rank is decidable. Its eigenvalues are computable and $-x \log_2 x$, extended continuously by zero, computes the entropy. The normalized data therefore have uniform rational approximants. Apply Proposition 2.2 and Lemma 2.1; finite endpoints give the eventual boundedness needed in the ratio comparison. No convergence modulus is asserted. □

3.1 Cell perturbations and content energy

Proposition 3.2 (Perturbed pair cells)**.** *Add Hermitian fields v, v' of norm at most η at the two endpoints of each pair. For $\eta < 1/4$ every pair has a unique ground state and gap at least $1 - 4\eta$. The cell still factors into pair and single-site blocks. Arbitrary pair fields preserve full Schmidt rank whenever*

$$\eta < \frac{1}{4 + 2\sqrt{d}}. \tag{14}$$

Mirror fields $v' = v^{\mathsf{T}}$ preserve full rank throughout $\eta < 1/4$. If the same specified pair fields are repeated, writing their pair entropy as $s(v, v')$ gives exactly $S(R_c) = m(h)s(v, v')$; for site-dependent fields it is the sum of the actual pair entropies.

Proof. The total pair perturbation has norm at most 2η. Weyl's inequalities separate the former zero eigenvalue from the rest by $1 - 4\eta$. Frozen single-site blocks have gap at least $1 - 2\eta$. Disjointness gives the cell factorization.

Write a pair ground vector as $a\Phi_d + b$, with $b \perp \Phi_d$, and write its energy as $E \le 2\eta$. With $Q_0 = I - |\Phi_d\rangle\langle\Phi_d|$, its eigenvalue equation gives

$$[(1 - E)I + Q_0 V Q_0]b = -aQ_0 V \Phi_d, \qquad \|b\| \le \frac{2\eta}{1 - 4\eta}|a|.$$

The coefficient matrix is $aI/\sqrt{d} + B$, with $\|B\|_{\mathrm{op}} \le \|b\|$. It is invertible under (14). In particular $a \neq 0$, since the displayed inequality otherwise forces the vector to vanish.

For mirror fields, vectorization makes the local field action $X \mapsto vX + Xv$. In an eigenbasis of v, all off-diagonal entries have energy at least $1 - 2\eta > E$. On the diagonal subspace the rank-one Bell term yields

$$X \text{ proportional to } [(1 - E)I + 2v]^{-1}.$$

All denominators are at least $1 - 4\eta > 0$, so every Schmidt coefficient is nonzero. Tensor products then multiply ranks and add the actual block entropies. Finite-dimensional continuity gives $s(v, v') \to \log_2 d$ as the fields vanish. Computable algebraic repeated fields give a computable positive s in the stated radii. □

The entropy factor is not determined by the norm bound. For $d = 2$, equal scalar fields leave the Bell entropy one, whereas fields ηZ on both endpoints give the ground vector of

$$\begin{pmatrix} 1/2 + 2\eta & -1/2 \\ -1/2 & 1/2 - 2\eta \end{pmatrix},$$

with entropy less than one for $\eta > 0$. Repeated specified fields rescale the cell entropy coefficient by s/b_0, while leaving its rank coefficient unchanged within the proved radius. This is a cell-level statement. Their energy shift can alter classical selection.

Proposition 3.3 (A design-dependent energetic competitor)**.** *Suppose a specified legal design admits a defect costing at most C which changes $m(h)$ independently priced content units by the favourable sign, without any other energy change. A field charging those units at magnitude $|\delta|$ makes this competitor favourable whenever $|\delta| m(h) > C$.*

Proof. Subtract the two expectation energies. The difference is at most $C - |\delta| m(h)$. □

An actual bounded-cost defect must be exhibited before applying this criterion. It is not a universal instability theorem. A logarithmically growing content cost does not satisfy the h^{-3} hypothesis of Lemma 2.3; no material stability window follows from the pair calculation alone.

4 Referee extraction and its clock-grid cost

Let $G = (X, Y, \mathcal{A}, \mathcal{B}, \pi, V)$ be finite. Each party has a system register, a clock, old question, answer and work registers. The initialization state of the old registers is

$$|\pi, 0\rangle = \sum_{x,y} \sqrt{\pi(x, y)} |x, y\rangle |0\rangle_{\text{ans,work}}.$$

The local circuits U_A, U_B read their own question without changing it; neither uses the other's question. Their coherent outputs are tested by the diagonal losing projection Π_{lose}. Put $Q = (U_A \otimes U_B)^* \Pi_{\text{lose}} (U_A \otimes U_B)$. Equation (3) is an exact assumption on a legal bipartite tensor product; the two clocks belong to their respective parties. Extra idle clocks can belong entirely to Alice. Product rewinding is the controlled local prefix circuit on each party's own clock and data. No nearest-neighbour realization of this whole initialization projector is implicit in the assumption.

On fresh external questions drawn with the original joint law π, run these circuits on the system with new local pure answer/work ancillas. All old registers and clocks are spectators. The resulting loss effect on the old register space is

$$F = I_{\text{old}} \otimes F_{\text{sys}}, \qquad 0 \le F \le I, \qquad [F, P] = 0, \qquad PFP = PQP. \tag{15}$$

The last identity follows by expanding the coherent old questions: preserved labels and the diagonal test kill cross terms, including when π is correlated. In particular $\langle Pv|F|Pv\rangle = \|QPv\|^2$. Keeping spectators preserves the shared Schmidt data. There is no postselection at a clock corner.

Lemma 4.1 (Elementary two-clock extraction)**.** *For two clocks, a legal unit vector of energy λ yields loss at most $5(1+\sqrt{2T})^2 M\lambda$ under the fresh-question strategy. If $\varepsilon_{\mathrm{opt}}$ is the least initialized loss with these circuits, $E_0(H) \le \varepsilon_{\mathrm{opt}}/M$.*

Proof. Write $W^*\Psi = \sum_v |v\rangle\psi_v$. Positivity gives

$$E_{\mathrm{prop}} = \sum_{\{v,w\}} \|\psi_v - \psi_w\|^2 \le \lambda, \quad \|(I-P)\psi_0\|^2 \le \lambda, \quad \|Q\psi_t\|^2 \le \lambda,$$

where $t = (T,T)$. A path of length at most $2T$ to either corner gives $\|\psi_v - \psi_c\| \le \sqrt{2T\lambda}$ by Cauchy–Schwarz. Set $a = 1+\sqrt{2T}$. Then $\|(I-P)\psi_v\| \le a\sqrt{\lambda}$ and $\|QP\psi_v\| \le 2a\sqrt{\lambda}$. By (15), the cross term between initialized and uninitialized components vanishes. Summing their losses over the M orthogonal clock components gives $M(4a^2+a^2)\lambda$. For the variational bound take the uniform clock with an initialized optimal system input. Propagation and initialization vanish; the final corner penalty is $\varepsilon_{\mathrm{opt}}/M$. □

Lemma 4.2 (Hilbert-valued resistance inequality)**.** *For a connected graph Laplacian L, a zero-sum real vector w, and Hilbert-space vectors ψ_v,*

$$\|\sum_v w_v\psi_v\|^2 \le (w^*L^+w) \sum_{\{v,z\}} \|\psi_v - \psi_z\|^2. \tag{16}$$

Proof. Diagonalize L on the orthogonal complement of constants. In that basis the left vector is the inner product of the scalar coefficients $L^{+1/2}w$ with the Hilbert-valued coefficients $L^{1/2}\psi$. Hilbert-space Cauchy–Schwarz gives (16). The constant mode drops out because w sums to zero. □

Proof of Theorem 1.2: energy and price. Let $m = M^{-1}\sum_v \psi_v$, $d_v = \psi_v - m$, and $G = G_k$. Using $w = \delta_c - M^{-1}\mathbf{1}$ in Lemma 4.2 gives $\|d_c\| \le \sqrt{G\lambda}$ at either corner. Poincaré's inequality gives $\sum_v \|d_v\|^2 \le \lambda/\mu_1$. The corner initialization and test imply

$$\begin{aligned} \|(I-P)m\| &\le (1+\sqrt{G})\sqrt{\lambda}, \\ \|QPm\| &\le \sqrt{\lambda} + \|(I-P)m\| + 2\sqrt{G\lambda}. \end{aligned}$$

Define $\ell(v) = \langle v|F|v\rangle$. Its square root is a seminorm, and (15) implies

$$\sqrt{\ell(m)} \le \|QPm\| + \|(I-P)m\| \le (3+4\sqrt{G})\sqrt{\lambda}.$$

Consequently

$$\sum_v \ell(\psi_v) \le 2M\ell(m) + 2\sum_v \|d_v\|^2 \le C_k M\lambda. \tag{17}$$

The shared state is exactly $W^*\Psi$, augmented only by local pure ancillas. Since W is a product unitary, its Schmidt coefficients, rank and entropy are unchanged. Compressing both local POVMs to their Schmidt supports yields a legitimate strategy of local dimension $\mathrm{SR}(\Psi)$, proving the dimension bound. Pure-state one-way distillation equals marginal entropy [14]; the definition (1) proves the entropy bound.

The quadratic-form inequality (17) is an operator inequality for a fixed fresh-question loss effect. It therefore holds after taking expectation in any legal density matrix. Local unitary invariance of $E_D^{A\to B}$ and (1) prove the mixed-state certificate. No entropy of a mixed marginal is being identified with its entanglement. □

4.1 The resistance trichotomy

The path eigenvalues and corner weights are

$$\mu_a = 4\sin^2\frac{\pi a}{2n}, \quad w_0 = 1/n, \quad w_a = \frac{2}{n}\cos^2\frac{\pi a}{2n} \quad (1 \le a < n).$$

Thus, exactly,

$$G_k = \sum_{\substack{0\le a_1,\dots,a_k<n\\(a_1,\dots,a_k)\neq 0}} \frac{w_{a_1}\cdots w_{a_k}}{\mu_{a_1}+\cdots+\mu_{a_k}}. \tag{18}$$

The first path eigenvalue satisfies $\mu_1 \ge 4/n^2$. For $k \ge 2$, $M\mu_1 \ge 4n^{k-2}$, proving the variance term in the theorem.

For a path, the centered potential solving $Lg = \delta_0 - n^{-1}\mathbf{1}$ has edge differences $g_j - g_{j+1} = (n-1-j)/n$. Hence its energy, equal to G_1, is

$$G_1 = \frac{1}{n^2}\sum_{j=1}^{n-1} j^2 = \frac{T(2T+1)}{6(T+1)}. \tag{19}$$

For two paths use $w_a \le 2/n$ and $\mu_a \ge 4a^2/n^2$ to obtain

$$G_2 \le \sum_{\substack{0\le a,b\le T\\(a,b)\neq(0,0)}} \frac{1}{a^2+b^2}.$$

The axes contribute at most $\pi^2/3$ and $(1,1)$ contributes $1/2$. Each remaining positive-index term is bounded by the integral of $(x^2+y^2)^{-1}$ over $[a-1,a]\times[b-1,b]$. Those cells lie inside the quarter annulus $1 \le \sqrt{x^2+y^2} \le \sqrt{2}T$. Therefore

$$G_2 \le \frac{\pi}{2}\ln T + \frac{\pi^2}{3} + \frac{1}{2} + \frac{\pi}{4}\ln 2 < \frac{\pi}{2}\ln T + 4.4 \qquad (T \ge 2). \tag{20}$$

To identify the leading constant, for $a, b \ge 1$ compare the exact summand in (18) with $4/[\pi^2(a^2+b^2)]$. Taylor bounds for sine and cosine, and $\sin x \ge 2x/\pi$ on $[0,\pi/2]$, bound the difference by $O(n^{-2})$ uniformly. Summing contributes $O(1)$; the axes are bounded. The quadrant sum of $(a^2+b^2)^{-1}$ is $(\pi/2)\ln n + O(1)$ by unit-square integration, proving $G_2 = (2/\pi)\ln T + O(1)$.

For three paths, (18) implies

$$G_3 \le \frac{2}{n}\sum_{a\neq 0}|a|^{-2} \le \frac{2}{n}\sum_{m=1}^{T}\left(3+\frac{3}{m}+\frac{1}{m^2}\right) < 14.$$

The shell $\max_i a_i = m$ has $3m^2+3m+1$ vectors. The last bound uses $1/m, 1/m^2 \le 1$ and $T/n < 1$. Thus

$$C_3 \le 2(3+4\sqrt{14})^2 + 1/2 < 647.$$

For any fixed $k > 3$ the same shell estimate is $O(T^{k-2})$ before the prefactor $2^{k-2}n^{2-k}$, so $G_k = O(1)$. This completes the resistance part of Theorem 1.2. The dimensional phenomenon is classical network theory [21, 28]; the application is the extraction inequality.

Proposition 4.3 (The logarithmic obstruction for all low-energy vectors)**.** *No constant independent of T can replace the two-clock extraction factor for all legal low-energy vectors in this general cell class.*

Proof. Let c be the losing corner, $g = L_2^+ \delta_c$, $G = g_c$, and $f = \mathbf{1} - g/G$. Then $f_c = 0$, $M^{-1} \sum_v f_v = 1$, and $f^* L_2 f = 1/G$. Take $P = I, Q = I$, with the always-losing scalar system and normalized clock $f/\|f\|$. The loss is one and the energy is $\lambda = 1/(G\|f\|^2)$, so

$$\frac{\text{loss}}{M\lambda} = G\frac{\|f\|^2}{M} \geq G.$$

Since $G \to \infty$, a uniform constant is impossible. A direct-sum zero-loss system sector can be added. This is an all-low-energy-state obstruction and makes no assertion about a special game's ground states alone. □

4.2 Approximate clock factorization

Proposition 4.4 (Normalized entropy-support estimate)**.** *For $k \geq 2$ put $u = \sqrt{M}\, m$ and $z = |\text{unif}\rangle \otimes \text{Pu}$. If $b = (3/2 + \sqrt{G_k})\sqrt{M\lambda} < 1$, then*

$$\|W^*\Psi - z\| \leq b, \qquad \|W^*\Psi - z/\|z\|\| \leq 2b.$$

Let t be the actual trace distance between the reduced Alice states of these two unit vectors. In the small-distance regime,

$$S_A(\Psi) \leq \log_2 d_{\text{sys},A} + \log_2 |X| + t \log_2(D_A - 1) + h_2(t), \qquad t \leq 2b. \tag{21}$$

D_A is an actual local support dimension of the compared states.

Proof. Poincaré gives $\|W^*\Psi - |\text{unif}\rangle \otimes \text{u}\| \leq \sqrt{\text{M}\lambda}/2$. Initialization bounds $\|(I - P)u\| \leq (1 + \sqrt{G})\sqrt{M\lambda}$. Their triangle inequality gives the first estimate. Since Ψ is unit, $|\|z\| - 1| \leq b$ and $z \neq 0$, giving the factor two upon normalization. The normalized reference consists of product-uniform local clocks, initialized questions and work, and a system vector. Its entropy is at most $\log_2 |X| + \log_2 d_{\text{sys},A}$. Local rewinding leaves entropy invariant. The trace distance is no larger than the vector distance; apply the finite-dimensional entropy-continuity bound of [4]. One may replace t by $2b$ only where that bound is monotone, for example $2b \leq 1 - 1/D_A$. □

There is no compulsory additive $\log_2(T + 1)$ in this reference entropy. There is also no exact clock separability at positive energy. For $P = I, Q = 0$, orthogonal path modes u (uniform) and v (first excited) give the entangled clock state $(u \otimes u + \delta v \otimes v)/\sqrt{1 + \delta^2}$, of energy $2\mu_1\delta^2/(1 + \delta^2)$ and entropy $h_2(\delta^2/(1 + \delta^2)) > 0$.

For a ground state with $M\lambda \leq \varepsilon_N$, an adequate continuity condition, if q counts work qubits, is

$$\sqrt{\varepsilon_N}(1 + \sqrt{G})[q + \log_2 d_{\text{sys},A} + \log_2(T + 1) + O(1)] \longrightarrow 0.$$

It is a condition on the number of qubits, not on its logarithm.

4.3 Perturbation and dimension certificates

Corollary 4.5 (Total-norm perturbations)**.** *For every legal-space Hermitian V with $\|V\| \leq \eta$ and every pure ground vector Ψ_η of $H + V$,*

$$S_A(\Psi_\eta) \geq d_G^{\rightarrow}(C_k[\varepsilon_{\text{opt}} + 2M\eta]), \quad \text{SR}(\Psi_\eta) \geq D_G(C_k[\varepsilon_{\text{opt}} + 2M\eta]).$$

Proof. The variational principle gives $\langle\Psi_\eta|H|\Psi_\eta\rangle \leq E_0(H) + 2\eta$. Use Lemma 4.1 and Theorem 1.2. □

Here η is the total cell operator norm. Summing local fields may produce a norm growing with the number of sites.

Proposition 4.6 (Register-limited frustration)**.** *In a fixed classical program block with n_s measured system qubits per party, let e be the cell ground energy. Then*

$$D_G(C_k M e) \le 2^{n_s}.$$

If $\log_2 D_G(\varepsilon) = \alpha \log_2(1/\varepsilon) + o(\log(1/\varepsilon))$, $\alpha > 0$, this implies

$$e \ge \frac{2^{-n_s/\alpha - o(n_s)}}{C_k M}. \tag{22}$$

Proof. After extraction, trace out old registers and clocks locally. The fresh measurements act only on the measured system, so its mixed state has the same loss and dimension 2^{n_s} on each side. Apply the definition of D_G. For each $\xi > 0$ the price is at least $(\alpha - \xi)\log_2(1/\varepsilon)$ at sufficiently small ε. Hence $C_k M e \ge 2^{-n_s/(\alpha-\xi)}$; outside this small-loss range the claimed asymptotic lower bound is weaker. Let ξ tend to zero. The argument is uniform over fixed classical program blocks. A quantum program used by the measurement must instead be counted in the system dimension. □

A matching upper frustration law needs a dimension-matched admitted strategy. Clock propagation does not by itself fix the distinct cell gap: different almost-winning system sectors can have arbitrarily close low eigenvalues. Honest uniform histories have energy loss$/M$, which is a variational scale, not a universal gap formula. A fixed Hamiltonian minimizes a linear functional of the state; its measurement encoding must therefore already be specified. Fixed circuits and classical program blocks suffice, without exhausting all possible encodings. Finally the elementary extraction factor shows that $\log T = o(\log(1/\varepsilon))$ is sufficient to preserve a logarithmic coefficient; it is not a necessary-and-sufficient theorem.

An extension to arbitrary physical states is possible if the legal projection factorizes across the cut, W has a product-unitary extension, the illegal space is invariant and has energy at least one, and the extraction processes its clock labels as spectators. The illegal weight is then bounded by its energy and its loss by its weight; since $C_k M \ge 1$, the same inequality holds. Block diagonality or a list of forbidden clock strings alone does not establish these hypotheses.

5 Strategy circuits, initialization rows, and padding

The halting-game finite model (7) is explicit but has long computation-history labels. A small system register does not imply a polynomial measurement circuit. The following resource bound is sufficient for the conditional composition.

Lemma 5.1 (Sequential orthogonality context)**.** *For projections E, F on $\mathbb{C}^d$, Alice can answer the context forbidding the simultaneous $E = F = 1$ outcome with the POVM*

$$E, \qquad (I-E)F(I-E), \qquad (I-E)(I-F)(I-E).$$

On a maximally entangled state, Bob using $E^{\mathsf{T}}, F^{\mathsf{T}}$ has zero E disagreement and F disagreement

$$\frac{3\,\mathrm{tr}(EF) - 2\,\mathrm{tr}(EFEF)}{d} \le 3\|EF\|_f^2, \qquad \|A\|_f^2 = d^{-1}\,\mathrm{tr}(A^*A). \tag{23}$$

Proof. The three positive effects sum to I. Measure E first and, after a negative answer, measure F; the listed effects are exactly the resulting measurement. Let $M_F = (I-E)F(I-E)$. The maximally entangled trace identity with Bob's transpose gives disagreement $\operatorname{tr}(M_F + F - 2M_F F)/d$. Expanding, $\operatorname{tr} M_F = \operatorname{tr} F - \operatorname{tr} EF$ and $\operatorname{tr}(M_F F) = \operatorname{tr} F - 2\operatorname{tr} EF + \operatorname{tr} EFEF$, proving the identity. The last trace is nonnegative because it is the trace of $(FEF)^2$ on $\operatorname{ran} F$. Also $\|EF\|_f^2 = \operatorname{tr}(EF)/d$. □

This coherent sequential POVM replaces the compiler's projection onto $(I-E)\operatorname{ran} F$. It changes only a constant in its loss bound, by at most a factor $3/2$. It is not an operator-norm approximation to that different projective measurement.

Proposition 5.2 (Halting-game circuit resources)**.** *For every fixed $\delta > 0$ and even N, the finite models of [26, Theorem 7.1], with the canonical compiler blocks of its Theorem 3.1 and Lemma 5.1, give near-winning measurement circuits on system dimension $c_0 d_N$, of loss at most $\varepsilon_N = C_1 16^{-N}$. Their size is at most $2^{(1+\delta)N}\operatorname{poly}(N)$ and their work space is $O(N^2)$ qubits. They have uniform classical descriptions generated within the same exponential scale, allowing an arbitrarily small fixed slack in δ for synthesis overhead. Finite-gate implementation error in the coherent measurement is at most 2^{-6N}.*

Proof. We describe the algorithm against the explicit source model; we use, rather than re-prove, the compiler's algebraic soundness and finite model identities. Its coordinate registers are N-bit address, fuel and time, and $k_N = O(N)$ optional slots. The flag is

$$f_N(x) = \sum_{e \in K_N} \mathbf{1}_{\{x \equiv 4^e \pmod{4^{e+1}}\}}.$$

The residue classes are disjoint. Computing the flag, digits, masks, fuel reset predicates, and partial counts is polynomial in N once K_N is known. The count $h_N(x) = \sum_{r<x} f_N(r)$ is a sum of residue counts, using division with remainder for each e; no scan over $r < x$ is needed. Compute the maximum N-window count defining k_N by a $2^N \operatorname{poly}(N)$ scan. Bounded simulation computes K_N in that time using polynomial space. These are classical preprocessing steps.

The diagonal geometric series for the source contractions B, C are truncated after $O(N)$ terms. Every translated slot is determined by the same residue counts and cyclic shifts. Address-clock dressing is the controlled slot rotation $D = \bigoplus_x R_0^{h_N(x)}$. Time increment, bit permutations and slot rotation have explicit involutive factorizations. For an increment coordinate, the maps $i \mapsto -i$ and $i \mapsto 1 - i$ multiply to increment; these maps are polynomial-time reversible arithmetic.

The only less immediate permutation is the dressed V. In the D^* gauge it increments address x, applying a fixed slot permutation $P_0 = R_0^{h_N(2^N)}$ only on wraparound. Given a slot string z, enumerate its orbit under P_0, of length $s \le k_N$; choose the least representative z_0 and express $z = P_0^j z_0$. The coordinate $i = x + 2^N j$ is a cycle coordinate modulo $2^N s$. Apply the two involutions above in this coordinate, convert back, and conjugate by D. Orbit search is over at most k_N rotations, not over an exponential cycle. An efficiently computable involution is implemented by compute, swap, and uncompute.

For a diagonal contraction r, the compiler uses

$$u(r) = 2r - 1 + 2i\sqrt{r(1-r)}.$$

Realification is a single controlled rotation. Its two reflection factors are the fixed diagonal reflection and the rotation-plane reflection. Diagonal labels are computed reversibly, the appropriate rotation or phase applied, and the label uncomputed. Projections use a Boolean phase; source flips use controlled bit flips.

The expensive labels have the form $\mathrm{Step}'^{t \bmod 2^j}(\mathrm{Init}(x))$. There are fewer than 2^N steps. The fixed counter machine has constantly many counters, each at most $N/2 + 2^N$, hence configurations use $O(N)$ bits. Direct computation costs $2^N \,\mathrm{poly}(N)$ time and $O(N)$ space. Reversible pebbling [13] implements compute, copy, and uncompute with time $2^{(1+\delta)N}\,\mathrm{poly}(N)$ and $O(N^2)$ work bits, for each fixed $\delta > 0$.

Precision must include square-root sensitivity at endpoints. The uniform inequality $|u(r)-u(s)| = O(\sqrt{|r-s|})$ on $[0,1]$ follows from the square-root inequality and the Lipschitz bound for $r(1-r)$. Use at least $16N + O(\log N)$ fractional bits for contraction values and discard a value 2^{-m} only beyond that precision. This yields source errors $O(2^{-8N})$ after square roots. For example discarding $r = 2^{-(8N+1)}$ would instead cause error $2\sqrt{r}$, too large for 2^{-6N}. Compute angles to a comparable linear precision. A polynomial number of register-controlled rotations is synthesized with polynomial precision overhead [23]; allocating $O(N + \log N)$ synthesis bits to each rotation gives total error at most 2^{-6N}. Arithmetic and reversible classical gates may be implemented exactly in a fixed universal algebraic gate set.

The remaining compiler operations have presentation-dependent constant size: realification, selector Pauli registers, controlled source words, and doubled reflection blocks. In the source's first conjugacy block these have forms $\mathrm{diag}(X_i, I)$ and $\left(\begin{smallmatrix} 0 & X_i \\ X_i & 0 \end{smallmatrix}\right)$; the second uses fixed-length products such as *ab*, *bab*, and *bc*. They require only controlled source reflections and a constant number of ancillary selector bits. Measure each commuting context with a constant number of controlled-reflection tests, and use Lemma 5.1 for each orthogonality context. Forbidden answers caused by gate approximation are mapped to a fixed allowed assignment, changing the losing probability by at most the implementation error. Every context has an allowed assignment by the imported compiler. The exact source defects $O(4^{-N})$ give loss $O(16^{-N})$; adding $O(2^{-6N})$ leaves that scale unchanged. All preprocessing and gate-label generation are covered by the stated bound, with slack absorbing fixed polynomial factors. □

This is an algorithmic resource construction. It is not a supplied executable implementation of the complete compiler, and does not establish polynomial-size measurement circuits. No classical circuit lower bound is inferred from the long history labels.

5.1 Distributed initialization: a conditional algebraic estimate

To specify what a physical layout must preserve, consider two clocks and a row $R_j = \{t_A = j\}$. Let u_{R_j} be uniform on that row and u uniform on the grid. Spectral separation in the Bob coordinate gives

$$(u_{R_j} - u)^* L_2^+ (u_{R_j} - u) = (L_{\mathrm{path}}^+)_{jj}/n \le 1/3. \tag{24}$$

For completeness the centered path diagonal is

$$(L_{\mathrm{path}}^+)_{jj} = \frac{1}{n^2}\left[\sum_{i=1}^{j} i^2 + \sum_{i=1}^{n-1-j} i^2\right],$$

the squared edge-flow sum for injection at j and uniform withdrawal. In particular the corner diagonal maximizes it and equals $(n-1)(2n-1)/(6n) < n/3$.

Suppose a proposed layout supplies q commuting rewound ancilla projections P_j, with a unit penalty on every vertex of their initialization row. Jensen's inequality and positivity give $\|(I - P_j)\,\mathrm{avg}_{R_j}\,\psi\| \le \sqrt{\lambda/n}$. Together with (24),

$$\|(I - P_j)m\| \le (1/\sqrt{n} + 1/\sqrt{3})\sqrt{\lambda}.$$

Since $I - \prod_j P_j \le \sum_j (I - P_j)$, initialization error for all ancillas is at most $q(1/\sqrt{n} + 1/\sqrt{3})^2\lambda$. Include a commuting corner question projection and set

$$B = (1 + \sqrt{G_2})^2 + q(1/\sqrt{n} + 1/\sqrt{3})^2.$$

The proof of (17) now gives the safe coefficient

$$C' = 2(1 + 2\sqrt{B} + 2\sqrt{G_2})^2 + 1/2 = O(q + G_2), \tag{25}$$

and the normalized reference distance is at most $2(1/2 + \sqrt{B})\sqrt{M\lambda}$. These estimates are conditional on exactly the indicated rewound row penalties; they do not build them.

Lemma 5.3 (Padding)**.** *Idle padding to $T \ge h^{3/2}$ makes the initialized variational energy at most h^{-3} for two clocks. If also $T \le h^C$ for a fixed C, then $G_2 = O(\log h)$.*

Proof. The variational input has loss at most one and energy at most $1/M$, where $M = (T+1)^2 > h^3$. The upper bound on T and (20) give the resistance estimate. A lower bound on T alone would not imply the latter conclusion. □

6 Conditional composition into a material

We isolate both assumptions needed to turn a referee cell into a translation-invariant material. Neither follows merely from small positive cell energy or from a general circuit-to-Hamiltonian citation.

Definition 6.1 (Local selection hypothesis)**.** For every classical configuration, intact cells support disjoint K_h and all remaining quantum terms are nonnegative. The area decomposition into wall segments has indicators v_s for divider corners and $i_s \le v_s$ for intact-cell corners, at pitch p_s. Clean matching runs have constant pitch and deficit greater than -1, with zero deficit for a closed run. Open and bad runs can be charged at most six times to each wall violation. Missing intact-cell corners can be charged at most six times to a wall violation and once to a tableau or flag violation. The rules specify these assignments locally.

Lemma 6.2 (Selection from the local hypothesis)**.** *Assume Definition 6.1 and $0 \le e(h) \le \min(1/64, ch^{-3})$. Perfect stripes are selected by Lemma 2.3; at fixed small off-tie g every non-minimizing block has eventual separation at least $7/16$.*

Proof. Write $e_s = e(p_s - 1)$, $e_{\max} = 1/64$ and $f_e^* = \min_p F_g(p) < 0$. The exact area bookkeeping and intact-cell lower bound give energy excess at least

$$\sum_s p_s(F_g(p_s) - f_e^*) + \sum_s (r + e_s)(v_s - 1/p_s) - \sum_s e_s(v_s - i_s) + V_w + V_t + \mathrm{Area}_{W\text{-free}}(-f_e^*).$$

The run hypothesis bounds the second term below by $-6(r+e_{\max})V_w$. The missing-corner hypothesis bounds the third by $-6e_{\max}V_w - e_{\max}V_t$. Thus the defect coefficients are at least $1 - 6r - 12e_{\max} = 7/16$ and $1 - e_{\max} = 63/64$. Clean wrong stripes have positive density separation; at sufficiently large commensurate factorial volume their energy excess exceeds $7/16$, as does the vacuum excess. □

This proves the implication from the local hypothesis. The hypothesis itself is an additional combinatorial rule-level requirement for positive-energy cells.

Definition 6.3 (Referee layout hypothesis)**.** A fixed finite alphabet and rational algebraic on-site/bond rule set must implement, inside each selected $h \times h$ cell, the following. The depth is $N = 2\lfloor (c/2)\log_2 h \rfloor$, the strategies are those of Proposition 5.2, and the padded clocks have $h^{3/2} \le T \le h^C$. There is an exact invariant Cartesian legal propagation space with product rewinding and either the corner initialization in (3) or the commuting row penalties used in (25). The loss and question predicates have fixed-size interface support; initialization, transport and folds have specified nearest-neighbour rules. A local isometric support description gives $\log_2 D_A = O(N^2 + \log T)$ for low-energy cell vectors and their reference states. Unused sites are frozen. The generated classical gate labels fit the cell's time and space budget. All nonlegal sectors have the claimed quantitative separation and the factorized projection/product extension required for any asserted physical-state extraction. The effective extraction constant C'_N is polynomial in N and the reference distance is $\mathrm{poly}(N)\sqrt{M\lambda}$.

For an assertion about Gibbs states one must additionally have a genuine bipartite tensor-product state space and a proved bounded cut norm. An abstract globally conditioned legal subspace alone does not supply these thermal hypotheses.

Theorem 6.4 (Conditional halting-game cell coefficient)**.** *Assume Definitions 6.1 and 6.3, with $c = 3/4$, the admitted dimension-matched halting-game strategies, and the specified competing/frozen margins. In every selected cell, every pure ground vector satisfies*

$$S_A = 4\alpha_K N + o(N) = 3\alpha_K \log_2(1/g) + o(\log(1/g)). \tag{26}$$

For any marginal ρ_c supported on that cell ground space,

$$E_D^{A\to B}(\rho_c) = 4\alpha_K N + o(N), \qquad S((\rho_c)_A) = 4\alpha_K N + o(N). \tag{27}$$

The coefficient $3\alpha_K = 3(3+\theta_K)/4$ is Turing-equivalent to K. This is a conditional cell theorem; no unconditional translation-invariant implementation is asserted.

Proof. The initialized test strategy has target loss budget $\varepsilon_N = C_1 16^{-N}$, so $Me(h) \le \varepsilon_N$ and Lemma 5.3 gives $e(h) \le h^{-3}$. Freeze the finitely many cells below the fixed threshold needed for $e(h) \le 1/64$. Lemma 6.2 and the assumed margins then give selected-block transfer. For every pure cell ground vector, extraction yields

$$S_A \ge d^{\rightarrow}_{G_K}(C'_N \varepsilon_N) = 4\alpha_K N + o(N),$$

because $\log_2 C'_N = O(\log N)$ and $\log_2(1/\varepsilon_N) = 4N + O(1)$. The actual optimal loss is only upper bounded by this target budget and is not identified with it. The normalized reference distance is $\mathrm{poly}(N)4^{-N}$ and the assumed support logarithm is polynomial. Proposition 4.4, or its row version, gives

$$S_A \le \log_2(c_0 d_N) + \log_2 |X| + o(1) = 4\alpha_K N + o(N).$$

This proves the pure squeeze.

For the mixed marginal, the loss operator inequality gives the same lower usable price. Uniformity of the projection estimate over the ground space bounds its weight outside the reference subspace by $\mathrm{poly}(N)16^{-N}$. Project and normalize the density matrix. The gentle projection bound gives trace distance $\mathrm{poly}(N)4^{-N}$; its reference Alice support has dimension at most $|X|c_0 d_N$. Entropy continuity with the same actual support bound gives

$$4\alpha_K N + o(N) \le E_D^{A\to B}(\rho_c) \le S((\rho_c)_A) \le 4\alpha_K N + o(N).$$

The middle inequality is the standard upper bound of distillation by local entropy. The last entropy need not itself be within-cell entanglement when ρ_c is mixed. This argument does not imply entropy additivity for arbitrary unions of cells.

Lemma 2.3 gives $\log_2 h = \log_2(1/g) + O(1)$; the even depth rounding changes N by at most two, so (26) follows. An oracle for K computes θ_K by finite sums and the geometric tail. Conversely subtract known contributions below e and put $w_e = 4^{-(e+1)}$. If $e \notin K$ the residual is at most $w_e/3$; if $e \in K$ it is at least w_e. Accuracy $w_e/6$ separates the two cases, inductively computing K. Rational affine rescaling preserves this Turing degree. □

The resource inequalities motivate the hypothesis but do not prove its rules. For $N = (3/4)\log_2 h + O(1)$ and small fixed $\delta < 1/3$, Proposition 5.2 gives generation time $o(h)$ after absorbing its polynomial factor. Its $O(N^2)$ work qubits and padded length $h^{3/2}$ use $h^{3/2}\,\mathrm{poly}(\log h) = o(h^2)$ time-space area. Exact legal propagation after folding, local endpoint access, distributed initialization, and label packing still require an actual rule table. The complete layout is a plausible open construction problem, stated once here. We do not extend this conditional result to every game with a limiting price coefficient.

7 The finite-dimensional correlation ceiling

Let $\mathcal{A}_X, \mathcal{A}_Y$ be disjoint quasi-local spin algebras, each the norm closure of finite-site matrix algebras, with finite local site dimension. A state may be mixed or an arbitrary infinite-volume state. Measurement families are finite and their effects belong to these quasi-local algebras.

Proposition 7.1 (Quasi-local correlations are approximable)**.** *Every such correlation belongs to $C_{qa} = \overline{C_q}$, the closure of finite-dimensional tensor-product correlations. If $\omega_{qa}(G) < 1$, its loss in G is at least $1 - \omega_{qa}(G)$. In a legal referee cell this implies*

$$e \geq \frac{1 - \omega_{qa}(G)}{C_k M}.$$

Proof. Choose increasing finite subsets of X and conditional expectations onto their matrix algebras, using normalized trace on the omitted sites. These maps are unital completely positive, preserve a POVM's positivity and sum, and converge in norm on every quasi-local observable. Do the same independently in Y. Restrict the state to the resulting finite tensor product of matrix algebras; it is a density matrix. Thus every approximating correlation is in C_q. Norm convergence of the finitely many effects gives convergence of all joint probabilities and proves membership in its closure. A Bell functional is continuous on this finite probability space, proving the loss floor. Compose the floor with extraction for the energy inequality. □

This standard approximation principle [48] does not depend on a pure-state infinite-volume tensor splitting. General commuting operator correlations need not admit these quasi-local measurements; the distinction between C_{qa} and C_{qc} is substantive [37]. The cell energy floor retains its normalization $C_k M$; it need not be bounded away from zero as the clock grows.

8 Whole-cell perturbations

All perturbations in this section are arbitrary Hermitian operators on the legal whole-cell space, bounded in total norm. A rank-one operator on a history is generally nonlocal in the physical lattice. No random or generic noise ensemble is being assumed.

First, there can be no universal saturation of entanglement for every norm-η perturbation. For example $V = \eta H/\|H\|$ preserves the ground space and all its entropies. A positive perturbation annihilating that space gives another example. The energy comparison of Corollary 4.5 remains a lower certificate for every perturbation, but does not by itself supply an upper bound.

Proposition 8.1 (A collapse perturbation)**.** *Suppose an honest uniform history Φ_0 of a product system input has loss $\ell_0 \leq 1$, so its entropy is at most $\log_2 |X|$ and its energy is ℓ_0/M. For $V = -\eta|\Phi_0\rangle\langle\Phi_0|$, every unit ground vector Ψ obeys*

$$|\langle\Phi_0, \Psi\rangle|^2 \geq 1 - \frac{\ell_0}{M\eta}. \tag{28}$$

If $L_A = \log_2 D_A$ is an actual local support logarithm and $M\eta/L_A^2 \to \infty$, then $S_A(\Psi) \leq \log_2 |X| + o(1)$.

Proof. Positivity of H and the test vector give

$$-\eta|\langle\Phi_0, \Psi\rangle|^2 \leq E_0(H + V) \leq \ell_0/M - \eta.$$

Rearrange to prove (28). The pure-state trace distance is at most $\sqrt{\ell_0/(M\eta)}$. Apply entropy continuity with the specified support dimension. Its error is bounded by that distance times L_A, plus binary entropy, and hence tends to zero. □

The condition involves dimension as well as M. In the conditional layout, $L_A = \mathrm{poly}(N)$ and M grows exponentially in N, so fixed $\eta > 0$ suffices. Mere divergence of M without dimension control would not suffice.

We next specify a different cell: a free-depth menu. For each admissible even depth $N_0 \leq j \leq N$, it contains a classical sector running the admitted dimension-matched strategy at depth j, all with the same padded clock volume M. Unused system registers are frozen locally, and consistent depth labels are locally available to both players. The extraction uses a product rewinding on the whole direct sum and a uniform coefficient C'_N polynomial in N. The menu has actual local support logarithm $L_A = \mathrm{poly}(N)$. Assume the uniform reference estimates and the depth-wise dimension law $\log_2 d_j = 4\alpha j + o(j)$; honest histories have entropy at most $4\alpha j + o(j) + O(1)$ and target loss $\varepsilon_j = C_1 16^{-j}$. The same fixed game has both prices with coefficient $\alpha > 0$. Implementing this menu in a local material is an extra layout obligation.

Define $S_{\mathrm{worst}}(N, \eta)$ as the infimum of $S_A(\Psi)$ over all allowed V with $\|V\| \leq \eta$ and all unit pure ground vectors Ψ of $H + V$.

Theorem 8.2 (Worst-case free-depth law)**.** *In this menu model, put $a = M\eta$ and $r_N = \log_2(1/\max(\varepsilon_N, a))$. In joint limits with $r_N/\log(N+2) \to \infty$,*

$$S_{\mathrm{worst}}(N, \eta) = \alpha r_N(1 + o(1)). \tag{29}$$

For every perturbation the lower bound holds. The saturation upper bound uses a chosen rank-one whole-cell perturbation. In the separate collapse regime $M\eta \gg L_A^2$, Proposition 8.1 gives the bounded upper entropy.

Proof. The deepest admitted history has energy at most ε_N/M. For every perturbed ground vector, the energy comparison and uniform extraction yield

$$S_A \geq d_G^{\rightarrow}\left(C'_N[\varepsilon_N + 2M\eta]\right).$$

In the stated regime the effective loss tends to zero and its negative logarithm is $r_N + O(\log N)$. The imported price therefore gives $S_A \geq \alpha r_N(1 - o(1))$.

We first justify the unperturbed upper bound for the menu. The system dimension argument of Proposition 4.6 in sector j gives

$$e_j \geq \frac{16^{-j+o(j)}}{\mathrm{poly}(N)M}.$$

Any fixed bounded sector also has positive fixed dimension-constrained loss, because the game's divergent price forbids a perfect strategy in that dimension. For each fixed $\delta > 0$, sectors $j \leq (1-\delta)N$ are eventually above ε_N/M. Thus every unperturbed minimizing sector has $j = N - o(N)$. Sector-wise normalized reference estimates squeeze its entropy to $4\alpha N + o(N)$. Superpositions of at most $O(N)$ locally distinguished sectors contribute at most $\log_2 O(N)$ additional entropy. Uniform continuity errors still vanish on the assumed support, proving the unperturbed upper bound. In particular one can always choose a pure unperturbed ground vector with this bound.

If $a \leq \varepsilon_N$, choose $V = 0$ for the desired upper bound. If $a > \varepsilon_N$, fix $p > 2$ and choose the *smallest* admissible even depth j satisfying

$$\varepsilon_j \leq a L_A^{-p}. \tag{30}$$

When such a depth exists, the logarithmic regime ensures that it eventually exceeds the fixed lower cutoff and

$$4j = \log_2(1/a) + p \log_2 L_A + O(1).$$

Take its honest history Φ_j and $V = -\eta|\Phi_j\rangle\langle\Phi_j|$. The same variational argument as (28) gives overlap at least $1 - L_A^{-p}$ with every perturbed ground vector. Entropy continuity costs $O(L_A^{1-p/2}) + h_2(L_A^{-p/2}) = o(1)$. Therefore

$$S_A \leq 4\alpha j + o(j) + O(1) = \alpha \log_2(1/a)(1 + o(1)),$$

since $\log L_A = O(\log N) = o(r_N)$. If (30) would require $j > N$, then $1 < a/\varepsilon_N < L_A^p$ up to a fixed endpoint factor. In that case $V = 0$ again gives the upper bound, as the difference in the relevant negative logarithms is only $O(\log N) = o(r_N)$. This handles the precision floor and completes the saturation proof. □

The effective variable is $M\eta$, together with the extraction coefficient. In a noise-dominated logarithmic regime, changing T at fixed total η and fixed k inserts the term $-k\alpha \log_2(T+1)$. This observation is confined to that regime and this menu model. Open-clock perturbation analyses such as [51] concern their specified dynamics and do not establish a local-noise version of (29).

9 Temperature, preparation time, and work

The mixed-state certificate (5) applies to any legal state, including a Gibbs state when the legal space is a genuine bipartite tensor product. Its composition with established physical inequalities gives necessary resource bounds. We set $k_B = \hbar = 1$. We use [20]

$$E_D^{A\to B}(\rho) \leq E_D(\rho) \leq E_{\mathrm{sq}}(\rho) \leq I(A:B)_\rho/2. \tag{31}$$

The last inequality uses the trivial extension in the definition of squashed entanglement. All quantities in (31) are in bits.

Theorem 9.1 (Necessary thermal, temporal, and work budgets)**.** *Suppose a finite bipartite state plays G at loss ℓ. For the Gibbs state of $H = H_A + H_B + H_\partial$ at inverse temperature β,*

$$\beta\|H_\partial\| \geq (\ln 2) d_G^{\rightarrow}(\ell). \tag{32}$$

For unitary preparation from a product state, with time-dependent cut interaction on fixed interacting dimensions, let d_∂ be the smaller interacting local dimension. Then

$$18 \int_0^t \|H_\partial(s)\| \log_2 d_\partial \, ds \geq d_G^{\rightarrow}(\ell). \tag{33}$$

For average-work isothermal preparation at temperature Θ from $\gamma_A \otimes \gamma_B$, with the noninteracting Hamiltonian $H_A + H_B$ restored at the endpoint and no uncharged resource,

$$\mathcal{W} \geq 2\Theta \ln 2 \, d_G^{\rightarrow}(\ell). \tag{34}$$

If $d_G^{\rightarrow}(\ell) = \alpha_G \log_2(1/\ell) + o(\log(1/\ell))$, the same imported coefficient appears in the right sides. These are lower bounds; no matching preparation protocol is asserted.

Proof. For the Gibbs state ρ, put $\sigma = \rho_A \otimes \rho_B$. Free-energy minimality gives

$$\beta^{-1} I_{\text{nats}}(A:B)_\rho \leq \operatorname{tr}[H(\sigma - \rho)] = \operatorname{tr}[H_\partial(\sigma - \rho)] \leq 2\|H_\partial\|.$$

The equality cancels local Hamiltonian terms using equal marginals. This is the thermal mutual-information bound of [56]. Equation (31) and $E_D^{A\to B}(\rho) \geq d_G^{\rightarrow}(\ell)$ prove (32), with the factor $\ln 2$ converting bits to nats.

For pure product input, the small incremental entangling theorem [53], with constant 18, bounds the entropy derivative by $18\|H_\partial(s)\| \log_2 d_\partial$, independent of spectator dimensions. Integrate from zero initial entropy. For mixed product input, purify each party locally, evolve with those purifying registers as spectators, and use monotonicity under local discarding. This bounds the final distillable entanglement by the same integrated entropy, and gives (33). If $\|H_\partial\| \leq J$, it yields $t \geq d_G^{\rightarrow}(\ell)/(18J \log_2 d_\partial)$. Pre-existing shared entanglement or an increasing interface is outside this hypothesis.

For work, the stipulated free-energy accounting gives

$$\mathcal{W} \geq F(\rho) - F(\gamma_A \otimes \gamma_B) = \Theta D(\rho \| \gamma_A \otimes \gamma_B),$$

with natural relative entropy. Its exact decomposition is

$$\begin{aligned} D(\rho \| \gamma_A \otimes \gamma_B) &= I_{\text{nats}}(A:B)_\rho + D(\rho_A \| \gamma_A) + D(\rho_B \| \gamma_B) \\ &\geq I_{\text{nats}}(A:B)_\rho. \end{aligned}$$

Apply (31). This is the standard correlation work bound [36], with the game's usable demand inserted. □

Subtracting a scalar from the cut interaction can improve its norm without changing the argument. The bounds apply to the true physical tensor product; a nonproduct conditioning on legality cannot silently replace that state space. The work statement is not a general equality between creation work and any local-reset heat convention.

Corollary 9.2 (Fixed-temperature and fixed-time coefficients)**.** *For Gibbs states with a uniform cut norm $\|H_\partial\| \leq J$, every nonnegative entanglement quantity $E \leq I/2$ satisfies*

$$0 \leq E \leq \beta J / \ln 2.$$

At fixed positive temperature its coefficient after division by $\log_2(1/g)$ is zero. The same conclusion holds at fixed preparation time from product input when the integrated interface rate is uniformly bounded. For a two-corner referee with $J \leq 2$, the thermal bound is $E \leq 2\beta/\ln 2$.

Proof. The first bound is the Gibbs mutual-information inequality. For preparation, local purifications give $I(A:B)/2 \le S(A \text{ with local purifier})$, since mutual information decreases under local discarding. The integrated rate therefore bounds every $E \le I/2$. Divide either uniform bound by the divergent logarithm. □

No gap assumption is needed here. Thermal marginal entropy includes local thermal entropy and is not covered. Nor are all possible mixed entanglement measures, growing interfaces, or fluctuating-wall Gibbs states without the specified cut and state convention. An uncomputable coefficient may appear in these necessary precision inequalities without being a proved optimal resource rate or the only observable that retains it.

9.1 A finite-dimensional Gibbs energy estimate

For any finite legal Gibbs Hamiltonian of dimension D and minimum energy e_0, free-energy minimality against a pure ground state gives

$$\operatorname{tr}(\rho_\Theta H) \le e_0 + \Theta \ln D. \tag{35}$$

Thus the extraction certificate has effective precision at most $C'[\varepsilon_N + M\Theta \ln D]$ when $Me_0 \le \varepsilon_N$. To retain a depth-N leading coefficient in this lower certificate it suffices that

$$-\log_2(C' M\Theta \ln D) \ge 4N - o(N)$$

and $\log_2 C' = o(N)$, or more restrictively $M\Theta \ln D = O(\varepsilon_N)$. Smallness relative only to an inverse polynomial of depth does not imply this exponential precision. No claim of a thermal material region follows without controlling the full physical dimension, illegal sectors, and wall distribution.

9.2 Thermal pair correlations

The Bell-pair cell provides an exact comparison at the level of mutual information. For a pair (12), let $d = 2^{b_0}$ and

$$\rho_\beta = \frac{I + (e^\beta - 1)|\Phi_d\rangle\langle\Phi_d|}{Z}, \qquad Z = e^\beta + d^2 - 1, \qquad F = e^\beta/Z.$$

Both marginals are I/d. The mutual information in bits is

$$i_{b_0}(\beta) = 2\log_2 d + F\log_2 F + (1-F)\log_2 \frac{1-F}{d^2-1}. \tag{36}$$

It is computable for computable β, positive for every $\beta > 0$, and tends to $2b_0$ as $\beta \to \infty$. Positivity follows because equality with the product of the maximally mixed marginals occurs only at $\beta = 0$. Independent disjoint pairs therefore give $I(R_c : R_c^c) = m(h) i_{b_0}(\beta)$ at every fixed positive inverse temperature. The normalized half-mutual-information coefficient is the programmed coefficient times $i_{b_0}(\beta)/(2b_0)$.

This is not all-temperature entanglement survival. These isotropic states are separable exactly when

$$F \le 1/d \quad \Longleftrightarrow \quad e^\beta \le d + 1. \tag{37}$$

For the separable direction, twirl product states by $U \otimes \overline{U}$; their Bell overlap ranges over $[0, 1/d]$ and determines the isotropic state. For the other direction, partial transpose has an antisymmetric eigenvalue $[1 - (e^\beta - 1)/d]/Z < 0$. This is consistent with the isotropic-state criterion in [35]. In particular, for $\Theta \ge 1/\ln(d+1)$, every pair and their product are separable despite nonzero mutual information. The comparison concerns total correlations. Its cut norm grows with the number of pairs and hence does not satisfy the bounded-interface hypothesis of Corollary 9.2.

10 Scope and further questions

The explicit gapped construction programs a logarithmic entropy coefficient of a growing sub-cell region, and a power exponent of its Schmidt rank. It does not classify every entanglement observable in every gapped spin system. The arithmetic ceiling requires effective algebraic cell data, a specified region, and a specified state rule. Our correlation ceiling requires norm-quasi-local measurement effects.

The referee theorem concerns an exact legal-space operator with local rewinding. A physical implementation must prove its legality, initialization, transport and interface properties. For positive cell energies it must also prove the local charging hypothesis. Standard history constructions make these requirements meaningful but do not automatically discharge them. The conditional theorem records exactly what the game price would then force, without asserting that this particular material has been constructed.

The perturbation theorem concerns a specified free-depth cell and arbitrary total-norm operators. The resource bounds are necessary inequalities. The thermal pair calculation retains mutual information; it supplies no all-temperature entanglement claim. General robust exponents, generic local-noise saturation, and matching physical preparation costs are not conclusions of this paper. Further questions include quantitative protection conditions for specified microscopic designs and sharper extraction constants under additional circuit or ground-state assumptions. No conjectural theorem is used in any proof.

11 How this work was done

The authors developed the mathematical argument with assistance from AI models from more than one developer. They take responsibility for the claims, the scope of the imports, and the distinction between a proof, a conditional implication, and a finite numerical check. The manuscript uses the released ruler and game theorems at the numbered locations stated in Section 1.1. Its new arguments are written above; those imported proofs are not reproduced.

The ancillary directory contains a sequential runner, source files, recorded outputs, a summary with measured peak memory, and a SHA-256 manifest. Four supplied checks are retained for the elementary two-clock inequality, perfect-stripe arithmetic and pair samples, two-clock resistance extraction, and three-clock extraction. Their numerical calculations use floating-point eigenvalues. A description in the pair-sampling script is restricted to the finite scope actually checked. Independent checks test rational staircase identities, finite-grid pseudoinverses, the extraction operator inequality for complex projection data, normalized reference estimates, mirror-field formulas, the depth-selection rule, and thermal-pair formulas.

Scripts run sequentially with numerical-library thread counts set to one and a monitored combined process-memory ceiling below one gigabyte. Passing these checks is finite consistency evidence. It does not certify an uncomputable limiting real, a translation-invariant rule table for the referee, the local positive-energy charging hypothesis, or any statistical claim about local noise. The analytic proofs and their stated hypotheses govern all infinite-volume conclusions.

References

[1] Dorit Aharonov, Daniel Gottesman, Sandy Irani, and Julia Kempe. The power of quantum systems on a line, 2009. Communications in Mathematical Physics 287, 41–65; arXiv:0705.4077.

[2] Dorit Aharonov, Wim van Dam, Julia Kempe, Zeph Landau, Seth Lloyd, and Oded Regev. Adiabatic quantum computation is equivalent to standard quantum computation, 2007. SIAM Journal on Computing 37, 166–194; arXiv:quant-ph/0405098.

[3] Serge Aubry. The twist map, the extended Frenkel–Kontorova model and the devil's staircase, 1983. Physica D 7, 240–258. doi:10.1016/0167-2789(83)90129-X.

[4] Koenraad M. R. Audenaert. A sharp continuity estimate for the von Neumann entropy, 2007. Journal of Physics A 40, 8127–8136; arXiv:quant-ph/0610146.

[5] Per Bak and Rob Bruinsma. One-dimensional Ising model and the complete devil's staircase, 1982. Physical Review Letters 49, 249–251. doi:10.1103/PhysRevLett.49.249.

[6] Paul Balister, Béla Bollobás, Robert Morris, and Paul Smith. The critical length for growing a droplet, 2025. Memoirs of the AMS 310, no. 1571; arXiv:2203.13808v3.

[7] Paul Balister, Béla Bollobás, Robert Morris, and Paul Smith. Universality for monotone cellular automata, 2026. Journal of the European Mathematical Society 28; arXiv:2203.13806v3.

[8] Johannes Bausch and Elizabeth Crosson. Analysis and limitations of modified circuit-to-hamiltonian constructions, 2018. Quantum 2, 94; arXiv:1609.08571v3. doi:10.22331/q-2018-09-19-94.

[9] Johannes Bausch, Toby S. Cubitt, Angelo Lucia, and David Pérez-García. Undecidability of the spectral gap in one dimension, 2020. Physical Review X 10, 031038. doi:10.1103/PhysRevX.10.031038.

[10] Johannes Bausch, Toby S. Cubitt, Angelo Lucia, David Pérez-García, and Michael M. Wolf. Size-driven quantum phase transitions, 2018. PNAS 115, 19–23. doi:10.1073/pnas.1705042114.

[11] Johannes Bausch, Toby S. Cubitt, and James D. Watson. Uncomputability of phase diagrams, 2021. Nature Communications 12, 452. doi:10.1038/s41467-020-20504-6.

[12] Aleksandrs Belovs and Ben W. Reichardt. Span programs and quantum algorithms for st-connectivity and claw detection, 2012. arXiv:1203.2603.

[13] Charles H. Bennett. Time/space trade-offs for reversible computation, 1989. SIAM Journal on Computing 18, 766–776. doi:10.1137/0218053.

[14] Charles H. Bennett, Herbert J. Bernstein, Sandu Popescu, and Benjamin Schumacher. Concentrating partial entanglement by local operations, 1996. Physical Review A 53, 2046–2052; arXiv:quant-ph/9511030.

[15] Sergey Bravyi. Upper bounds on entangling rates of bipartite hamiltonians, 2007. Physical Review A 76, 052319; arXiv:0704.0964.

[16] Nikolas P. Breuckmann and Barbara M. Terhal. Space-time circuit-to-hamiltonian construction and its applications, 2014. Journal of Physics A 47, 195304; arXiv:1311.6101.

[17] Libor Caha, Zeph Landau, and Daniel Nagaj. Clocks in Feynman's computer and Kitaev's local hamiltonian: bias, gaps, idling, and pulse tuning, 2018. Physical Review A 97, 062306; arXiv:1712.07395.

[18] Laura Castilla-Castellano and Angelo Lucia. Instability of the undecidable behavior of the spectral gap in 1d, 2026. arXiv:2607.08686v1.

[19] Laura Castilla-Castellano and Angelo Lucia. Undecidability of the spectral gap in rotationally symmetric hamiltonians, 2026. Physical Review Research 8, 013033; arXiv:2410.13589v2.

[20] Matthias Christandl and Andreas Winter. Squashed entanglement: an additive entanglement measure, 2004. Journal of Mathematical Physics 45, 829–840; arXiv:quant-ph/0308088.

[21] József Cserti. Application of the lattice Green's function for calculating the resistance of an infinite network of resistors, 2000. American Journal of Physics 68, 896–906; arXiv:cond-mat/9909120.

[22] Toby S. Cubitt, David Pérez-García, and Michael M. Wolf. Undecidability of the spectral gap, 2022. Forum of Mathematics, Pi 10, e14; full proof of the 2015 Nature announcement. doi:10.1017/fmp.2021.15.

[23] Christopher M. Dawson and Michael A. Nielsen. The Solovay–Kitaev algorithm, 2006. Quantum Information and Computation 6, 81–95; arXiv:quant-ph/0505030.

[24] Seth Douglas and Nidhal Mghirbi. The price of precision: usable entanglement in nonlocal games and the geometry of groups, 2026. Version 1.1.0. Zenodo. doi:10.5281/zenodo.23753349.

[25] Seth Douglas and Nidhal Mghirbi. Uncomputable critical exponents, 2026. Version 2.0.0. Zenodo. doi:10.5281/zenodo.23757361.

[26] Seth Douglas and Nidhal Mghirbi. The uncomputable price of precision, 2026. Version 4.0.0. Zenodo. doi:10.5281/zenodo.23753741.

[27] Rodney G. Downey, Carl G. Jockusch, Jr., and Paul E. Schupp. Asymptotic density and computably enumerable sets, 2013. Journal of Mathematical Logic 13; arXiv:1307.0040.

[28] Peter G. Doyle and J. Laurie Snell. Random walks and electric networks, 1984. Mathematical Association of America; arXiv:math/0001057.

[29] Bruno Durand, Andrei Romashchenko, and Alexander Shen. Fixed-point tile sets and their applications, 2012. Journal of Computer and System Sciences 78, 731–764; arXiv:0910.2415v7.

[30] Hamza Fawzi, Omar Fawzi, and Samuel O. Scalet. Certified algorithms for equilibrium states of local quantum hamiltonians, 2023. arXiv:2311.18706.

[31] Richard P. Feynman. Quantum mechanical computers, 1986. Foundations of Physics 16, 507–531. doi:10.1007/BF01886518.

[32] Michael E. Fisher and Walter Selke. Infinitely many commensurate phases in a simple Ising model, 1980. Physical Review Letters 44, 1502–1505. doi:10.1103/PhysRevLett.44.1502.

[33] Daniel Gottesman and Sandy Irani. The quantum and classical complexity of translationally invariant tiling and hamiltonian problems, 2013. Theory of Computing 9, 31–116; arXiv:0905.2419. doi:10.4086/toc.2013.v009a002.

[34] Michael Hochman and Tom Meyerovitch. A characterization of the entropies of multidimensional shifts of finite type, 2010. Annals of Mathematics 171, 2011–2038. doi:10.4007/annals.2010.171.2011.

[35] Michał Horodecki and Paweł Horodecki. Reduction criterion of separability and limits for a class of protocols of entanglement distillation, 1999. Physical Review A 59, 4206–4216; arXiv:quant-ph/9708015.

[36] Marcus Huber, Martí Perarnau-Llobet, Karen V. Hovhannisyan, Paul Skrzypczyk, C. Klöckl, Nicolas Brunner, and Antonio Acín. Thermodynamic cost of creating correlations, 2015. New Journal of Physics 17, 065008; arXiv:1404.2169.

[37] Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. MIP* = RE, 2026. arXiv:2001.04383v4, 16 October 2026; first posted 2020.

[38] Tosio Kato. Perturbation theory for linear operators, 1995. Springer, reprint of second edition. doi:10.1007/978-3-642-66282-9.

[39] Alexei Yu. Kitaev, Alexander H. Shen, and Mikhail N. Vyalyi. Classical and quantum computation, 2002. Graduate Studies in Mathematics 47, American Mathematical Society. doi:10.1090/gsm/047.

[40] Tom Meyerovitch. Growth-type invariants for $\mathbb{Z}^d$ subshifts of finite type and arithmetical classes of real numbers, 2011. Inventiones Mathematicae 184, 567–589. doi:10.1007/s00222-010-0296-1.

[41] Jacek Miękisz. Stable quasicrystalline ground states, 1993. arXiv:cond-mat/9301017v2.

[42] Jacek Miękisz. Quasicrystal problem: on rigidity of non-periodic structures from statistical mechanics point of view, 2024. arXiv:2412.19594.

[43] Ramis Movassagh and Peter W. Shor. Supercritical entanglement in local systems: counterexample to the area law for quantum matter, 2016. PNAS 113, 13278–13282; arXiv:1408.1657.

[44] V. L. Pokrovsky and A. L. Talapov. Ground state, spectrum, and phase diagram of two-dimensional incommensurate crystals, 1979. Physical Review Letters 42, 65–67. doi:10.1103/PhysRevLett.42.65.

[45] James Purcell, Zhi Li, and Toby S. Cubitt. Chaitin phase transition, 2024. arXiv:2410.02600v1.

[46] Olof Salberger, Takuma Udagawa, Zhao Zhang, Hosho Katsura, Israel Klich, and Vladimir Korepin. Deformed Fredkin spin chain with extensive entanglement, 2017. Journal of Statistical Mechanics, 063103; arXiv:1611.04983.

[47] Christof Schmidhuber. Chaitin's Omega and an algorithmic phase transition, 2022. Physica A 586, 126458; arXiv:1909.09231v3.

[48] Volkher B. Scholz and Reinhard F. Werner. Tsirelson's problem, 2008. arXiv:0812.4305.

[49] William Slofstra and Thomas Vidick. Entanglement in non-local games and the hyperlinear profile of groups, 2018. Annales Henri Poincaré 19, 2979–3005; arXiv:1711.10676.

[50] Siamak Taati and Jacek Miękisz. Quasicrystalline gibbs states in 4-dimensional lattice-gas models with finite-range interactions, 2025. arXiv:2512.24436v1.

[51] David G. Tempel and Alan Aspuru-Guzik. Feynman's clock for open quantum systems, 2014. New Journal of Physics 16, 113066; arXiv:1406.5631.

[52] Naïri Usher, Matty J. Hoban, and Dan E. Browne. Non-unitary quantum computation in the ground space of local hamiltonians, 2017. Physical Review A 96, 032321; arXiv:1703.08118.

[53] Karel Van Acoleyen, Michaël Mariën, and Frank Verstraete. Entanglement rates and area laws, 2013. Physical Review Letters 111, 170501; arXiv:1304.5931.

[54] James D. Watson and Johannes Bausch. The complexity of approximating critical points of quantum phase transitions, 2021. arXiv:2105.13350.

[55] James D. Watson, Emilio Onorati, and Toby S. Cubitt. Uncomputably complex renormalisation group flows, 2022. Nature Communications 13, 7618. doi:10.1038/s41467-022-35179-4.

[56] Michael M. Wolf, Frank Verstraete, Matthew B. Hastings, and J. Ignacio Cirac. Area laws in quantum systems: mutual information and correlations, 2008. Physical Review Letters 100, 070502; arXiv:0704.3906.

[57] Xiangling Xu, Matthias Schötz, Jie Wang, Victor Magron, Igor Klep, Omar Fawzi, and Marc-Olivier Renou. The bulk spectral gap is certifiable from above but uncomputable from below, 2026. arXiv:2606.03836v4, 15 October 2026.

[58] Zhao Zhang, Amr Ahmadain, and Israel Klich. Novel quantum phase transition from bounded to extensive entanglement, 2017. PNAS 114, 5142–5146; arXiv:1606.07795.

[59] Xizhong Zheng and Klaus Weihrauch. The arithmetical hierarchy of real numbers, 2001. Mathematical Logic Quarterly 47, 51–65.