Non-soficity of OpenAI's torsion-free hyperbolic group

Seth Douglas Nidhal Mghirbi

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Abstract

Is every word-hyperbolic group sofic? OpenAI recently constructed a torsion-free word-hyperbolic group G that is not residually finite, with a Lean formalization. We show that G is not sofic. Their rank argument against finite quotients uses a quotient only through its permutation matrices, and it survives approximation: a relation that fails on a small fraction of points costs a small rank error, and a word with few fixed points costs a small trace error. So approximate permutation models are excluded at a fixed positive accuracy, while soficity would supply them at every accuracy; moreover, one fixed nonidentity element of G dies in every sofic quotient. We formalize the conclusion in Lean on top of OpenAI's development. Sandhu and Thom independently found a different non-sofic torsion-free hyperbolic group by the same kind of rank count. The same rank count, with group elements in place of permutation matrices, excludes Linnell division rings: if K is a division ring of the characteristic q fixed by the construction and Λ is any group containing G, then $K[\Lambda]$ has no Linnell division ring. In such a division ring, the defining independence conditions inside certain free subgroups of G would give the relevant matrices full rank, while OpenAI's finite-field interpolation identity forces their rank to collapse; no zero divisor is exhibited. OpenAI's torsion-free groups whose group algebras over $\mathbb{F}_2$ have zero divisors, or are not directly finite, already show that Linnell division rings need not exist in characteristic two; here the group is G itself. We also show that G is the free product of a free group and a torsion-free hyperbolic group that again has no Linnell division ring in characteristic q, and we prove barriers to extending the argument to hyperlinearity. The group and every input to the non-soficity argument are OpenAI's.

The authors developed this work with AI models from more than one developer and take responsibility for every claim. How the work was done, and what is machine-checked, is described at the end of the note.

1 Introduction

A group is *sofic* if every finite piece of its multiplication table can be modelled by permutations of a nonempty finite set, with nonidentity elements moving almost every point [20, 57, 13]. Residually finite groups and amenable groups are sofic. Non-sofic groups were first constructed in 2026 [44], building on Kun's expander decomposition for property (T) groups [26] and the centralizer obstruction of Kun and Thom [27]; the argument of this note uses neither property (T) nor expanders. Further examples followed [17, 28, 51], among them OpenAI's finitely presented groups whose group algebras over a field of characteristic two are not directly finite [42, 41], which are not sofic by a theorem of Elek and Szabó [12, Corollary 4.7], as noted there. These works do not establish a non-sofic word-hyperbolic group. Chapman and Peled showed that a non-sofic hyperbolic group exists if random two-dimensional Linial–Meshulam complexes in a middle range of triangle densities typically have linear cocycle stability in permutations [9]. The hyperbolic group admitting a nonsofic probability-measure-preserving action noted in [28, Section 1], obtained from [44] and [2,

Corollary E], is a different assertion: nonsoficity of an action does not establish nonsoficity of its acting group. Whether every word-hyperbolic group is sofic remained open: Ozawa asked whether every hyperbolic group is hyperlinear [45, Introduction and Section 7], and Pestov records the question for soficity as well [46, Open question 9.2]. Arzhantseva conjectured that all hyperbolic groups are residually finite if and only if all hyperbolic groups are sofic [5, Conjecture 2.8]; see also [6, Conjecture 6.6].

OpenAI then constructed a torsion-free word-hyperbolic group G that is not residually finite [39, Theorem 1.1], with a Lean formalization of that theorem [43]. We show that the same group is not sofic. Independently, Sandhu and Thom constructed a different torsion-free word-hyperbolic group that is not sofic [49]; see the end of this introduction.

Theorem 1.1. *The torsion-free word-hyperbolic group G of [39] is not sofic.*

The proof in [39] assumes a finite quotient Q in which a finite set $\mathcal{R}$ of *rectangular words* survives. It then compares two ranks of matrices built from the regular representation of Q: a lower bound from traces, and an upper bound from the defining relations and a finite-field interpolation identity. The comparison contradicts a fixed numerical budget. That argument never uses subgroups, kernels, orders of elements or any other structure of Q. It uses only that the regular representation is a homomorphism into permutation matrices, and that the rectangular words act without fixed points. Both facts have approximate versions, and every step of the comparison degrades by an amount we control. Passing from sofic approximations to normalized ranks over a division ring goes back to Elek and Szabó, who used it to embed the group algebra of a sofic group in a regular ring with a rank function and to deduce stable finiteness [12, Theorem 1 and Corollary 4.7]. Theorem 1.2 states the result with explicit constants.

Write $\mathrm{mov}(\alpha)$ for the fraction of points moved by a permutation α of a finite set, and $\mathrm{fix}(\alpha) = 1 - \mathrm{mov}(\alpha)$. The group G has a finite presentation whose relators are *triangle relators* θ_t together with two relators that the argument never uses (Section 2). The construction depends on a prime q and an integer $r < q$.

Theorem 1.2. *Assign a permutation of a nonempty finite set Ω to every generator of G, and extend the assignment to words. If every rectangular word fixes at most a fraction $1/(r-1)^2$ of Ω, then some triangle relator moves at least a fraction ε_0 of Ω. Here $\varepsilon_0 = 1/(4r^2N(q-1))$ for any data satisfying the budget (2.1), where N is the number of line labels, and $\varepsilon_0 = 1/(6rq)$ for the data of [39, Lemma 2.2].*

Theorem 1.1 follows because soficity supplies such assignments with both defects arbitrarily small (Lemma 4.1). The same argument applied to a sofic image of G gives a stronger conclusion.

Corollary 1.3. *Some fixed rectangular word $w_0 \neq 1$ of G lies in the kernel of every homomorphism from G to a sofic group. In particular every quotient of G in which all rectangular words survive is non-sofic.*

Corollary 1.3 extends a consequence drawn in [39, Corollary 1.2]: there, one fixed nonidentity element of G is killed by every finite-dimensional linear representation over a commutative field. Finitely generated linear groups are residually finite [33] (see also [38, Introduction and §3]), hence sofic [46, Example 4.2], so they are among the targets here. Since LEF, residually amenable and initially subamenable groups are sofic [46, Example 4.5 and the paragraph before it], Theorem 1.1 shows that G is none of these. For LEF this already follows from [39, Theorem 1.1], since G is finitely presented and finitely presented LEF groups are residually finite [56]; and for finitely presented groups, initial subamenability is equivalent to residual amenability [5, proof of Proposition 2.7].

Division rings. The same matrices and the line certificate also obstruct division rings. Let K be a division ring and Λ a group. A *Linnell division ring* for $K[\Lambda]$ is a division ring generated by a copy of $K[\Lambda]$ in which, for every subgroup $C \leq \Lambda$, elements of Λ in distinct right cosets of C are left linearly independent over the division subring generated by $K[C]$ [24, Section 2.3], [16, Definition 2.2]. Those sources state the definition for torsion-free Λ; we use the same condition for every group, and it fails for every group with torsion, since $K[\Lambda]$ then has zero divisors. The independence conditions go back to Hughes [21], and the name to a division ring that Linnell constructed from the group von Neumann algebra [29]; see [24, Section 2.3] and [16, Remark 2.3]. Section 6 recalls the definitions in full. Jaikin-Zapirain and Linton conjectured that for every field K and every torsion-free group Λ, $K[\Lambda]$ has a Linnell division ring, unique up to $K[\Lambda]$-isomorphism [24, Conjecture 1]. Fisher and Sánchez-Peralta state the same conjecture for crossed products over division rings [16, Conjecture 1.2].

Theorem 1.4. *Let K be a division ring of characteristic q, and let Λ be a group containing G. Then $K[\Lambda]$ has no Linnell division ring. Moreover, no division $K[\Lambda]$-ring is strongly Hughes-free in the sense of Linnell [30], as recalled in [24, Section 2.3], and no epic division $K[\Lambda]$-ring satisfies condition (HF) of [16, Definition 2.2(1)].*

Hughes-freeness itself is defined only for locally indicable groups, and we do not know whether G is locally indicable; Theorem 1.4 excludes the independence condition (HF) on which that notion rests. Since G is torsion-free, and data as in Section 2 exist for every sufficiently large prime q [39, Lemma 2.2], the existence part of both conjectures fails in every sufficiently large prime characteristic q, each time for the group G built from that prime.

These conjectures were already refuted in positive characteristic. A Linnell division ring contains $K[\Lambda]$, so $K[\Lambda]$ would have no zero divisors and would be directly finite. OpenAI constructed a finitely presented torsion-free group Λ with nonzero $\alpha, \beta \in \mathbb{F}_2[\Lambda]$ such that $\alpha\beta = 0$ [40, Theorem 1.1], and another for which $\mathbb{F}_2[\Lambda]$ is not directly finite [41, Theorem 1.1]. Either one refutes the existence part of both conjectures over $\mathbb{F}_2$. For every odd prime p, Ivanov constructed finitely presented torsion-free groups whose group algebras over $\mathbb{F}_p$ have zero divisors [23, Theorem 1.1], which refutes them over $\mathbb{F}_p$. In a companion note, building on [40, 41], we show that Kaplansky's group-ring conjectures fail for torsion-free word-hyperbolic groups in every positive characteristic [10]. In particular there are such groups with zero divisors over every $\mathbb{F}_p$; for $p = 2$ the group is that of [40], for a suitable choice of its random data, and for odd p the construction uses its machinery. For the same reason these groups have no Linnell division ring in characteristic p. They are different groups from G. Theorem 1.4 concerns G itself, which is not sofic by Theorem 1.1, and its proof counts ranks instead of exhibiting a zero divisor.

The proof (Section 6) uses the same matrices $Y_i = (x_{iuv})_{u,v\in[r]}$ as Theorem 3.1 below, now with entries in $K[G]$. For each label i, the elements $x_{iuv}x_{i00}^{-1}$ with $(u, v) \neq (0, 0)$ freely generate a subgroup A_i of G, a *line subgroup* (Lemma 6.2). In a Linnell or strongly Hughes-free division ring, left linear independence of the powers of one element of A_i over the division closure of a normal subgroup with infinite cyclic quotient forces every Y_i to have full rank r (Lemma 6.3), while the line certificate forces $\sum_i \operatorname{rank} Y_i < Nr/4$. Whether $\mathbb{F}_q[G]$ embeds in some division ring is open. If it does, $\mathbb{F}_q[G]$ embeds in a division ring but has no Linnell division ring. If it does not, G is a further counterexample to [16, Conjecture 1.1], which predicts such an embedding for every torsion-free group; that conjecture already fails for the groups of [40], [23] and [10], whose group algebras have zero divisors.

Section 6.3 shows that G is the free product of a free group and a finitely presented torsion-free hyperbolic group $G^\sharp$, and that $K[\Lambda]$ has no Linnell division ring whenever Λ contains $G^\sharp$ and K has characteristic q. Consequently neither $G^\sharp$ nor G is bi-orderable. An Euler characteristic count

shows that G embeds in no fundamental group of a two-complex with nonpositive immersions in the sense of Wise [58]. Section 5.1 makes precise what the argument says about hyperlinearity.

Independent work. Sandhu and Thom [49], based on an AI-generated construction obtained by H. Sandhu, give an explicit torsion-free word-hyperbolic group that is not sofic; their argument and ours were found independently. They take from [39] the marked lines, the triangle relations built from Cartesian products, the interpolation identity, and the comparison of real and finite-field ranks. Their real-rank bound is the one in Step 2 of Theorem 3.1, from [4, Theorem 6], and [39, Lemma 2.4] passes it to $\mathbb{F}_q$ [49, Lemmas 3.2 and 3.3]. In place of the random incidence graph they use the explicit graph $D(5, q)$ of Lazebnik and Ustimenko, of girth at least ten, with one block at each point, and they give the complex a locally CAT(-1) metric [49, Theorem A]. Their approximate rank inequality for permutation assignments [49, Theorem 3.1] and their passage from soficity [49, Section 5] follow the same mechanism as Sections 3 and 4. Through a theorem of Kapovich and Wise [25], they also obtain a nontrivial torsion-free word-hyperbolic group with no nontrivial sofic quotient [49, Corollary B], which goes further than Corollary 1.3.

What is ours. The group, its construction, its hyperbolicity and torsion-freeness, the nontriviality of the rectangular words, the trace bound on the real rank, the block factorization, the interpolation certificate, the rank-reduction lemma and the exact rank comparison against finite quotients are all from [39], and all are theorems of the Lean development [43]. We contribute the approximate version of the rank comparison (Section 3), the passage from soficity (Section 4), the explicit constants, and a Lean formalization of the headline statement; Sandhu and Thom found the approximate rank comparison and the passage from soficity independently, for their own group [49, Theorem 3.1 and Section 5]. We also contribute Theorem 1.4 and the other results of Sections 5.1 and 6. The freeness of the line subgroups follows from [39, Theorem 2.1 and Lemma 2.3] by the argument that gives the nontriviality of the rectangular words there. Appendix A lists earlier sources for the inputs taken from [39]. Section 8 records exactly what was checked, and how.

2 The group and the facts we use

We recall the data of [39, Section 2]. The Lean structure `MarkedLineData` records the marked lines, the exceptional set, the degree and girth conditions and the budget; the two-block partitions and their coordinates are then chosen from these data (`MarkedLineData.partition`).

Data. A prime q, the integer $h = 20$, and an integer r with $r < q$. A subset $S \subset \mathbb{F}_q$ of size $\lfloor q/100 \rfloor$. The point set $P = \mathbb{F}_q^h$ has $m = q^h$ points. A set I of N line labels: label i has a mark $a_i \in S^h$, with distinct labels having distinct marks, and a nonzero direction; its punctured line is $\ell_i^\circ = \{a_i + tv_i : t \in \mathbb{F}_q^\times\}$, with $q - 1$ points. For $z \in P$ put $I(z) = \{i : z \in \ell_i^\circ\}$. An exceptional set $E \subset P$. At each regular point $z \notin E$, the set $I(z)$ is split into two disjoint blocks, and each block b carries a bijection $i \mapsto (\alpha_b(i), \beta_b(i))$ from I_b onto $\mathbb{Z}/s_b \times \mathbb{Z}/t_b$ with $s_b, t_b \geq r$. Moreover $|I(z)| \geq 4r^2$ for $z \notin E$, and the bipartite graph on $P \sqcup I$ with an edge $z \sim i$ whenever $z \in \ell_i^\circ$ has no cycle of length at most 12 [39, Lemma 2.2]; these two conditions are used only through (F1), (F4) and [39, Lemma 2.3], and in Proposition 6.7. The data satisfy the strict budget [39, (2.2)]

$$q > r, \qquad \frac{Nr}{4} > 2m + r|E|. \tag{2.1}$$

We write $\Gamma = Nr - 4(2m + r|E|)$, a positive integer.

Presentation. Let $[r] = \{0, \dots, r-1\}$. The generators are x_{iuv} for $i \in I$ and $u, v \in [r]$; $L_{b,u,j}$ for $j \in \mathbb{Z}/s_b$; and $R_{b,v,k}$ for $k \in \mathbb{Z}/t_b$, over all blocks b. For every block b, label $i \in I_b$ and $u, v \in [r]$ there is a triangle relator

$$\theta_{b,i,u,v} = L_{b,u,\alpha_b(i)+v}\; x_{iuv}\; R^{-1}_{b,v,\beta_b(i)+u}. \tag{2.2}$$

Two further relators kill the edges of a spanning tree of the three-vertex graph underlying the triangle complex of [39]; the argument below never uses them. The group G is the group of this presentation, which is π_1 of that complex. We index triangle relators by $t = (b, i, u, v)$; their number is

$$T = r^2 \sum_{z \notin E} |I(z)| \le r^2 N(q-1). \tag{2.3}$$

Rectangular words. For $i \in I$, $u \ne u'$ and $v \ne v'$ in $[r]$,

$$w_{i,u,u',v,v'} = x_{iuv}\, x^{-1}_{iu'v}\, x_{iu'v'}\, x^{-1}_{iuv'}.$$

Let $\mathcal{R}$ be the finite set of these words.

Facts from [39]. Each is a theorem of the Lean development; Lean names are in the namespace `OAI.Release075`.

(F1) Every rectangular word is nontrivial in G [39, (2.6)] (`MarkedLineData.rectangular_nontriviality`).

(F2) *Line certificate.* For arbitrary matrices Y_i ($i \in I$) over $\mathbb{F}_q$, all of one size,

$$\sum_{i \in I} \operatorname{rank} Y_i \le \sum_{z \in P} \operatorname{rank}\Big(\sum_{i \in I(z)} Y_i \Big).$$

This is [39, (2.13)–(2.14)]: product Lagrange interpolation on S^h [52, Lemma 1.3, Theorem 2.3] gives matrices $H(z)$ of rank at most one with $\sum_{z \in \ell_i^\circ} H(z) = -E_{ii}$, because entries restricted to a line have degree below $q-1$, and a polynomial of degree below $q-1$ sums to zero over $\mathbb{F}_q$ [53, Chapter I, §2.1, Lemma], [52, Theorem 3.3(i)]; hence $-\bigoplus_i Y_i = \sum_z H(z) \otimes \sum_{i \in I(z)} Y_i$ (`grid_line_certificate`, `rank_budget_of_line_certificate`). No group is involved.

(F3) *Rank under reduction* [39, Lemma 2.4]. If M is an integer matrix whose rows have Euclidean norm at most $\sqrt{r}$ and $q > r$ is prime, then $\operatorname{rank}_{\mathbb{F}_q} M \ge \frac{1}{2} \operatorname{rank}_{\mathbb{R}} M$ (`rank_reduction_half`). The same bound follows from Soulé's bound on the torsion of a cokernel [54, Lemma 1], part of an argument that Soulé credits to Gabber; see also [50, Lemma 3.1].

(F4) G is torsion-free and word-hyperbolic [39, Theorem 2.1, Lemma 2.3] (`MarkedLineData.presentation_torsionFree`, `MarkedLineData.presentation_wordHyperbolic`).

3 Approximate permutation models

For a permutation π of a nonempty finite set Ω with $|\Omega| = D$, let $P(\pi)$ be its permutation matrix, $P(\pi)e_c = e_{\pi(c)}$. Then $P(\pi\rho) = P(\pi)P(\rho)$, $P(\pi)^{\mathsf{T}} = P(\pi^{-1})$, and $\operatorname{tr} P(\pi) = D \operatorname{fix}(\pi)$.

A *model* assigns a permutation $\sigma(s)$ of Ω to each generator s. It extends to words by $\sigma(s^{-1}) = \sigma(s)^{-1}$ and multiplication; this extension is a homomorphism from the free group. A model has *relator defects* $\delta_t = \operatorname{mov} \sigma(\theta_t)$ and *rectangle defect* $\eta = \max_{w \in \mathcal{R}} \operatorname{fix} \sigma(w)$.

Theorem 3.1. *Every model with $\eta \le 1/(r-1)^2$ satisfies*

$$\sum_t \delta_t \;\ge\; \frac{Nr}{4} - 2m - r|E| \;=\; \frac{\Gamma}{4}.$$

Proof. For each label i let Y_i be the $rD \times rD$ matrix of $D \times D$ blocks $P(\sigma(x_{iuv}))$, $u, v \in [r]$. It has entries 0 and 1, and each row has exactly r ones.

Step 1: traces. Put $A_i = Y_i Y_i^{\mathsf{T}}$. Then $\operatorname{tr} A_i = \sum_{u,v} \operatorname{tr}(PP^{\mathsf{T}}) = r^2 D$, and

$$\operatorname{tr} A_i^2 = \sum_{u,u',v,v'} \operatorname{tr} P(\sigma(x_{iuv} x_{iu'v}^{-1} x_{iu'v'} x_{iuv'}^{-1})).$$

If $u = u'$ or $v = v'$ the word cancels freely, so the term is exactly D; there are $2r^3 - r^2$ such index choices. The other $r^2(r-1)^2$ terms are $D \operatorname{fix} \sigma(w)$ for rectangular words w, each at most ηD. So $\operatorname{tr} A_i^2 \le (2r^3 - r^2)D + r^2(r-1)^2 \eta D$.

Step 2: real rank. Since A_i is positive semidefinite, Cauchy–Schwarz on its eigenvalues gives $\operatorname{rank} A_i \ge (\operatorname{tr} A_i)^2 / \operatorname{tr} A_i^2$, so

$$\operatorname{rank}_{\mathbb{R}} Y_i \;\ge\; \frac{r^2 D}{2r - 1 + (r-1)^2 \eta} \;\ge\; \frac{rD}{2} \qquad \text{when } (r-1)^2 \eta \le 1.$$

This is the bound of [4, Theorem 6] for the 0/1 matrix Y_i, whose rows and columns each contain r ones: four-cycles in its bipartite graph are counted by fixed points of rectangular words, their number $\square$ satisfying $4\square = D \sum \operatorname{fix} \sigma(w)$ over the $r^2(r-1)^2$ index choices with $u \ne u'$ and $v \ne v'$ in Step 1. The inequality $\operatorname{rank} A \ge (\operatorname{tr} A)^2 / \operatorname{tr} A^2$ is classical; see [4, Section 3] and the proof of [3, Lemma 2.2].

Step 3: rank over $\mathbb{F}_q$. The rows of Y_i have norm $\sqrt{r}$ and $q > r$, so (F3) gives $\operatorname{rank}_{\mathbb{F}_q} Y_i \ge rD/4$.

Step 4: local factorization with errors. Fix a block b at a regular point z and a triangle $t = (b, i, u, v)$, and abbreviate $L = L_{b,u,\alpha_b(i)+v}$, $x = x_{iuv}$, $R = R_{b,v,\beta_b(i)+u}$. A point p is fixed by $\sigma(\theta_t) = \sigma(L)\sigma(x)\sigma(R)^{-1}$ exactly when the column $c = \sigma(R)^{-1}(p)$ of $P(\sigma(x))$ agrees with that of $P(\sigma(L))^{-1} P(\sigma(R))$. Hence

$$P(\sigma(x)) = P(\sigma(L))^{-1} P(\sigma(R)) + \Delta_t, \qquad \operatorname{rank} \Delta_t \le \delta_t D,$$

since Δ_t has at most $\delta_t D$ nonzero columns. For fixed u, v, as i runs over I_b the pair $(\alpha_b(i)+v, \beta_b(i)+u)$ runs over $\mathbb{Z}/s_b \times \mathbb{Z}/t_b$ exactly once. Therefore

$$\sum_{i \in I_b} Y_i = \Big(\sum_j P(\sigma(L_{b,u,j}))^{-1}\Big)_{u \in [r]} \Big(\sum_k P(\sigma(R_{b,v,k}))\Big)_{v \in [r]} + \Delta_b,$$

a block column times a block row, with inner dimension D, plus a matrix Δ_b whose (u, v) block is $\sum_{i \in I_b} \Delta_{(b,i,u,v)}$. This is the factorization [39, (2.11)] with error Δ_b, and $\operatorname{rank} \Delta_b \le D \sum_{t \in b} \delta_t$. Each regular point has two blocks, so $Z(z) = \sum_{i \in I(z)} Y_i$ satisfies

$$\operatorname{rank} Z(z) \le 2D + D \sum_{t \text{ at } z} \delta_t \quad (z \notin E), \qquad \operatorname{rank} Z(z) \le rD \quad (z \in E),$$

where the second bound is the size of the matrix.

Step 5: certificate. By (F2) over $\mathbb{F}_q$, $\sum_i \operatorname{rank} Y_i \le \sum_z \operatorname{rank} Z(z)$. With Steps 3 and 4,

$$\frac{NrD}{4} \le 2(m - |E|)D + r|E|D + D \sum_t \delta_t.$$

Divide by D. $\square$

When every $\delta_t = 0$ and $\eta = 0$, which is the case of the regular representation of a finite quotient in which $\mathcal{R}$ survives, Theorem 3.1 contradicts (2.1); this is the argument of [39, Section 2].

Proof of Theorem 1.2. Suppose $\eta \le 1/(r-1)^2$ and every $\delta_t < \varepsilon_0$. Theorem 3.1 gives $\Gamma/4 \le \sum_t \delta_t < T\varepsilon_0$.

For general data, $\Gamma \ge 1$ and $T \le r^2 N(q-1)$ by (2.3), so $T\varepsilon_0 \le 1/4$ when $\varepsilon_0 = 1/(4r^2N(q-1))$, a contradiction.

For the data of [39, Lemma 2.2], fixed there for a sufficiently large prime q, its proof gives $N \ge cm/2$ and $r|E| \le m$, where $c = 100^{-20}$ and $cr \ge 100$. Then $3m \le 6N/c \le 0.06\,Nr$, so $\Gamma/4 \ge Nr/4 - 3m \ge 0.19\,Nr$. With $\varepsilon_0 = 1/(6rq)$, $T\varepsilon_0 \le r^2N(q-1)/(6rq) < Nr/6 < 0.19\,Nr$, again a contradiction. □

The threshold is astronomically small, since $r \ge 100^{21}$ and q is a much larger prime, but it is fixed and positive, and it does not depend on $|\Omega|$.

4 From soficity to models

Let $d(\alpha, \beta) = \mathrm{mov}(\alpha^{-1}\beta)$ be the normalized Hamming distance on $\mathrm{Sym}(\Omega)$; it is bi-invariant. We use the standard definition: a group H is sofic if for every finite $F \subset H$ and every $\varepsilon > 0$ there are a nonempty finite set Ω and a map $\varphi : H \to \mathrm{Sym}(\Omega)$ with

$$d(\varphi(gh), \varphi(g)\varphi(h)) \le \varepsilon \quad (g, h \in F), \qquad d(\varphi(g), 1) \ge 1 - \varepsilon \quad (g \in F \setminus \{1\}).$$

This is [13, Definition 1.2], up to the normalization $\varphi(1) = 1$ made there. It is equivalent to the forms with separation only bounded away from 0: the proof of [13, Theorem 1] amplifies separation by tensor powers; see also [46, Theorem 3.5 and its proof]. Equivalently, for a presentation $H = F/K$, soficity says that finitely many words in K can be made to fix almost every point while finitely many words outside K fix almost none, by permutations assigned to the generators; Lemma 4.1 is the instance we need, with explicit constants.

Lemma 4.1. *Let $\pi : G \to H$ be a homomorphism to a sofic group such that $\pi(w) \ne 1$ for every $w \in \mathcal{R}$. Then for every $\varepsilon > 0$ there is a model with every $\delta_t \le 5\varepsilon$ and $\eta \le 8\varepsilon$.*

Proof. Let $F \subset H$ consist of 1, the images of the generators and their inverses, and the images of all prefixes of the triangle relators and the rectangular words. Take φ as in the definition for F and ε, and set $\sigma(s) = \varphi(\pi(s))$ for each generator s.

From $\varphi(1)\varphi(1) \approx_\varepsilon \varphi(1)$ we get $d(\varphi(1), 1) \le \varepsilon$, and then $d(\varphi(\pi s^{-1}), \varphi(\pi s)^{-1}) \le 2\varepsilon$. For a word $w = s_1^{e_1} \cdots s_k^{e_k}$ whose prefixes map into F, telescoping over prefixes and bi-invariance give

$$d(\varphi(\pi w), \sigma(w)) \le (k-1)\varepsilon + 2k_-\varepsilon,$$

where k_- is the number of inverse letters. A triangle relator has $k = 3$ and $k_- = 1$, and $\pi(\theta_t) = 1$, so $\delta_t = d(\sigma(\theta_t), 1) \le 4\varepsilon + d(\varphi(1), 1) \le 5\varepsilon$. A rectangular word has $k = 4$ and $k_- = 2$, and $d(\varphi(\pi w), 1) \ge 1 - \varepsilon$ because $\pi(w) \ne 1$; so $\mathrm{fix}\,\sigma(w) \le \varepsilon + 7\varepsilon = 8\varepsilon$. □

Proof of Theorem 1.1. If G were sofic, Lemma 4.1 with $H = G$, $\pi = \mathrm{id}$ and (F1) would give models with every $\delta_t \le 5\varepsilon$ and $\eta \le 8\varepsilon$ for every $\varepsilon > 0$. Taking ε with $8\varepsilon \le 1/(r-1)^2$ and $5\varepsilon < \varepsilon_0$ contradicts Theorem 1.2. □

Proof of Corollary 1.3. Suppose each $w \in \mathcal{R}$ survives under some homomorphism π_w to a sofic group H_w. Finite direct products and subgroups of sofic groups are sofic [14, Theorem 1(1)], so the image H of G in $\prod_{w\in\mathcal{R}} H_w$ is sofic, and every $w \in \mathcal{R}$ survives in H. Lemma 4.1 and Theorem 1.2 give a contradiction. So some $w_0 \in \mathcal{R}$ dies under every homomorphism to a sofic group, and $w_0 \neq 1$ by (F1). □

5 Beyond permutation models

5.1 Hyperlinearity

Countable sofic groups are hyperlinear [13, Theorem 2], so if G is hyperlinear, it separates soficity from hyperlinearity and answers [46, Open question 3.4] in the negative. Non-hyperlinear groups were constructed recently [1, Corollary C], [51, Theorem 1.4]; see also the conditional construction of [55]. The present argument does not determine whether G is hyperlinear, and it breaks in three places. The rank reduction (F3) needs integer matrices; the certificate (F2) needs a finite field; and in unitary models a relator that holds approximately in normalized Hilbert–Schmidt norm need not hold off a low-rank error, so Step 4 fails. The three results below locate these breaks. With bounded denominators every step survives (Proposition 5.1). In the group von Neumann algebra of G the trace and factorization steps hold exactly, while the inequality of the certificate is reversed (Proposition 5.2). And a hyperlinear quotient of G that is not sofic exists as soon as unitary approximations keep every rectangular word away from the identity (Proposition 5.3). None of them decides whether G is hyperlinear.

A *unitary model* assigns a unitary matrix $U(s)$ of some size d to every generator s, and extends to words as in Section 3. Write $\tau_d = \frac{1}{d}\operatorname{tr}$ and $\|A\|_2 = \tau_d(A^*A)^{1/2}$ for the normalized trace and Hilbert–Schmidt norm. For a permutation π, $\tau_d(P(\pi)) = \operatorname{fix}(\pi)$ and $\|P(\pi) - I\|_2^2 = 2\operatorname{mov}(\pi)$.

Proposition 5.1 (Bounded denominators)**.** *Let ν be a positive integer with $\nu^2 r < q$, and let U be a unitary model all of whose matrix entries lie in $\nu^{-1}\mathbb{Z}[i]$. If $\operatorname{Re}\tau_d(U(w)) \le 1/(r-1)^2$ for every rectangular word w, then*

$$\frac{\nu^3}{2}\sum_t \|U(\theta_t) - I\|_2^2 \;\ge\; \frac{Nr}{4} - 2m - (r-2)|E| \;\ge\; \frac{\Gamma}{4}.$$

Proof. We follow the proof of Theorem 3.1, with $Y_i = (U(x_{iuv}))_{u,v\in[r]}$, a matrix of size rd.

Steps 1 and 2. As there, $\operatorname{tr}(Y_iY_i^*) = r^2 d$, and

$$\operatorname{tr}\big((Y_iY_i^*)^2\big) = \sum_{u,u',v,v'} \operatorname{tr} U\big(x_{iuv}x_{iu'v}^{-1}x_{iu'v'}x_{iuv'}^{-1}\big)$$

is real. The $2r^3 - r^2$ terms with $u = u'$ or $v = v'$ equal d, and the real parts of the other $r^2(r-1)^2$ terms are at most $d/(r-1)^2$. So $\operatorname{tr}\big((Y_iY_i^*)^2\big) \le 2r^3 d$ and $\operatorname{rank}_{\mathbb{C}} Y_i \ge (r^2d)^2/(2r^3d) = rd/2$.

Step 3. Let $\rho(M) = \left(\begin{smallmatrix} \operatorname{Re}M & -\operatorname{Im}M \\ \operatorname{Im}M & \operatorname{Re}M \end{smallmatrix}\right)$ be the realification of a complex matrix; it is additive and multiplicative, and $\operatorname{rank}_{\mathbb{R}}\rho(M) = 2\operatorname{rank}_{\mathbb{C}} M$. The integer matrix $\rho(\nu Y_i)$ has real rank at least rd, and each of its rows has norm $\nu\sqrt{r}$, since a row of Y_i consists of r rows of unitary matrices. By (F3) with $\nu^2 r$ in place of r, its reduction $\widehat{Y}_i$ modulo q has rank at least $rd/2$ over $\mathbb{F}_q$.

Step 4. For a triangle $t = (b, i, u, v)$ with sides L, x, R as in Step 4 of Theorem 3.1, put $\Delta_t = U(x) - U(L)^*U(R) = U(L)^*(U(\theta_t) - I)U(R)$. Its entries lie in $\nu^{-2}\mathbb{Z}[i]$, and the factorization of Step 4 reads

$$\nu^2 \sum_{i\in I_b} Y_i = \Big(\nu\sum_j U(L_{b,u,j})^*\Big)_{u\in[r]} \Big(\nu\sum_k U(R_{b,v,k})\Big)_{v\in[r]} + \nu^2\Delta_b$$

over $\mathbb{Z}[i]$, where the (u,v) block of Δ_b is $\sum_{i\in I_b}\Delta_{(b,i,u,v)}$. Apply ρ and reduce modulo q. Since ν is invertible modulo q and reduction does not increase the rank of an integer matrix, $\operatorname{rank}\sum_{i\in I_b}\widehat{Y}_i \le 2d + 2\sum_{t\in b}\operatorname{rank}\Delta_t$. A nonzero column of Δ_t is $a-a'$ with unit vectors $a\in(\nu^{-1}\mathbb{Z}[i])^d$ and $a'\in(\nu^{-2}\mathbb{Z}[i])^d$. Then $\operatorname{Re}\langle a,a'\rangle$ lies in $\nu^{-3}\mathbb{Z}$ and is less than 1, so $\|a-a'\|^2 = 2-2\operatorname{Re}\langle a,a'\rangle \ge 2\nu^{-3}$. Hence

$$\operatorname{rank}\Delta_t \le \frac{\nu^3}{2}\operatorname{tr}(\Delta_t^*\Delta_t) = \frac{\nu^3 d}{2}\|U(\theta_t)-I\|_2^2.$$

Step 5. By (F2) over $\mathbb{F}_q$, with $\widehat{Z}(z)=\sum_{i\in I(z)}\widehat{Y}_i$ of size $2rd$,

$$\frac{Nrd}{2} \le \sum_z \operatorname{rank}\widehat{Z}(z) \le 4d(m-|E|) + 2rd|E| + \nu^3 d\sum_t \|U(\theta_t)-I\|_2^2.$$

Divide by $2d$. □

For a permutation model, which has $\nu=1$, Proposition 5.1 is Theorem 3.1 in the form $\sum_t \delta_t \ge Nr/4 - 2m - (r-2)|E|$, which Step 5 of its proof also gives. Since $\Gamma\ge 1$ and there are $T\le r^2N(q-1)$ triangle relators, some triangle relator has $\|U(\theta_t)-I\|_2^2 \ge 1/(2\nu^3r^2N(q-1))$, whatever the dimension. So unitary models with denominators bounded by a fixed $\nu<(q/r)^{1/2}$ cannot make every triangle defect small while every rectangular word has $\operatorname{Re}\tau_d \le 1/(r-1)^2$. This is a barrier for bounded denominators only, and it says nothing about hyperlinearity: approximating arbitrary unitary matrices needs unbounded denominators.

A *tracial von Neumann algebra* is a von Neumann algebra with a faithful normal tracial state; it need not be finite-dimensional. Let λ be the left regular representation of G on $\ell^2(G)$, let $L(G)$ be the von Neumann algebra it generates, with trace $\tau(a)=\langle a\delta_1,\delta_1\rangle$, and put $\tau_r = \operatorname{Tr}_r\otimes\tau$ on $M_r(L(G))$. Assign $\lambda(s)$ to every generator s, and let $Y_i=(\lambda(x_{iuv}))_{u,v\in[r]}$ and $Z(z)=\sum_{i\in I(z)}Y_i$ in $M_r(L(G))$.

Proposition 5.2 (The group von Neumann algebra)**.** *(a) Every defining relator of G maps to 1, and $\tau(\lambda(w))=0$ for every rectangular word w.*

(b) For $Y\in M_r(L(G))$ let $\operatorname{rank}_\tau Y=\tau_r(p)$, where p is the projection onto the closure of the range of Y in $\ell^2(G)^r$. Then $\operatorname{rank}_\tau Y_i \ge r^2/(2r-1)$ for every label i, while $\operatorname{rank}_\tau Z(z)\le 2$ for $z\notin E$ and $\operatorname{rank}_\tau Z(z)\le r$ for $z\in E$. Hence

$$\sum_i \operatorname{rank}_\tau Y_i \;\ge\; \frac{Nr^2}{2r-1} \;>\; \frac{Nr}{4} \;>\; 2m+r|E| \;\ge\; \sum_z \operatorname{rank}_\tau Z(z),$$

so rank_τ reverses the inequality of the certificate (F2).

Proof. (a) holds because λ is a representation of G, and $w\ne 1$ by (F1).

(b) Put $A=Y_iY_i^*$. Its range is dense in that of Y_i, so p is also the range projection of A, and $pA=A$. Expanding as in Step 1 of Theorem 3.1 and using (a) gives $\tau_r(A)=r^2$ and $\tau_r(A^2)=2r^3-r^2$. The Cauchy–Schwarz inequality for the positive form $(a,b)\mapsto\tau_r(b^*a)$ gives $\tau_r(A)^2=\tau_r(pA)^2\le\tau_r(p)\,\tau_r(A^2)$, so $\operatorname{rank}_\tau Y_i\ge r^4/(2r^3-r^2)$. The factorization of Step 4 holds exactly in $M_r(L(G))$, so for $z\notin E$ we have $Z(z)=\Phi_z\Psi_z$ with Φ_z an $r\times 2$ matrix over $L(G)$. The range projection of $Z(z)$ is at most that of Φ_z, which has the same trace as the range projection of Φ_z^*, by polar decomposition; the latter is a projection in $M_2(L(G))$, so its trace is at most 2. For $z\in E$ the bound $\tau_r(1)=r$ is trivial. The chain of inequalities follows from (2.1). □

Part (a) makes $L(G)$ a countermodel: every statement about unitary tuples that holds in all tracial von Neumann algebras, whatever further elements, traces or projections it involves, holds in $L(G)$, where all defining relators hold exactly and every rectangular word has trace 0. So no such statement implies that exact defining relators force $\max_w \operatorname{Re}\tau(w) > 0$. In a matrix algebra they do: an assignment of unitary matrices satisfying all defining relators exactly is a representation of G, and some rectangular word lies in its kernel, because the proof of [39, Theorem 1.1] puts a rectangular word in the finite residual of G, which every finite-dimensional linear representation kills [39, Corollary 1.2]; Corollary 1.3 gives this again. A proof that G is not hyperlinear must therefore use a property of matrix algebras, or of their ultraproducts, that fails in $L(G)$; Proposition 5.1 uses one, bounded denominators. Part (b) locates the failure in the argument of Theorem 3.1: Steps 1, 2 and 4 hold exactly in $L(G)$, and the certificate is the step without a tracial analogue.

We use the characterization of hyperlinearity in [46, Theorem 3.6], which Pestov traces to Rădulescu [47]: a group H is hyperlinear if and only if for every finite $F \subset H$ and $\varepsilon > 0$ there is a map ϑ from F to the unitary group of some size with $\|\vartheta(g)\vartheta(h) - \vartheta(gh)\|_2 < \varepsilon$ whenever $g, h, gh \in F$, $\|\vartheta(1) - I\|_2 < \varepsilon$ if $1 \in F$, and $\|\vartheta(g) - \vartheta(h)\|_2 \geq 1/4$ for distinct $g, h \in F$.

Proposition 5.3 (A criterion)**.** *The following are equivalent.*

(i) There are unitary models U_n, of sizes d_n, such that $\|U_n(\theta) - I\|_2 \to 0$ for every defining relator θ and $\liminf_n \|U_n(w) - I\|_2 > 0$ for every rectangular word w.

(ii) G has a hyperlinear quotient in which every rectangular word survives.

A quotient as in (ii) is a hyperlinear group that is not sofic, by Corollary 1.3, and in (i) necessarily $d_n \to \infty$.

Proof. (i)⇒(ii). Fix a free ultrafilter ω on $\mathbb{N}$, and for $g \in G$ choose a word w_g representing it. Two words representing the same element differ by a product of conjugates of relators, so by unitary invariance of $\|\cdot\|_2$ their values under U_n have distance tending to 0. Hence $\tau_\omega(g) = \lim_\omega \tau_{d_n}(U_n(w_g))$ is well defined, and $\mathcal{N} = \{g : \lim_\omega \|U_n(w_g) - I\|_2 = 0\}$ is a normal subgroup of G. Let $\bar{G} = G/\mathcal{N}$; every rectangular word survives in $\bar{G}$. For $g \notin \mathcal{N}$ we have $\operatorname{Re}\tau_\omega(g) < 1$ and $|\tau_\omega(g)| \leq 1$, so $|(1 + \tau_\omega(g))/2| < 1$.

Let $V_{n,k}(s) = (U_n(s) \oplus I_{d_n})^{\otimes k}$, a unitary model of size $e = (2d_n)^k$. For every word w, $\tau_e(V_{n,k}(w)) = \big((1 + \tau_{d_n}(U_n(w)))/2\big)^k$ and $\|V_{n,k}(w) - I\|_2 \leq k\|U_n(w) - I\|_2$. Given a finite $F \subset \bar{G}$ and $\varepsilon > 0$, choose a preimage g of each $\bar{g} \in F$ and put $\vartheta(\bar{g}) = V_{n,k}(w_g)$. If $\bar{g}, \bar{h}, \bar{g}\bar{h} \in F$, then $\|\vartheta(\bar{g})\vartheta(\bar{h}) - \vartheta(\bar{g}\bar{h})\|_2 \leq k\|U_n(w_g w_h w_{gh}^{-1}) - I\|_2$, whose ω-limit is 0, and similarly for $\vartheta(1)$. For distinct $\bar{g}, \bar{h} \in F$, $\|\vartheta(\bar{g}) - \vartheta(\bar{h})\|_2^2 = 2 - 2\operatorname{Re}\tau_e(V_{n,k}(w_h^{-1} w_g))$, and since $h^{-1}g \notin \mathcal{N}$, the ω-limit of $|\tau_e(V_{n,k}(w_h^{-1} w_g))|$ is $|(1 + \tau_\omega(h^{-1}g))/2|^k$, which is below $1/2$ for large k. Choosing k first and then n in a set belonging to ω gives the maps required by [46, Theorem 3.6], so $\bar{G}$ is hyperlinear.

(ii)⇒(i). Let $\pi\colon G \to \bar{G}$ be such a quotient. Apply [46, Theorem 3.6] with $\varepsilon = 1/n$ to the finite set consisting of 1, the images of the generators and their inverses, and the images of all prefixes of the defining relators and of the rectangular words, and put $U_n(s) = \vartheta(\pi(s))$. The argument of Lemma 4.1, with $\|\cdot\|_2$ in place of the Hamming distance, bounds $\|U_n(w) - \vartheta(\pi(w))\|_2$ by a constant times $1/n$ for these words. So the relator defects tend to 0, and every rectangular word stays at distance at least $1/4 - O(1/n)$ from I.

Finally, if $d_n = d$ for infinitely many n in (i), compactness of the unitary group gives a limit model that satisfies all defining relators exactly, that is, a representation of G by $d \times d$ unitary matrices, in which no rectangular word maps to I. This contradicts [39, Corollary 1.2], since the proof of [39, Theorem 1.1] puts a rectangular word in the finite residual of G. □

5.2 Linear soficity

In models by invertible matrices over $\mathbb{F}_q$ with the normalized rank metric, as in the linear sofic approximations of Arzhantseva and Păunescu [7], Steps 4 and 5 survive but Steps 1–3 do not: there are no 0/1 entries and no real traces. So the argument says nothing about linear soficity, or about weak soficity in the sense of Glebsky and Rivera [18].

5.3 Where approximation is needed

The argument needs the rectangle defect η below $1/(r-1)^2$. A sofic approximation whose separation is only bounded below by a constant, such as the $1/4$ of [46, Theorem 3.5], would not suffice directly; it must first be amplified by tensor powers, as in the proof of [13, Theorem 1], to the standard form with separation $1-\varepsilon$ that Lemma 4.1 uses.

6 Division rings

Throughout this section K is a division ring of characteristic q. Its prime field $\mathbb{F}_q$ is central in K, and hence in every group ring $K[\Lambda]$. For a label i put

$$Y_i = (x_{iuv})_{u,v\in[r]} \in M_r(K[G]);$$

the matrices of Section 3 are its images under permutation models.

Let $\phi\colon K[\Lambda] \to D$ be a ring homomorphism to a division ring. For a subgroup $C \le \Lambda$ let D_C be the division closure of $\phi(K[C])$ in D, the smallest subring of D that contains $\phi(K[C])$ and is closed under inverting its nonzero elements; it is a division ring. For subgroups $C \le C'$, D is (C, C')-*free* if $\sum_j d_j\phi(g_j) \neq 0$ whenever $g_1, \dots, g_n \in C'$ lie in distinct right cosets of C and $d_1, \dots, d_n \in D_C$ are nonzero; it is then (C, C'')-free for every C'' with $C \le C'' \le C'$. Following [24, Section 2.3] and [16, Definition 2.2], D is *epic* if $D_\Lambda = D$. An epic D is *Linnell* if it is (C, Λ)-free for every subgroup C, and *strongly Hughes-free*, a notion due to Linnell [30], if it is (C, C')-free whenever C is normal in C'. Condition (HF) asks that whenever $C' \le \Lambda$ is finitely generated, C is normal in C' and C'/C is infinite cyclic, generated by tC, the powers $\phi(t)^k$, $k \in \mathbb{Z}$, be left linearly independent over D_C. In [16, Definition 2.2(1)] a Hughes-free division ring is an injective epic one satisfying (HF), and the notion is defined only for locally indicable groups.

6.1 Rank collapse

A *Sylvester matrix rank function* on a ring R assigns to every matrix M over R a real number $\operatorname{rk}(M) \ge 0$, with $\operatorname{rk}(M) = 0$ for zero matrices, $\operatorname{rk}(1) = 1$, $\operatorname{rk}(M_1M_2) \le \min\{\operatorname{rk} M_1, \operatorname{rk} M_2\}$, $\operatorname{rk}(M_1 \oplus M_2) = \operatorname{rk} M_1 + \operatorname{rk} M_2$, and $\operatorname{rk}\left(\begin{smallmatrix} M_1 & M_3 \\ 0 & M_2 \end{smallmatrix}\right) \ge \operatorname{rk} M_1 + \operatorname{rk} M_2$ [16, Section 2.5]. A ring homomorphism ϕ from R to a division ring D gives one, $\operatorname{rk}(M) = \operatorname{rank}_D \phi(M)$, and by a theorem of Malcolmson every integer-valued one arises in this way [35]; see [16, Section 2.5].

Theorem 6.1. *Every Sylvester matrix rank function on $K[G]$ satisfies*

$$\sum_{i\in I} \operatorname{rk}(Y_i) \;\le\; 2(m - |E|) + r|E| \;<\; \frac{Nr}{4}.$$

Proof. The axioms give $\operatorname{rk}(M_1 + M_2) \le \operatorname{rk} M_1 + \operatorname{rk} M_2$, since $M_1 + M_2 = \begin{pmatrix} I & I \end{pmatrix} (M_1 \oplus M_2) \begin{pmatrix} I \\ I \end{pmatrix}$; $\operatorname{rk}(M) \le k$ for a matrix M with k columns, since $M = MI_k$ and $\operatorname{rk}(I_k) = k$; and $\operatorname{rk}(-M) = \operatorname{rk}(M)$.

Local factorization. The triangle relators give $x_{iuv} = L_{b,u,\alpha_b(i)+v}^{-1} R_{b,v,\beta_b(i)+u}$ in G. As in Step 4 of Theorem 3.1, for each block b and fixed u, v, the pair $(\alpha_b(i)+v, \beta_b(i)+u)$ runs over $\mathbb{Z}/s_b \times \mathbb{Z}/t_b$ exactly once as i runs over I_b, so

$$\sum_{i \in I_b} Y_i = \Big(\sum_j L_{b,u,j}^{-1}\Big)_{u\in[r]} \Big(\sum_k R_{b,v,k}\Big)_{v\in[r]},$$

a column times a row. At a regular point z the two blocks give $Z(z) = \sum_{i\in I(z)} Y_i = \Phi_z \Psi_z$ with Φ_z of size $r \times 2$, so $\operatorname{rk} Z(z) \le 2$. For $z \in E$, $\operatorname{rk} Z(z) \le r$.

Certificate. In (F2), $H(z) = f(z) f(z)^{\mathsf{T}}$ for a column $f(z) \in \mathbb{F}_q^N$ [39, Section 2.5], and the identity $-\bigoplus_i Y_i = \sum_z H(z) \otimes Z(z)$ uses only that the Kronecker product is $\mathbb{F}_q$-bilinear; so it holds over $K[G]$. Since $H(z) \otimes Z(z) = (f(z) \otimes I_r)\, Z(z)\, (f(z)^{\mathsf{T}} \otimes I_r)$,

$$\sum_i \operatorname{rk}(Y_i) = \operatorname{rk}\Big(-\bigoplus_i Y_i\Big) \le \sum_{z\in P} \operatorname{rk} Z(z) \le 2(m - |E|) + r|E|,$$

which is less than $Nr/4$ by (2.1). □

6.2 Freeness and full rank

Fix a label i, and put $y_{uv} = x_{iuv} x_{i00}^{-1}$ and $A_i = \langle y_{uv} : u, v \in [r] \rangle \le G$; note that $y_{00} = 1$. We call A_i a *line subgroup*.

Lemma 6.2. *A_i is free on $\{y_{uv} : (u,v) \ne (0,0)\}$.*

Proof. Let X be the triangle complex of [39, Section 2.3], with vertices O, V, W, edges $L_{b,u,j} \colon O \to V$, $x_{iuv} \colon V \to W$ and $R_{b,v,k} \colon O \to W$, and one triangle for each triangle relator. The presentation of Section 2 identifies G with $\pi_1(X, O)$: if γ_V, γ_W are the paths from O in the spanning tree, the generator x_{iuv} is the class of $\gamma_V x_{iuv} \bar{\gamma}_W$. So y_{uv} is the class of $\gamma_V\, x_{iuv} \bar{x}_{i00}\, \bar{\gamma}_V$. Let $\Theta_i \subset X$ be the graph with vertices V, W and the r^2 edges x_{iuv}. Its fundamental group $\pi_1(\Theta_i, V)$ is free on the loops $x_{iuv}\bar{x}_{i00}$ with $(u,v) \ne (0,0)$, so it suffices that $\pi_1(\Theta_i, V) \to \pi_1(X, V)$ is injective.

A nontrivial element of $\pi_1(\Theta_i, V)$ is freely homotopic in Θ_i to a nonempty cyclically reduced closed edge path. Each of its turns is at V or W and compares the directions of two distinct edges x_{iuv} and $x_{iu'v'}$ with the same label i. By [39, Lemma 2.3] their link distance is at least $14\pi/5$ or infinite, so by the closed-path assertion of [39, Theorem 2.1], proved as [39, Proposition 3.3], the path is not nullhomotopic in X. This is the argument that gives (F1) in [39, (2.6)], applied to all closed paths in Θ_i. □

Let $\delta_i \colon A_i \to \mathbb{Z}$ be the homomorphism with $\delta_i(y_{uv}) = -1$ for $u \ne v$ and $\delta_i(y_{uu}) = 0$, let $A_i^0 = \ker \delta_i$, and let $t_i = y_{01}^{-1}$, so that $\delta_i(t_i) = 1$.

Lemma 6.3. *Let Λ be a group containing G, and $\phi \colon K[\Lambda] \to D$ a ring homomorphism to a division ring such that the powers $\phi(t_i)^k$, $k \in \mathbb{Z}$, are left linearly independent over $D_{A_i^0}$. Then $\phi(Y_i)$ is invertible in $M_r(D)$.*

Proof. Write $\hat{t} = \phi(t_i)$ and $D_0 = D_{A_i^0}$. Conjugation by $\hat{t}$ is an automorphism of D that maps $\phi(K[A_i^0])$ onto itself, because A_i^0 is normal in A_i; hence it maps D_0 onto itself. Call its restriction γ. By hypothesis the sum $R = \sum_{k\in\mathbb{Z}} D_0 \hat{t}^k$ is direct, and $\hat{t} d = \gamma(d)\hat{t}$ for $d \in D_0$, so R is a subring of D, a skew Laurent polynomial ring over D_0. For $f = \sum_k a_k \hat{t}^k \ne 0$ let $\deg f$ be the largest k with $a_k \ne 0$. The coefficient of $\hat{t}^{\deg f + \deg g}$ in fg is $a_{\deg f} \gamma^{\deg f}(b_{\deg g}) \ne 0$, so $\deg(fg) = \deg f + \deg g$.

The ring R satisfies the left Ore condition: nonzero $a, b \in R$ have nonzero $f, g \in R$ with $fa = gb$. Indeed, multiplying a and b on the left by powers of the unit $\hat{t}$, we may assume that their lowest exponents are 0; let their degrees be α and β, and let $R_n = \bigoplus_{k=0}^{n} D_0 \hat{t}^k$, a left D_0-space of dimension $n+1$. For $n \geq \alpha+\beta$ the left D_0-linear maps $f \mapsto fa$ on $R_{n-\alpha}$ and $g \mapsto gb$ on $R_{n-\beta}$ are injective with values in R_n, and $(n-\alpha+1)+(n-\beta+1) > n+1$, so their images meet nontrivially. Consequently the elements $s^{-1}f$ of D with $s, f \in R$ and $s \neq 0$ form a division subring D', in which finitely many elements have a common left denominator. (For products, $fs^{-1} = s_1^{-1} f_1$ whenever $s_1 f = f_1 s$ with $s_1 \neq 0$; for sums, use a common left multiple of the denominators.)

Put $Y' = (y_{uv})_{u,v\in[r]} = Y_i\, x_{i00}^{-1}$. Each entry is $y_{uv} = n_{uv} t_i^{\delta_i(y_{uv})}$ with $n_{uv} \in A_i^0$, so $\phi(y_{uv}) = \phi(n_{uv})\hat{t}^{\delta_i(y_{uv})}$ lies in R, with a unit of D_0 as its only coefficient; its degree is 0 if $u = v$ and -1 otherwise. Suppose that $c\,\phi(Y') = 0$ for a nonzero row c over D'. Clearing a common left denominator, we may take c over R. Let M be the largest degree of a nonzero entry of c, attained at c_{u^*}, and consider entry u^* of $c\,\phi(Y')$, namely $\sum_u c_u \phi(y_{uu^*})$. The term $u = u^*$ has degree M, and every other nonzero term has degree at most $M-1$. So this entry is nonzero, a contradiction. The rows of $\phi(Y')$ are therefore left linearly independent over the division ring D', so $\phi(Y')$ is invertible over D', hence over D, and so is $\phi(Y_i) = \phi(Y')\phi(x_{i00})$. □

Proof of Theorem 1.4. Let $\phi\colon K[\Lambda] \to D$ be a ring homomorphism to a division ring. If the powers of $\phi(t_i)$ were left linearly independent over $D_{A_i^0}$ for every i, Lemma 6.3 would make every $\phi(Y_i)$ invertible, so that $\sum_i \operatorname{rank}_D \phi(Y_i) = Nr$. Restricting ϕ to $K[G]$ and applying Theorem 6.1 gives $\sum_i \operatorname{rank}_D \phi(Y_i) < Nr/4$. So for some i the powers of $\phi(t_i)$ are left linearly dependent over $D_{A_i^0}$.

The elements t_i^k lie in distinct right cosets of A_i^0. So this dependence contradicts (A_i^0, Λ)-freeness, which every Linnell division ring has; (A_i^0, A_i)-freeness, which every strongly Hughes-free division ring has, as A_i^0 is normal in A_i; and condition (HF) for the finitely generated subgroup A_i, as A_i/A_i^0 is infinite cyclic, generated by $t_i A_i^0$. □

6.3 The free factor and further consequences

Let $X^\sharp$ be obtained from X by splitting O into one vertex O_b for each block b, V into V_u and W into W_v for $u, v \in [r]$, with edges $L_{b,u,j}\colon O_b \to V_u$, $x_{iuv}\colon V_u \to W_v$ and $R_{b,v,k}\colon O_b \to W_v$. The triangle of $\theta_{b,i,u,v}$ runs from O_b to V_u to W_v and back to O_b, so every triangle of X closes in $X^\sharp$, and identifying all O_b, all V_u and all W_v recovers X. Let $B = 2(m - |E|)$ be the number of blocks.

Proposition 6.4. *$X^\sharp$ is connected, and $G^\sharp = \pi_1(X^\sharp)$ is finitely presented, torsion-free and word-hyperbolic. The quotient map $X^\sharp \to X$ induces an isomorphism of $G^\sharp$ onto a free factor of G, and $G \cong G^\sharp * F_{B+2r-3}$. Both X and $X^\sharp$ are aspherical.*

Proof. By (2.1), $r|E| < Nr/4$, and $N \leq m$ because the marks are distinct points of S^h; so $m - |E| > 3N/4$ and $B \geq 2$. Every O_b is joined to every V_u and every W_v, so $X^\sharp$ is connected.

Every corner of a triangle at O lies in one block, every corner at V joins some $L_{b,u,j}$ to some x_{iuv} with the same u, and every corner at W joins some $R_{b,v,k}$ to some x_{iuv} with the same v. Hence the link of X at O is the disjoint union of the links of $X^\sharp$ at the vertices O_b, with the same angles, and similarly at V and W. So every link of $X^\sharp$ is a union of connected components of a link of X, and [39, Lemma 2.3] holds verbatim for $X^\sharp$, including the bound on link distances between x-directions with the same label. By [39, Theorem 2.1], $G^\sharp$ is torsion-free and word-hyperbolic, and it is finitely presented because $X^\sharp$ is finite.

Add to $X^\sharp$ a forest T of $B + 2r - 3$ edges joining one O_b to all the others, V_0 to every V_u and W_0 to every W_v. Collapsing each tree of T to a point is a homotopy equivalence from $X^\sharp \cup T$ to X,

and since $X^\sharp$ is connected, $X^\sharp \cup T$ is homotopy equivalent to $X^\sharp$ with $B + 2r - 3$ circles attached at one point. This gives $G \cong G^\sharp * F_{B+2r-3}$, with $G^\sharp$ included by the map induced by $X^\sharp \to X$.

Finally, X and $X^\sharp$ are finite piecewise Euclidean complexes, hence complete geodesic spaces [8, Theorem I.7.50], and every injective loop in one of their vertex links has length at least $12\pi/5$ [39, Lemma 2.3]. Their barycentric subdivisions are simplicial complexes with the same links at the original vertices, and every injective loop in the link of a new vertex has length at least 2π [39, Section 3.1]. So they are nonpositively curved [8, Theorem II.5.2 and Lemma II.5.6], their universal covers are CAT(0) [8, Theorem II.4.1(2)], and CAT(0) spaces are contractible [8, Corollary II.1.5]. □

Corollary 6.5. *If a group Λ contains $G^\sharp$, then $K[\Lambda]$ has no Linnell division ring.*

Proof. Choose paths in $X^\sharp$ from the base vertex to every vertex, and for an edge e let $g_e \in G^\sharp$ be the class of the loop that runs along the path to the start of e, along e, and back along the path from its end. The triangles give $g_{x_{iuv}} = g_L^{-1} g_R$ for the sides L, R of the triangle (b, i, u, v), as in (2.2), so Theorem 6.1 holds for the matrices $Y_i^\sharp = (g_{x_{iuv}})_{u,v\in[r]}$ over $K[G^\sharp]$, with the same proof. Let $\phi\colon K[\Lambda] \to D$ be a Linnell division ring. It suffices to show that every $\phi(Y_i^\sharp)$ is invertible, since then $\sum_i \operatorname{rank}_D \phi(Y_i^\sharp) = Nr$, against Theorem 6.1.

Fix i and write $g_{uv} = g_{x_{iuv}}$. The x-edges of label i form a complete bipartite graph $\Theta_i^\sharp \subset X^\sharp$ between the V_u and the W_v, with a spanning tree formed by x_{i00}, the x_{iu0} and the x_{i0v}. As in Lemma 6.2, using the link-distance bound for $X^\sharp$ from Proposition 6.4, $\pi_1(\Theta_i^\sharp, W_0)$ injects into $\pi_1(X^\sharp, W_0)$. Its standard basis consists of the loops $\bar{x}_{iu0} x_{iuv} \bar{x}_{i0v} x_{i00}$ with $1 \le u, v \le r-1$, so the elements $e_{uv} = g_{u0}^{-1} g_{uv} g_{0v}^{-1} g_{00}$, which are these loops moved to the base point, freely generate a subgroup F_i of $G^\sharp$. Scaling by units,

$$\operatorname{diag}(g_{u0}^{-1})\, Y_i^\sharp\, \operatorname{diag}(g_{0v}^{-1} g_{00}) = \begin{pmatrix} 1 & \mathbf{1}^\mathsf{T} \\ \mathbf{1} & (e_{uv}) \end{pmatrix},$$

and subtracting the first row from the others leaves $\left(\begin{smallmatrix} 1 & \mathbf{1}^\mathsf{T} \\ 0 & M \end{smallmatrix}\right)$ with $M = (e_{uv} - 1)_{1\le u,v\le r-1}$. So it suffices that $\phi(M)$ is invertible.

We show by induction on k that $\phi(M_k)$ is invertible, with inverse entries in D_{F_k}, where $M_k = (e_{uv} - 1)_{1\le u,v\le k}$ and $F_k = \langle e_{uv} : u, v \le k\rangle$. For $k = 1$, e_{11} and 1 lie in distinct cosets of the trivial subgroup, so $(1, \Lambda)$-freeness gives $\phi(e_{11}) - 1 \neq 0$. For $k > 1$ write

$$M_k = \begin{pmatrix} M_{k-1} & b \\ c & e_{kk} - 1 \end{pmatrix}, \qquad C = \langle e_{uv} : u, v \le k,\ (u, v) \neq (k, k)\rangle.$$

The entries of $\phi(b)$, $\phi(c)$ and $\phi(M_{k-1})^{-1}$ lie in D_C, because $F_{k-1} \le C$. The Schur complement is $s = \phi(e_{kk}) + d$ with $d = -1 - \phi(c)\,\phi(M_{k-1})^{-1}\phi(b) \in D_C$. Since the e_{uv} form a free basis, $e_{kk} \notin C$, so $Ce_{kk} \neq C$, and (C, Λ)-freeness gives $s \neq 0$; if $d = 0$, then $s = \phi(e_{kk})$ is a unit. By block elimination $\phi(M_k)$ is invertible, and its inverse has entries in D_{F_k}. □

If G or $G^\sharp$ is locally indicable, then for every field K of characteristic q the Hughes-free division ring of $K[G]$, respectively of $K[G^\sharp]$, does not exist. For G this is the (HF) part of Theorem 1.4. For $G^\sharp$ it follows from Corollary 6.5 and a theorem of Gräter [19], as stated in [24, Theorem 2.5]: the Hughes-free division ring of the group algebra of a locally indicable group, if it exists, is a Linnell division ring. Such a group would refute, in characteristic q, the conjecture recalled in [24, Section 2.3] that the Hughes-free division ring exists for every locally indicable group and every field; see [22] for what is known. Whether G or $G^\sharp$ is locally indicable is open.

Corollary 6.6. *Neither $G^\sharp$ nor G is bi-orderable.*

Proof. Since $G^\sharp$ is a subgroup of G, it suffices to treat $G^\sharp$. Suppose that $\Lambda = G^\sharp$ has a total order invariant under left and right multiplication, and let $K = \mathbb{F}_q$. The Malcev–Neumann series $\sum_{g\in\Lambda} k_g g$ with well-ordered support form a division ring $K((\Lambda))$ containing $K[\Lambda]$ [34, 37], and for a subgroup $C \leq \Lambda$ those supported in C form a division subring $K((C))$ [48, Section 1.3]. Let D be the division closure of $K[\Lambda]$ in $K((\Lambda))$; it is epic, and $D_C \subseteq K((C))$ for every subgroup C. If $g_1, \dots, g_n$ lie in distinct right cosets of C and $d_1, \dots, d_n \in D_C$ are nonzero, then $d_j g_j \neq 0$ is supported in Cg_j, and these sets are disjoint, so $\sum_j d_j g_j \neq 0$. Thus D is a Linnell division ring for $K[\Lambda]$; this is the argument of [24, Proposition 2.4], given there for right-invariant orders and the Dubrovin ring. This contradicts Corollary 6.5. □

Proposition 6.7. *The Euler characteristics satisfy*

$$\chi(G) = 3 - Nr^2 - r\sum_b (s_b + t_b) + T \;>\; 3 + Nr^2(3r^2 - 7), \qquad \chi(G^\sharp) = \chi(G) + B + 2r - 3.$$

Neither G nor $G^\sharp$, nor any subgroup of finite index in either, is isomorphic to a subgroup of the fundamental group of a two-complex with nonpositive immersions.

Proof. By Proposition 6.4, $\chi(G) = \chi(X)$ and $\chi(G^\sharp) = \chi(X^\sharp)$. The complex X has 3 vertices, $Nr^2 + r\sum_b(s_b + t_b)$ edges and T triangles, which gives the formula; $X^\sharp$ has $B + 2r$ vertices and the same edges and triangles. Since $s_b, t_b \geq r$, we have $r(s_b + t_b) \leq 2s_b t_b = 2|I_b|$, so $r\sum_b(s_b + t_b) \leq 2T/r^2$ and $\chi(G) \geq 3 - Nr^2 + (r^2 - 2)T/r^2$. By the degree condition of Section 2, $T/r^2 = \sum_{z\notin E} |I(z)| \geq 4r^2(m - |E|) > 3Nr^2$, using $m - |E| > 3N/4$ from the proof of Proposition 6.4. Hence $\chi(G) > 3 - Nr^2 + 3Nr^2(r^2 - 2)$.

Let G' be a subgroup of index d in G or in $G^\sharp$. It is infinite, and it is the fundamental group of a finite aspherical two-complex, a d-sheeted cover of X or $X^\sharp$, with positive Euler characteristic $d\chi(G)$ or $d\chi(G^\sharp)$. The Euler–Poincaré formula for its universal cover and the vanishing of the zeroth L^2-Betti number of an infinite group [32, Theorem 2.7(ix),(x)] give $b_2^{(2)}(G') - b_1^{(2)}(G') > 0$. If G' were isomorphic to a subgroup of the fundamental group of a two-complex with nonpositive immersions in the sense of Wise [58], it would satisfy $b_2^{(2)}(G') \leq b_1^{(2)}(G')$, since it is finitely generated [24, Proposition 2.9]; the proof there follows Louder and Wilton [31, Corollary 1.6]. □

7 Open questions

(1) Is G hyperlinear? If it is, it is a hyperlinear group that is not sofic. Proposition 5.1 excludes approximations with bounded denominators. By Proposition 5.2, a proof that G is not hyperlinear cannot use only properties shared by all tracial von Neumann algebras, since $L(G)$ is one; hyperlinearity of G is equivalent to an embedding of $L(G)$ in an ultrapower of the hyperfinite II_1 factor [46, Theorem 8.5]. By Proposition 5.3 it would suffice, for a hyperlinear quotient that is not sofic, to find unitary approximations that keep every rectangular word away from the identity.

(2) Does $\mathbb{F}_q[G]$ embed in a division ring? If it does, $\mathbb{F}_q[G]$ embeds in a division ring but has no Linnell division ring; if it does not, G is a further counterexample in characteristic q to [16, Conjecture 1.1], which the zero divisors of [40], [23] and [10] already refute.

(3) Is G, or $G^\sharp$, locally indicable? If so, the Hughes-free division ring of its group algebra in characteristic q does not exist (Section 6.3), contrary to the conjecture recalled in [24, Section 2.3].

Is G left-orderable? If G has a right-invariant order, then by [24, Proposition 2.4] and Theorem 1.4 the Dubrovin ring of $K[G]$, for a field K of characteristic q, is not a division ring, contrary to a conjecture of Dubrovin [11] recalled in [24, Section 2.3].

8 Verification

The inputs. We rebuilt OpenAI's Lean development `https://github.com/openai/math` at commit

`adc7f1241b42e322a6451854ab7e4b4c146bf78a`,

with Lean 4.34.1 and Mathlib at tag `v4.34.1` (commit `d13f23b`). All 29 modules of `OAI/GroupTheory/Hyperbolic` compile without errors, warnings or `sorry`, and all 29 passed Lean's kernel checker. (The first build did not capture the exit code of `Model`; a later recompile of that module exits 0 with no output.) A second, complete rebuild from scratch with the repository's scripts repeated every compilation and kernel check, including those of the formalization described below; all 66 steps exit 0. The main theorem `OAI.Release075.main` and each declaration cited in (F1)–(F4), together with `BlockPresentation.triangle_relation` and the budget field of `MarkedLineData`, depend only on the axioms `propext`, `Classical.choice` and `Quot.sound`. The definitions of torsion-free and word-hyperbolic (the latter by thin geodesic triangles in a Cayley graph) agree with the repository's challenge statement by definitional equality. The Lean group is the presented group of Section 2; the identification with π_1 of the triangle complex and the general criterion [39, Theorem 2.1] are not formalized, and neither is [39, Corollary 1.2]. None of these is used here. Build instructions are in the accompanying repository.

The transfer. Sections 3 and 4 give complete written proofs. They were derived twice, independently: once from the paper and the Lean source, and once by a separate derivation that had no access to the first. Both reach Theorem 1.1 by the same mechanism.

The formalized headline. The transfer is also formalized in Lean on top of OpenAI's development, in the folder `lean/ext` of the accompanying repository. The formal theorem is

$$\texttt{exists_torsionFree_wordHyperbolic_not_sofic} : \exists\, G,\ \texttt{TorsionFree}\, G \wedge \texttt{WordHyperbolic}\, G \wedge \neg\, \texttt{IsSofic}\, G,$$

with OpenAI's definitions of the first two properties. Mathlib has no definition of soficity, so `IsSofic` is defined there exactly as in Section 4: maps to permutations of $\{0,\dots,n-1\}$, normalized Hamming distance, multiplicativity error at most ε and separation at least $1-\varepsilon$. The theorem depends only on the axioms `propext`, `Classical.choice` and `Quot.sound`, and Lean's kernel checker re-checked the three new modules. The formal proof follows Sections 3 and 4: it proves Lemma 4.1 with the same constants 5 and 8, and an obstruction of the form of Theorem 3.1 with cruder constants, a rectangle fraction of $1/r^2$ and a single threshold $1/(8r^2Nq)$.

What is machine-checked is therefore the headline statement, not every statement of this note. The sharper constants of Theorem 1.2, and Corollary 1.3 and its consequences, are proved here in writing only; the equivalence of this definition of soficity with other common forms is taken from the literature [13, Theorem 1 and its proof], [46, Theorem 3.5].

Division rings and the further results. Theorem 1.4 and the results of Sections 5.1 and 6 are proved here in writing only; none of them is formalized in Lean. Besides (F2), Theorem 1.4 uses the freeness of the line subgroups (Lemma 6.2), which rests on the written proofs of [39, Theorem 2.1, Proposition 3.3 and Lemma 2.3] and is not formalized; its consequence for the conjectures also uses the torsion-freeness in (F4). Theorem 1.4 was proved twice, independently and by different routes: by the skew Laurent argument of Lemma 6.3, and by an induction on Schur complements that also gives the strongly Hughes-free and (HF) assertions.

How this work was done

This note is the work of its two authors together with AI models from more than one developer. The authors set the direction throughout: they chose the questions, decided which lines to pursue and which to drop, judged and selected the results, and take full responsibility for every claim. The group and every input to the non-soficity argument are AI-generated work of OpenAI [39]. For the rest, the models did most of the exploration, derivation, computation, formalization and drafting.

Results were checked by independent derivation, and the results new in this version were also given to reviewers who had not seen our proofs and were asked to prove or refute them. The approximate rank argument of Sections 3 and 4 was derived twice independently, and its conclusion was formalized in Lean (Section 8). For Theorem 1.4, an instance that had not seen our proof found the proof given here, and a refutation-first review proved the theorem again by a different route; neither found an error in the argument. A priority search then found the characteristic-two refutations [40, 41], which our first statement of the result had missed, and the reviews led to the scoped wording about Hughes-free division rings. The results of Sections 5.1 and 6.3 were first obtained in separate model sessions and then proved again by instances that had not seen those proofs; those checks corrected four of the seven statements, and the text gives the corrected versions. The numbered citations added in this version were checked against the cited texts; where only a preprint version could be opened, the reference says which. The proofs are given in full: the checks support them but do not replace them.

Acknowledgments

We thank Andreas Thom and Harjas Sandhu. Their note [49] drew our attention to [9], [50] and [54], which version 1.0.0 of this note did not cite. We thank Mahan Mj and Ian Agol for proposing the references discussed at the end of Appendix A.

A Sources of the inputs

For each input that this note takes from [39], the list gives where it appears there and earlier sources that we checked, with result numbers. Items 1 and 7, and Theorem II.5.2 and Lemma II.5.6 of [8] in item 2, are cited in [39]; the other sources are not. An earlier source is not a claim of priority, and a missing citation is not evidence of copying.

1. *Random marked lines, then deletion of the labels on short cycles* [39, Lemma 2.2]. The alteration method of Erdős [15, proof of (4), pp. 35–37], where every edge on a short circuit of a random graph is deleted. The marked lines and the exceptional set are not in [15].
2. *Link criterion for curvature* [39, Theorem 2.1, Lemma 2.3]. [8, Theorem II.5.2, Theorem II.5.4 and Lemma II.5.6]. [39] proves the form it uses.

3. *Closed paths with large turns are essential; torsion-freeness* [39, Propositions 3.3 and 3.4]. For the CAT(0) universal cover, [8, Proposition II.1.4(2) and Corollary II.2.8(1)]; [39] argues directly with reduced diagrams and shortest loops.
4. *Reduced disc diagrams* [39, Lemma 3.2]. [36, Lemmas 2.16 and 2.17].
5. *Combinatorial Gauss–Bonnet formula* [39, (3.1)]. [36, Definition 4.4 and Theorem 4.6].
6. *Hyperbolicity from strict angle excess* [39, Lemma 4.1, Proposition 4.3]. Standard routes: [8, Theorem III.H.1.5] for the CAT(0) universal cover, or [8, Proposition III.H.1.2] for a piecewise hyperbolic metric, with [8, Proposition I.8.19 and Theorem III.H.1.9]. [39] gives a direct diagram argument.
7. *Stability of quasi-geodesics* [39, Lemma 4.2]. [8, Theorem III.H.1.7].
8. $\operatorname{rank} A \geq (\operatorname{tr} A)^2 / \operatorname{tr} A^2$ *for symmetric* A [39, Section 2.4]. Classical; the proof of [3, Lemma 2.2], and [4, Section 3].
9. *Real rank of a* 0/1 *matrix from its degrees and four-cycles* [39, (2.8)–(2.9) and the display after them]. [4, Theorem 6]; see Step 2 of Theorem 3.1.
10. *Rank of a bounded integer matrix modulo a large prime* [39, Lemma 2.4]. [54, Lemma 1], in an argument credited there to Gabber, and [50, Lemma 3.1]. Lemma 2.4 of [39] follows from Soulé's lemma; the two statements differ.
11. *Product Lagrange interpolation on* S^h [39, Section 2.5]. [52, Lemma 1.3 and Theorem 2.3].
12. *A polynomial of degree below* $q-1$ *sums to zero over* $\mathbb{F}_q$ [39, (2.13)]. [53, Chapter I, §2.1, Lemma], whose argument [39] gives as the standard power-sum identity, using the root bound in place of the cyclicity of $\mathbb{F}_q^{\times}$; [52, Theorem 3.3(i)].
13. *Block coordinates, coupled cyclic shifts and the block factorization* [39, (2.3)–(2.5), (2.11)]. Not traced to an earlier source.
14. *Interpolation turned into a rank certificate* [39, (2.13)–(2.14)]. The ingredients are in items 9–12; the assembly was not traced to an earlier source. The same device, matrices $x_v x_v^{\mathsf{T}}$ of rank at most one built from low-degree interpolation that sum to zero on every full affine line, appears on the same date in [42, Lemma 2.2 and Proposition 2.3].

Sources proposed in discussion. For item 13, three references were suggested. A Cartesian decomposition of a finite set [59, Definition 2.1, Theorem 3.3] and the rectangles of species theory [61, Definitions 2.1 and 2.2] give the language of a product structure on a finite set, but not the construction; [62, Theorem 4], on biclique covers and partitions, is not a source for this step. The labelling itself is elementary, since a set of st elements is in bijection with $\mathbb{Z}/s \times \mathbb{Z}/t$. For items 12 and 14, Dvir's bound for Kakeya sets in finite fields [60, Theorem 3] uses a related polynomial argument on lines, but contains neither the identity of item 12 nor the certificate.

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