Perfect play has different arithmetical complexity in semifinite and type III algebras

Seth Douglas Nidhal Mghirbi

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Abstract

For finite rational nonlocal games, existence of perfect normal play in a type II_1 algebra, a type II_∞ algebra, or any semifinite algebra is Σ_2^0-complete. For each fixed $\lambda \in [0,1]$, existence in a type III_λ factor is equivalent to commuting perfection and is Π_1^0-complete. Existence of commuting perfect play together with absence of every normal semifinite perfect strategy is Π_2^0-complete. Its hardness holds on a computable family whose every member is commuting perfect and has finite-dimensional quantum value one.

The lower bound combines a critical binary-tree selector, an effective group embedding, and a Boolean compiler on arbitrary faithful normal supports. The selector finds a finite terminal depth even in diffuse semifinite algebras. A finite-corner overlap criterion gives the upper bound. The perfect-play languages for type I and type II_1 are incomparable, and both are proper subsets of the type II_∞ language. The type-I complexity used here is established in the companion paper. Two directional differences are $D_2(\Sigma_2^0)$-complete: perfect play in type I but not in type II_1 (I-ONLY), and perfect spatial play attainable only with infinite entropy (INF-ENTROPY-ONLY). For semifinite-but-not-type-I play (II-ONLY) and its finite-entropy variant, completeness remains open; both are Σ_2^0-hard and Π_2^0-hard.

AI systems assisted with this work. The authors take responsibility for its claims; see How this work was done. This preprint has not been peer reviewed.

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1 Perfect play and host type

A finite game can have perfect play in one kind of operator algebra and none in another. We classify the difficulty of deciding which host types allow perfect play. Semifinite hosts share one complexity level; every type-III subtype has the commuting language. The inclusion picture nevertheless branches: type I and finite type II are incomparable, and their directional differences have a finer classification.

A finite game has finite classical question and answer sets, a rational question distribution, and a Boolean winning predicate. Perfection means zero probability for every losing event of positive question weight. All reductions are computable many-one reductions of such finite encodings. For a host class X, $\mathrm{PERF}(X)$ denotes existence of a perfect strategy normal on an ambient von Neumann algebra of class X; Bob acts in its commutant. The ambient algebra need not be generated by the measurements. The state may be chosen along with the host. These are existence languages, not tests of a prescribed algebra, state, or full correlation table.

Write $\mathrm{PERF}_{\mathrm{fin}}$ for finite-algebra hosting and $\mathrm{PERF}_{\mathrm{sf}}$ for semifinite hosting. Factor hosting gives the same existence languages as hosting in algebras purely of the indicated type. We use separable hosts when decomposing. For commuting input this follows from the joint GNS representation of the finitely generated measurement C^*-algebra. For finite or semifinite input the finite-corner overlap construction of Section 2 gives a separable host in the same class: generate the two unital C^*-algebras by the finitely many effects and the finite band projection, and take the closure of their orbit of the trace vector. It is separable; Lemma 2.1 preserves a nonzero perfect component and proves its represented host finite or semifinite. One may then standardize Bob and decompose over the center. Every losing probability is an integral of nonnegative probabilities, so the finitely many zero integrals vanish on a common conull set of factor fibres. A finite or semifinite fibre remains in that class; the amplifications in (4) give the required pure type-II factor. For type I use its direct-integral decomposition into spatial factor strategies. The type-III constructions below start with an unrestricted commuting factor fibre. Thus no group algebra constructed below is implicitly assumed to be a factor.

Theorem A (classification). *Perfect normal play in a finite algebra, a type* II_1 *algebra, or a semifinite algebra is* Σ_2^0*-complete. The same holds for the type* II_∞ *and type* II *existence languages. For every fixed* $\lambda \in [0,1]$*, the type* III_λ *language equals commuting perfection and is* Π_1^0*-complete. Commuting perfection without semifinite perfection is* Π_2^0*-complete, with hardness on a family of commuting-perfect games all having* $\omega_q = 1$*.*

For context, the table combines these new classifications with the previous finite-dimensional, spatial, entropy and value-one results. The known rows and their sources are identified immediately below.

Language	Complexity
Finite-dimensional perfect play	Σ_1^0-complete
PERF(I); perfect spatial play with finite marginal entropy	Σ_2^0-complete
$\mathrm{PERF}(\mathrm{II}_1)$	Σ_2^0-complete
$\mathrm{PERF}(\mathrm{II}_\infty) = \mathrm{PERF}(\mathrm{II}) = \mathrm{PERF}_{\mathrm{sf}}$	Σ_2^0-complete
$\mathrm{PERF}(\mathrm{III}_\lambda) = \mathrm{PERF}_{\mathrm{co}}$, fixed $0 \le \lambda \le 1$	Π_1^0-complete
III-ONLY $:= \mathrm{PERF}_{\mathrm{co}} \setminus \mathrm{PERF}_{\mathrm{sf}}$	Π_2^0-complete
$\{G : \omega_q(G) = 1\}$	Π_2^0-complete

The first row uses finite-dimensional semialgebraic feasibility and the perfect-completeness reduction of [16, Theorem 12.6]. The type-I and finite-entropy rows are from the companion paper [10]. The commuting classification is due to Slofstra [27], with another proof in [20, Theorem 6.4]. The quantum-value classification is [20, Theorem 6.9]; commuting value one is

attained by compactness. The new rows are proved below, completing the question left open in [8, Theorem 7.6].

A standard-form strategy for a faithful normal state φ on M uses its natural-cone vector ξ_φ in $L^2(M)$, Alice effects in M, and Bob effects in $M' = \mathcal{J}M\mathcal{J}$. Nonfaithful states are interpreted on their support corners. A general normal-marginal commuting strategy has this form without changing probabilities: after joint GNS, compress Bob to $\overline{M\psi}$ using the isometry $x\xi_\varphi \mapsto x\psi$. The compression lies in the standard commutant and is still a POVM. Semifiniteness passes to corners. This is [8, Lemma 1.6]; the argument includes mixed states.

Perfectly matched binary effects are sharp centralizer projections with mirror $\mathcal{J}E\mathcal{J}$. Indeed different matched classes satisfy $E_cF_d\xi_\varphi = 0$. Summing gives $E_c\xi_\varphi = F_c\xi_\varphi$ and $E_cE_d\xi_\varphi = 0$ for $c \neq d$. Separation gives orthogonality, and completeness gives idempotence. The self-adjoint mirror gives $\varphi(E_cx) = \varphi(xE_c)$. A fine positive effect below sharp marginals is their joint projection. These facts replace any finite-dimensional rigidity assumption.

Symbol	Meaning
$M, \varphi, \sigma_t^\varphi$	host, faithful state, modular group
$\tau, h; \mathcal{J}, \xi_\varphi$	trace and density; standard conjugation and state vector
$P, Q, C; B_0, B_1$	terminal, initial and depth labels; tree edges
$\mathcal{T}, E_n$	tree transfer map and depth projections
$\mathcal{R}_{\mathrm{dec}}, \mathcal{A}_{\mathrm{dec}}$	recursive predicate and its Σ_2^0 set
$\mathcal{D}_{\mathrm{good}}, \mathcal{G}_e, \mathcal{K}_n$	good depths, finite presentation, HNN extension
$a, b, c; T, L$	group words; flip and level reader
ε_m	group-model basis vector (not a scaffold corner)
$e_0, e_g, w_g, v, v^{(k)}$	source, selected corners, bridges, one-bridge and chain transport
$a_g, \mathsf{M}_g, Z_k^{\mathrm{sc}}$	bridge activity, magic support, scaffold normalizer
$\Lambda, V_{g,\ell_{\mathrm{lab}}}$	transported unitary encodings and bridge labels
$\mathfrak{c}_x, \mathfrak{r}_y; u_x, v_y$	calibrated rays and maps onto them
$X_x, Y_y; \kappa$	pulled-back signs; positive corner multiplier
F_i, G_{ij}, F_{ij}, H_G	qutrit frame, graphs, pair support, auxiliary graph
$\Pi_\triangle, B_{ab}^{ij}, \Pi_t^{\mathrm{row}}$	triangle, parameter joint partition, row partition
$\mathsf{S}_{\mathrm{row}}, S_{\mathrm{graph}}$	rational projection row and lifted graph row
$\ell_{\mathrm{row}}, \ell_e, d_e, N_e$	row denominator, input denominator, slot size, count
$R_\infty, R_\lambda, R_{\mathrm{hyp}}$	injective III_1, III_λ, II_1 factors
$N_{\mathrm{in}}, N_{\mathrm{abs}}, N_\Gamma, t_\lambda$	input, absorbed and crossed-product factors; time step
$\mathsf{q}, \mathsf{E}, \mathsf{I}, \mathsf{J}, \mathsf{U}, \mathsf{C}$	six existence languages (Section 9)
$R_*, T_*, K_{\mathrm{Cat}}, Q_*$	fixed games, distinct from operator labels

Table 1: Notation. The group involution j is distinct from $\mathcal{J}$.

$$
\begin{array}{ccc}
\text{finite-corner overlap} & \longrightarrow & \Sigma_2^0 \text{ upper bounds} \\
\text{critical tree + group embedding} & \longrightarrow & \text{semifinite selector reduction} \\
 & & \downarrow \text{ one bridge + Boolean rows} \\
\text{KMS and finite-tree models} & \longrightarrow & \Sigma_2^0 \text{ hardness; } \Pi_2^0 \text{ III-ONLY} \\
\text{expectations + crossed products} & \longrightarrow & \mathrm{PERF}(\mathrm{III}_\lambda) = \mathrm{PERF}_{\mathrm{co}}.
\end{array}
$$

Figure 1: Dependencies. The new semifinite lower bound does not use the companion paper's Weyl-clock reduction.

Nearest operational work is the commuting self-test for exact embezzlement of Paddock–Schmidt [23], and Harris's Cuntz-algebra self-testing and factor classification [13]. These concern exact embezzlement rather than the arithmetical classification of host-existence languages. The FMS geometry is due to Mančinska [18]. The algebraic compilers are from

[9, Lemma 4.2 and Theorem B]. Group embeddings originate with Higman [14]; the effective form follows Chiodo [1, Theorem 2.2]; subtype transfer uses Connes–Takesaki flow-of-weights theory [6] and the argument inside Haagerup–Musat [12, Lemma 3.8]. MNY [20], Lin's gapped commuting result [17], Fu–Miller–Slofstra [11], and MIP* = RE [16] provide the computability context. They do not replace the normal-host compiler in Section 5.

To our knowledge, the semifinite completeness classifications and the directional-difference classifications proved here are new. This priority statement is provisional and rests on the cited literature and targeted searches. It is not an exhaustive bibliographic certificate.

Directional perfection. Section 9 gives the full six-model table of "perfect in X but not in Y." Perfect play in type I but not in type II_1 (I-ONLY), and perfect spatial play attainable only with infinite entropy (INF-ENTROPY-ONLY), are $D_2(\Sigma_2^0)$-complete, as are both directions of the incomparable pairs and $\mathrm{PERF}_{\mathrm{sf}} \setminus \mathrm{PERF}(\mathrm{II}_1)$. The finite-dimensional exclusion column is Σ_2^0-complete for the four middle models. The commuting exclusion row is Π_2^0-complete, except for exclusion of finite-dimensional perfection, which is Π_1^0-complete even under $\omega_q = 1$. Only semifinite-but-not-type-I play (II-ONLY) and its finite-entropy variant retain an open completeness question; we prove both Σ_2^0- and Π_2^0-hardness and a D_2 upper bound. We keep PERF strictly for attainment. Nezhadi's 2025 thesis [21] uses "perfectly winnable" for the MNY value-one Π_2^0 problem, so its terminology has a different scope.

2 Finite-corner compactness

Lemma 2.1 (Finite-corner overlap)**.** *Let $\mathcal{A}, \mathcal{B}$ be commuting unital C^*-algebras containing a game's effects. Suppose unit vectors ψ, η and $e = e^2 = e^* \in \mathcal{A}$ satisfy: ψ is perfect, $e\eta = \eta$, $\langle \eta, \psi \rangle \neq 0$, and the vector state of η is tracial on $e\mathcal{A}e$. Then the game has a perfect normal semifinite strategy. If $e = 1$, it has a finite-algebra strategy.*

Proof. Let $\mathcal{H}_\eta = \overline{\operatorname{span} \mathcal{A}\mathcal{B}\eta}$. It reduces both algebras because they are self-adjoint and commute. Put $M = (\mathcal{A}|_{\mathcal{H}_\eta})''$ and $p = e|_{\mathcal{H}_\eta}$. The vector functional of η is normal on M and tracial on pMp: Kaplansky density approximates each corner element by a bounded net from $p\mathcal{A}p$, and separate ultraweak continuity extends $\tau_\eta(xy) = \tau_\eta(yx)$ one variable at a time. Moreover

$$p\mathcal{H}_\eta = \overline{\operatorname{span} \mathcal{B}(pMp)\eta}. \tag{1}$$

Indeed $pab\eta = bpap\eta$, and bounded strong approximation supplies the reverse closure. If $x \geq 0$ in pMp has $\tau_\eta(x) = 0$, then

$$\|x^{1/2}a\eta\|^2 = \tau_\eta(a^*xa) = \tau_\eta(x^{1/2}aa^*x^{1/2}) \leq \|a\|^2\tau_\eta(x) = 0.$$

Since x commutes with $\mathcal{B}$, (1) gives $x = 0$. Thus pMp has a faithful normal finite trace. If a central projection z of M annihilates p, it annihilates η, and hence $\mathcal{A}\mathcal{B}\eta$ and $\mathcal{H}_\eta$; so $z = 0$. Thus p has full central support. A type-III summand has no nonzero finite projection, so its intersection with p is zero and fullness excludes that summand. Consequently M is semifinite.

The orthogonal projection onto $\mathcal{H}_\eta$ commutes with both parties. $\zeta = P_{\mathcal{H}_\eta}\psi$ is nonzero by its overlap with η. Each losing positive product annihilates ψ and hence ζ. Normalize ζ. Its state is normal on M and Bob's compressed effects commute with M. If $e = 1$, the faithful finite trace is on M. □

Conversely standardize a normal semifinite strategy and write $\psi = h^{1/2} \in L^2(M, \tau)$, with $\tau(h) = 1$. Some band $e = 1_{[a,b]}(h)$, $0 < a < b < \infty$, is nonzero. It has $0 < \tau(e) \leq 1/a$. For $\eta = e/\sqrt{\tau(e)}$,

$$\langle \eta, \psi \rangle = \frac{\tau(eh^{1/2}e)}{\sqrt{\tau(e)}} \geq \sqrt{a}\sqrt{\tau(e)} > 0. \tag{2}$$

Its corner state is tracial. In a finite algebra take a faithful normalized trace and $e = 1$; then $\tau(h^{1/2}) > 0$. Adjoin e to Alice's algebra if necessary.

Theorem 2.2. $\mathrm{PERF}_{\mathrm{fin}}$ *and* $\mathrm{PERF}_{\mathrm{sf}}$ *belong to* Σ_2^0.

Proof. For $k \geq 1$ seek moments $m_{ij}(w) = \langle \xi_i, w\xi_j \rangle$, where $\xi_0 = \psi, \xi_1 = \eta$. Letters are the self-adjoint effects and Alice's extra projection e. Impose positivity for all polynomial vector lists, localizing positivity for each effect and its complement, POVM normalization, Alice–Bob commutation *including* e, $e^2 = e$, $e\eta = \eta$, unit norms, zero losing expectations on ψ, and

$$\mathrm{Re}\langle \eta, \psi \rangle \geq 1/k, \qquad \langle \eta, eue\, eve\, \eta \rangle = \langle \eta, eve\, eue\, \eta \rangle$$

for all Alice words u, v, including e. A vector equality means vanishing squared norm. These form a recursive list of closed rational polynomial constraints. Impose $|m_{ij}(w)| \leq 1$ directly. All genuine contraction moments obey this bound, and the ambient product of closed disks is compact. Localizing positivity gives bounded GNS effects $0 \leq E \leq 1$. One can also use the certificate $E - E^2 = (I - E)E(I - E) + E(I - E)E$.

At stage m take the first m constraints and their coordinates. Feasibility $F_{\mathrm{sf}}(G, k, m)$ is a non-strict rational semialgebraic question, decidable by real quantifier elimination. Its solution set is a closed cylinder in the compact product of all moment disks. The cylinders are nested because stages retain previous constraints. If every stage is feasible, their intersection is nonempty: otherwise their open complements have a finite subcover, already making one stage empty. A point of the intersection gives a positive matrix-valued functional. Quotient polynomial pairs by its null space and complete. Left multiplication yields the bounded GNS effects and the two vectors. Relations hold on the polynomial dense space and extend by boundedness. Corner traciality holds on the linear span of eue and then in norm on $e\mathcal{A}e$. Apply Lemma 2.1. Conversely (2) supplies some k for every semifinite strategy. Thus

$$G \in \mathrm{PERF}_{\mathrm{sf}} \iff \exists k\, \forall m\, F_{\mathrm{sf}}(G, k, m). \tag{3}$$

Set $e = 1$ for the finite-algebra version. □

Tensoring a finite factor host with R_{hyp} gives a II_1 host. Tensoring a semifinite factor host with $R_{\mathrm{hyp}} \bar{\otimes} B(\ell^2)$ gives a II_∞ host. Product-state expectations, or standard-form tensoring, preserve the probabilities. Factor decomposition as in Section 1 therefore gives

$$\mathrm{PERF}_{\mathrm{fin}} = \mathrm{PERF}(\mathrm{II}_1), \qquad \mathrm{PERF}_{\mathrm{sf}} = \mathrm{PERF}(\mathrm{II}) = \mathrm{PERF}(\mathrm{II}_\infty). \tag{4}$$

These are equalities of existence languages.

3 A critical tree selects a finite depth

Work on a faithful normal support. Let P, Q be centralizer projections, $0 \leq C \leq I$ a centralizer contraction, and let B_0, B_1 satisfy

$$B_i^* B_j = \delta_{ij}(I - P), \qquad \sum_i B_i B_i^* = I - Q, \tag{5}$$

$$\sigma_t^{\varphi}(B_i) = 2^{-it} B_i, \tag{6}$$

$$CQ = Q, \qquad CB_i = \tfrac{1}{2} B_i C, \qquad [C, P] = 0. \tag{7}$$

Lemma 3.1 (Critical-tree selector)**.** *In a normal semifinite realization,* $P \neq 0$, $\ker C = 0$, *and* $E_n = 1_{\{2^{-n}\}}(C)$ *satisfy* $\sum_{n \geq 0} E_n = I$ *strongly. Some finite* n *has* $PE_n \neq 0$.

Proof. The modular eigenoperators are entire analytic. KMS gives $\varphi(B_iB_i^*) = \varphi(B_i^*B_i)/2$. Summing gives

$$\varphi(P) = \varphi(Q). \tag{8}$$

If $P = 0$, faithfulness makes $Q = 0$, giving a full Cuntz pair. Write $\varphi(x) = \tau(hx)$, h injective, $\tau(h) = 1$. The covariance is $h^{it}B_i = B_i(h/2)^{it}$, hence $f(h)B_i = B_if(h/2)$ for bounded Borel f. For bounded $g \geq 0$ with compact support in $(0, \infty)$, $hg(h)$ is positive integrable, and

$$\varphi(g(h)) = \sum_i \tau(B_i^*hg(h)B_i) = \tfrac{1}{2}\sum_i \tau(hg(h/2)) = \varphi(g(h/2)).$$

The trace exchange is $\tau(T^{1/2}vv^*T^{1/2}) = \tau(v^*Tv)$ for positive integrable T. There is no permutation of arbitrary unbounded products. Use indicators of $(2^k, 2^{k+1}]$, or monotone approximations to them. All these windows have equal φ-mass and partition $(0, \infty)$, whose mass is one. Countable additivity is contradicted. Thus $P \neq 0$.

Let $z = 1_{\{0\}}(C)$. Covariance and its adjoint give $zB_i = B_iz$ by functional calculus at zero. Also $Qz = 0$ and $[P, z] = 0$. Since z is a centralizer projection, zMz is modularly invariant. The KMS calculation there gives $\varphi(Pz) = 0$, hence $Pz = 0$. If $z \neq 0$, the normalized restriction of φ is faithful normal on a semifinite corner with a full critical Cuntz pair. The same window argument excludes this. Hence $z = 0$.

Put $\mathcal{T}(X) = \sum_i B_iXB_i^*$. Telescoping gives

$$I = \sum_{j=0}^{m-1} \mathcal{T}^j(Q) + \mathcal{T}^m(I). \tag{9}$$

For a length-m word v, $CB_v = 2^{-m}B_vC$. Bounded spectral calculus puts its range in $1_{[0,2^{-m}]}(C)\mathcal{H}$. Thus

$$0 \leq \mathcal{T}^m(I) \leq 1_{[0,2^{-m}]}(C) \longrightarrow 0$$

strongly by injectivity of C. Each $\mathcal{T}^j(Q)$ is supported in the 2^{-j} eigenspace of C. These positive terms sum to I and have mutually orthogonal spectral supports. Compressing the sum to each eigenspace gives

$$E_j = \mathcal{T}^j(Q), \qquad \sum_{j\geq 0} E_j = I. \tag{10}$$

There is no residual spectral subspace since the sum is I. As $[P, C] = 0$ and $P \neq 0$, some PE_n is nonzero. In particular $Q = E_0$. □

3.1 Finite models and a small example

For words w of length at most n, let B_i append i unless w is a leaf, when it vanishes. Let Q select the empty word and P the depth-n words. Put

$$C|w\rangle = 2^{-|w|}|w\rangle, \qquad h_n|w\rangle = \frac{2^{-|w|}}{n+1}|w\rangle. \tag{11}$$

Each nonleaf has two distinct children and each nonroot a unique parent. This proves (5), including the cross term. The child-to-parent density ratio is $1/2$, proving (6); the same ratio for C proves (7). Each depth has state mass $1/(n+1)$, whence

$$\varphi_n(P) = \varphi_n(Q) = \frac{1}{n+1}, \qquad S(h_n) = \log_2(n+1) + \frac{n}{2}. \tag{12}$$

For $n = 2$, the seven word densities are $1/3$, two copies of $1/6$, and four copies of $1/12$. Their C labels are $1, 1/2, 1/4$. The leaf rank is four but its state mass is $1/3$. A finite host's normalized trace gives $\tau(P) \geq 1/2$ from (5); this does not bound $\varphi(P)$ away from zero uniformly over states.

4 A finite presentation verifies the depth

Fix a total recursive predicate $\mathcal{R}_{\mathrm{dec}}(e,n,s)$ and $\mathcal{A}_{\mathrm{dec}} = \{e : \exists n\, \forall s\, \mathcal{R}_{\mathrm{dec}}(e,n,s)\}$. With $[x,y] = xyx^{-1}y^{-1}$ set

$$\begin{aligned}\Gamma_e = \langle a,b,c \mid\ & w_n^2 = 1\ (n \geq 0),\\ & w_n = 1 \text{ if } \exists s\, \neg\mathcal{R}_{\mathrm{dec}}(e,n,s)\rangle, \qquad w_n = [b, a^n c a^{-n}].\end{aligned} \tag{13}$$

Lemma 4.1. *w_n is nontrivial in Γ_e exactly when $\forall s\, \mathcal{R}_{\mathrm{dec}}(e,n,s)$; a nontrivial w_n has order two. Uniformly there is a finite presentation $\mathcal{G}_e$ and explicit words for embedded a,b,c preserving this property.*

Proof. Let $\mathcal{D}_{\mathrm{good}} \subseteq \mathbb{N}$ be the good integers. Take two copies of $V = \bigoplus_{m\in\mathbb{Z}} \mathbb{F}_2 \varepsilon_m$ and $W = \bigoplus_{k\in\mathcal{D}_{\mathrm{good}}} \mathbb{F}_2 z_k$. Define $\beta(\varepsilon_m, \varepsilon_\ell) = z_{\ell-m}$ if $\ell - m \in \mathcal{D}_{\mathrm{good}}$, zero otherwise. On $V \oplus V \oplus W$ put

$$(x,y,z)(x',y',z') = (x+x', y+y', z+z'+\beta(x,y')). \tag{14}$$

Bilinearity makes the third coordinate in either association $z + z' + z'' + \beta(x,y') + \beta(x,y'') + \beta(x',y'')$. The identity is $(0,0,0)$; the inverse of (x,y,z) is $(x,y,z+\beta(x,y))$. Simultaneously shifting both V indices preserves β; adjoin this automorphism as a. Put $b = (\varepsilon_0, 0, 0)$ and $c = (0, \varepsilon_0, 0)$. Multiplication gives $w_n = (0,0,z_n)$ for $n \in \mathcal{D}_{\mathrm{good}}$, identity otherwise. All relators hold, and every good word has a nontrivial order-two image. Bad words are killed by listed relators, proving the assertion.

For precision we give a uniform Craig conversion [7] of the c.e. relator list to a decidable literal-word list. Chiodo's convention already permits recursive enumeration, so this extra normalization is optional for the cited embedding theorem. Here is a Craig conversion that also preserves the old generator words. Adjoin d with relator $d = 1$. For an enumerated relator r at time t, replace that occurrence by the literal word rd^k, where distinct increasing codes k are chosen effectively with $k \geq t$. Given a word of length ℓ, membership in the resulting literal relator set is decidable by simulating the first ℓ enumeration steps, constructing those coded words of length at most ℓ, and comparing literally. Use freely reduced old words r, which contain no d; the final d^k therefore survives free reduction. Include d itself as a relator. Modulo $d = 1$, the coded relations are exactly the original r. The group is therefore unchanged and the old generator images remain explicit. Higman embedding [14], in the effective uniform formulation of [1, Theorem 2.2], gives a finite presentation with an explicit injective embedding. Injectivity preserves the nontrivial w_n. The model (14) need not itself be computable. □

Represent the generators of $\mathcal{G}_e$ by centralizer unitaries satisfying its finite relators. Add centralizer unitaries T, L. Require every group generator and T to commute with B_0, B_1, C. Impose

$$LQ = cQ, \quad LB_i = B_i a L a^*, \quad [L,C] = [L,P] = 0, \tag{15}$$

and the centralizer polynomial

$$P\big(T[b,L]T^* + [b,L]\big) = 0. \tag{16}$$

Lemma 4.2 (Level reader)**.** *In every semifinite realization $LE_n = a^n c a^{-n} E_n$. The terminal equation then implies that some n is good.*

Proof. Group generators commute with C, hence with $Q = E_0$. In particular $a^*Q = Qa^*$; this is needed in the induction. For a word v of length n, the endpoint equation and commutation of a with the B_i give $LB_vQ = B_v a^n L a^{-n} Q = B_v a^n c a^{-n} Q$. By (10) these ranges span $E_n\mathcal{H}$, proving the level identity. A unitary commuting with B_i also commutes with B_i^*: commute its inverse, then take adjoints. It therefore commutes with P. Thus PE_n reduces the group

generators, T and L. Choose a nonzero such sector by Lemma 3.1. On it (16) becomes $Tw_nT^* = -w_n$. A bad n has relator $w_n = I$, giving $I = -I$ on a nonzero space. This contradiction proves goodness. □

Lemma 4.3 (Finite realization)**.** *A good n gives a finite-algebra realization of all relations.*

Proof. In $\mathcal{G}_e \times \langle j \mid j^2 = 1\rangle$, the subgroup $\langle w_n, j\rangle$ is C_2^2. The map $w_n \mapsto jw_n$, $j \mapsto j$ is its automorphism. Form the HNN extension $\mathcal{K}_n$ with stable letter T:

$$Tw_nT^{-1} = jw_n, \qquad TjT^{-1} = j. \tag{17}$$

The HNN normal-form theorem [15] embeds the base group. Consequently $j \neq 1$, and j commutes with the base and with T, so is central in $\mathcal{K}_n$. This is the group-extension mechanism used in the nonlocal-game setting of [27]. In $L(\mathcal{K}_n)$, $p_- = (1 - \lambda(j))/2$ is central and has trace $1/2$. The corner $N = p_-L(\mathcal{K}_n)$ with trace 2τ is finite. Compressed group unitaries are unitaries of N, and (17) becomes $Tw_nT^* = -w_n$.

Tensor N with the depth-n tree, with density $h_n \otimes 1$ relative to matrix trace times the normalized trace of N. Group generators and T are constant across vertices. Set $L = a^d c a^{-d}$ on depth d. It is a unitary centralizer label commuting with P, C. On a nonleaf, $LB_i = B_i a L a^*$ because depth increases by one; both sides vanish on a leaf. At the root $LQ = cQ$. The terminal equation is the sign-flip identity on the leaves and is vacuous elsewhere. The tree relations hold by (11). Tensor with R_{hyp} and select a factor fibre to obtain a II_1 host. The presentation itself never needs to find the good n. □

For a small example, at good depth $n = 1$ the tree is $\{\emptyset, 0, 1\}$ and the blocks of L are c, aca^{-1}, aca^{-1}. The terminal word is $w_1 = [b, aca^{-1}]$. The model (14) with good set $\{1\}$ sends it to $z_1 \neq 1$, while w_0, w_2 vanish. The HNN step flips the chosen involution after compression to $j = -1$. One T is not required to flip all good words.

5 One bridge compiles the normal-host reduction

We extend the label refinement and rational-row tools of [9, Lemma 4.2 and Theorem B] from spatial strategies to normal commuting strategies. The finite predicates are retained. The proofs below specify where separation replaces Schmidt-range cancellation.

5.1 The calibrated normal bridge

Here FMS means *flagged magic square*; Fu–Miller–Slofstra are cited by name. A question or projection is *synchronized* when a binary event is perfectly matched with its opposite-party copy; on faithful support it is a centralizer projection with its standard commutant mirror. Here is the finite predicate, in zero-based indices. Alice asks R_0, R_1, R_2, T_0, T_1 and Bob asks C_0, C_1, C_2, U_0, U_1. A row answer is a flag f or a triple of product $+1$. A column answer is f or a triple of product $+1, +1, -1$ in the respective columns. At (R_i, C_j) flags must match; nonflags agree on their intersecting cell. Choose $\gamma_0 = \rho_0 = (+++)$ and $\gamma_1 = \rho_1 = (+--)$. At T_x replace f, γ_x by signs $+, -$ and retain the other three column-zero triples; U_y does the same with f, ρ_y in row zero. At (T_x, C_0) signs may match only f, γ_x, and retained triples must equal Bob's triple. Impose the symmetric rule at (R_0, U_y). At (T_x, U_y) two signs a, b must obey $ab = (-1)^{xy}$; pairs with a retained triple are accepted. Unspecified pairs are accepted.

Question pair	Winning rule
(R_i, C_j)	flags match; nonflags agree at cell (i, j)
(T_x, C_0)	a sign pairs only with f, γ_x; retained triples match
(R_0, U_y)	symmetric calibration at ρ_y
(T_x, U_y)	two signs obey $ab = (-1)^{xy}$; a retained triple is accepted
Conditional copy	both inactive or both active; activity, flags and selected events match their binary names
Other pairs	all valid answer pairs accepted

Conditional copies also have an inactive answer; reject exactly one inactive answer, and bind activity, flags and the stated magic events to named synchronized binary questions.

The Pauli square of Mermin–Peres [19, 25] is

$$\begin{pmatrix} XI & IX & XX \\ IZ & ZI & ZZ \\ XZ & ZX & YY \end{pmatrix}.$$

Its four column-zero outcome projections are $m_{00}, m_{01}, m_{10}, m_{11}$ in the order $(+++), (+--), (-+-), (--+)$. On Alice, read m_{00} from the retained answer of T_1 and the other three from T_0; on Bob read them from C_0. Bind each chosen occurrence to the opposite binary name. Also bind positive ZI from Alice's R_1 component 1 and Bob's C_1 component 1, and positive IX from Alice's R_0 component 1 and Bob's C_1 component 0. If $M_{\mathrm{mag}} = \sum m_{ab}$, put

$$x = I - 2M_{\mathrm{mag}} + 2E_{ZI,+}, \qquad z = I - 2M_{\mathrm{mag}} + 2E_{IX,+}.$$

These are centralizer reflections, identity off the magic support. They transport m_{00} to m_{ab} by $x^a z^b$.

For clarity we recall the arbitrary-normal bridge argument of [8, Lemma 2.2]. Matching makes the cells sharp centralizer effects. Write the upper-left four cell reflections as a, b, c, d. The row/column relations give $[a, b] = [a, c] = [b, d] = [c, d] = 0$, $(ac)(bd) = -(ab)(cd)$, and $[ac, bd] = 0$. Canceling the outside a, d gives $cb = -bc$; substitution into the last commutator gives $ad = -da$. The two commuting Pauli pairs therefore generate centralizer M_4. In $\mathbb{C}^2 \otimes \mathbb{C}^2$ the calibrated unit vectors are $\mathfrak{c}_x = |+\rangle \otimes |x\rangle$ and $\mathfrak{r}_y = |+\rangle \otimes |(-1)^y\rangle$, where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|(-1)^y\rangle = (|0\rangle + (-1)^y|1\rangle)/\sqrt{2}$. Choose a minimal magic projection q and centralizer partial isometries u_x, v_y with initial projection q and ranges the projections onto $\mathfrak{c}_x, \mathfrak{r}_y$ (tensored with the multiplicity identity). Choose their phases so that their overlap is $u_x^* v_y = (-1)^{xy} q/\sqrt{2}$. If Δ_x^A, Δ_y^B are the original sign differences, their pull-backs on the standard corner $(f+q)\mathcal{J}(f+q)\mathcal{J}L^2(M)$ are

$$X_x = (f + u_x^*)\Delta_x^A(f + u_x), \qquad Y_y = \mathcal{J}(f + v_y^*)\mathcal{J}\Delta_y^B\mathcal{J}(f + v_y)\mathcal{J}.$$

They are commuting self-adjoint contractions. Their joint-support vector is

$$\theta_{xy} = (f + (-1)^{xy} q/\sqrt{2})\xi_\varphi.$$

Forbidden sign events give $X_x\theta_{xy} = (-1)^{xy} Y_y\theta_{xy}$ and $X_x Y_y \theta_{xy} = (-1)^{xy}\theta_{xy}$. The multiplier $f \pm q/\sqrt{2}$ is bounded invertible in the corner centralizer, so θ_{xy} separates both corner algebras. Commuting the opposite contraction through the first identity gives $X_x^2\theta_{xy} = \theta_{xy}$ and $Y_y^2\theta_{xy} = \theta_{xy}$; separation makes them reflections. Put $\kappa = f + q/\sqrt{2}$. The three positive cells give $X\kappa\xi_\varphi = Y\kappa\xi_\varphi$ and identify $X_0 = X_1 = X$ and $Y_0 = Y_1 = Y$ by separation on $\kappa\xi$. The fourth vector is $Z\kappa\xi$, $Z = f - q$. Its equation is $XZ\kappa\xi = -YZ\kappa\xi = -ZY\kappa\xi = -ZX\kappa\xi$, since opposite algebras commute. Separation gives $XZ = -ZX$. Thus $X = v + v^*$, $v = qXf$, $v^*v = f$, $vv^* = q$. The mirror equation means that X centralizes $\omega(\cdot) = \varphi((f + q/2)\cdot)$. The bounded

centralizer perturbation formula of Pedersen–Takesaki [24, Lemma 4.5] $\sigma_t^{\omega}(y) = k^{it}\sigma_t^{\varphi}(y)k^{-it}$, $k = f + q/2$, gives on the q–f block

$$\sigma_t^{\varphi}(v) = 2^{it}v, \qquad \varphi(q) = 2\varphi(f). \tag{18}$$

The last equality is KMS. Multiplication by the magic matrix units gives all four transports. If one sector vanished, the off-diagonal reflection would kill the other; hence on a nonzero active corner both sectors are nonzero. No finite multiplicity is used.

Lemma 5.1 (Commuting label refinement)**.** *Retain the old FMS questions. For a synchronized projection Z_0, add $T_0[Z_0]$ with answers (a, z_0), use the old rule after forgetting z_0 against unchanged U_0, U_1, C_0, and match z_0 to Bob's binary Z_0. Add the symmetric refined $U_0[Z_0]$ against unchanged T_0, T_1, R_0. Retain activity matching. In every perfect normal commuting strategy the coarse effects equal the original effects and commute with Z_0. Conversely, commuting labels give perfect refinements. Separate labels need not commute.*

Proof. Calibration fixes the retained effects and the sign support. Pull the new coarse sign difference X' to the same corner. Against the unchanged opposite sign it satisfies $X'\theta_{0y} = Y_y\theta_{0y} = X_0\theta_{0y}$. The separating vector above gives $X' = X_0$. Thus its coarse effects E_a are the old projections. For the fine effects Q_{a,z_0}, label matching gives $\sum_a Q_{a,1} = Z_0$ and $\sum_a Q_{a,0} = I - Z_0$ on faithful support. Positivity gives $0 \le Q_{a,1} \le E_a, Z_0$ and $0 \le Q_{a,0} \le E_a, I - Z_0$. Hence $E_aZ_0 = Q_{a,1} = Z_0E_a$, using completeness of each marginal. This proves commutation, and the symmetric argument handles Bob. Conversely use the joint products $E_aZ_0, E_a(I - Z_0)$. Each label has its own refined question with the same unchanged opposite signs, so there is no cross-label commutation constraint. □

5.2 Graphs turn products into rational rows

All projections here are synchronized, hence centralizer projections. For orthogonal F_i, F_j, $F_{ij} = F_i + F_j$, impose

$$G + K_G = F_{ij}, \quad H_G + K_H = F_{ij}, \quad 5H_G + F_{ij} = 3F_i + 4G. \tag{19}$$

On F_{ij}, $Z = F_i - F_j$ and $X = 2G - F_{ij}$ are reflections. Since $2H_G - F_{ij} = (3Z + 4X)/5$, squaring gives $ZX + XZ = 0$. Therefore

$$X = \begin{pmatrix} 0 & u_{ij} \\ u_{ij}^* & 0 \end{pmatrix}, \qquad u_{ij}^*u_{ij} = F_j, \quad u_{ij}u_{ij}^* = F_i.$$

For a triple partition $F_1 + F_2 + F_3 = I$, impose these graphs on all pairs and add a projection $\Pi_{\triangle}$ with $3\Pi_{\triangle} = 2(G_{12} + G_{23} + G_{13}) - I$. Its diagonal blocks are $I/3$, off-diagonal blocks $u_{ij}/3$. The $(1, 3)$ block of $\Pi_{\triangle}^2 = \Pi_{\triangle}$ gives $u_{13} = u_{12}u_{23}$. Conversely this identity makes the entire block matrix a projection. A reference identity triangle gives the scalar M_3 frame. We call this the *qutrit frame*: its three diagonal projections and identity graph projections specify the matrix units of a common scalar M_3, so all application operators act on its multiplicity space. Triangles $(U, 1, U)$ and $(1, U, U)$ identify all placements of a unitary; a triangle (U, V, W) then certifies $W = UV$. An inverse is a product equal to the identity. Take proper prefixes of the finite input word list to make all required products explicit.

To bind a prescribed synchronized projection E, introduce projections EF_i, NF_i with $EF_i + NF_i = F_i$, $\sum_i EF_i = E$, $\sum_i NF_i = I - E$. For each ij put $F_{ij} = F_i + F_j$ and add a joint partition $B_{00}^{ij} + B_{01}^{ij} + B_{10}^{ij} + B_{11}^{ij} = F_{ij}$ with marginals $EF_i + EF_j$ and $G_{ij}(1)$; the fifth, inactive outcome is $I - F_{ij}$. Define $G_{ij}(s_E) = B_{11}^{ij} + B_{00}^{ij}$, the equal-bit event inside F_{ij}. Sharp

marginals make these actual joint products. They force E to commute with the frame and give $s_E = 2e - I$ when $E = I_3 \otimes e$. Thus the parameter is the specified old event.

For a real rational polynomial $p = \sum_w q_w U_w$, the row

$$2\sum_w q_w G_{12}(U_w) - \left(\sum_w q_w\right)(F_1 + F_2) = 0 \tag{20}$$

has off-diagonal blocks p, p^*, so enforces even a non-self-adjoint equation $p = 0$. These arguments are bounded operator identities in the centralizer, which is finite: $\varphi|_{M_\varphi}$ is a faithful normal trace.

Lemma 5.2 (Rational rows on normal commuting hosts)**.** *Given finitely many synchronized projections and real rational linear identities among them, a finite Boolean extension enforces those identities on faithful normal commuting supports. Every realization extends after finite matrix amplification in standard form, preserving the old nonsynchronous tests and the host type.*

Proof. We give the finite extension and both directions. Certify scalar $M_{d_{\max}}$ by synchronized magic squares, with $d_{\max}$ a power of four. Joint questions between axes of distinct squares force their commutation; the Pauli calculation above identifies the tensor matrix algebra. Joint questions between every old projection and every axis force the old projections into its commutant. Thus old projections are $I \otimes e_i$ and scalar matrix units are centralizer elements. Bind global X and Z bit measurements to the axes.

For a row $\mathsf{S}_{\text{row}} = \alpha_0 I + \sum_i \alpha_i e_i$, clear denominators with ℓ_{row}. Using $s_i = 2e_i - I$, write $2\ell_{\text{row}}\mathsf{S}_{\text{row}}$ as a list of signed reflections $V_j \in \{\pm I, \pm s_i\}$ with sum $2\ell_{\text{row}}\mathsf{S}_{\text{row}}$ and first slot $V_0 = I$. Add cancelling $I, -I$ pairs as needed and pad to a power of four d. Its length can be even because the coefficient sum $2\ell_{\text{row}}\alpha_0 + 2\sum_i \ell_{\text{row}}\alpha_i$ is even. For each $j \neq 0$, joint scalar parity questions certify

$$\Pi_j^{\pm} = \tfrac{1}{2}(|0\rangle \pm |j\rangle)(\langle 0| \pm \langle j|).$$

Explicitly use X^j and a basis of the commuting Z parities $j^\perp$; select zero syndrome and the indicated sign. Joint questions of e_i with this three-outcome scalar measurement certify the graph coarse events $G_j^{\pm} = \frac{1}{2}\left(\begin{smallmatrix} I & \pm V_j \\ \pm V_j^* & I \end{smallmatrix}\right)$ on coordinates $0, j$. Here V_j is self-adjoint. The displayed 2×2 matrix acts on $\operatorname{span}\{|0\rangle, |j\rangle\}$ tensored with the old space and is zero on the remaining scalar coordinates.

Add a synchronized partition $(\Pi_t^{\text{row}})_{t \in \mathbb{F}_2^{\log_2 d}}$. Forbid Π_t^{row} with G_j^- when $\chi_t(j) = (-1)^{t \cdot j} = 1$, and with G_j^+ otherwise. A forbidden pair with mirror $\mathcal{J}G\mathcal{J}$ gives $\Pi_t^{\text{row}} G\xi_\varphi = 0$, since $\mathcal{J}G\mathcal{J}\xi_\varphi = G\xi_\varphi$; separation gives $\Pi_t^{\text{row}} G = 0$. Thus a vector in $\operatorname{ran}\Pi_t^{\text{row}}$ has coordinates $v_j = \chi_t(j) V_j v_0$. Define isometries

$$I_t u = d^{-1/2}(\chi_t(j) V_j u)_j, \qquad P_t = I_t I_t^*.$$

Character orthogonality gives $I_t^* I_u = \delta_{tu} I$ and $\sum_t P_t = I$, without commutation of distinct V_j. Hence $\Pi_t^{\text{row}} \leq P_t$; the two partitions sum to I, so $\Pi_t^{\text{row}} = P_t$. Finally forbid Π_0^{row} with the scalar projection $\Pi_d = |+_d\rangle\langle +_d|$, the all-positive global X event. Then

$$I_0^* \Pi_d I_0 = d^{-2}\Big(\sum_j V_j\Big)^*\Big(\sum_j V_j\Big)$$

vanishes exactly when $\mathsf{S}_{\text{row}} = 0$.

Conversely amplify by $M_{d_{\max}}$ with normalized trace. Use the actual scalar Paulis, joint products and projections P_t. All are centralizer projections. Bob uses $\mathcal{J}_{\text{new}} X \mathcal{J}_{\text{new}}$ for each

projection X. For a centralizer word xy, $\mathcal{J}(xy)\mathcal{J}\xi_\varphi = (xy)^*\xi_\varphi$; adjunction supplies the reversed order. Thus Bob's products are the actual transferred products, even when the old projections do not commute. Every structural forbidden event is zero, and the final event is zero precisely for a zero row. Old measurements act trivially on the new matrix factor, preserving all old probabilities. Finite matrix amplification preserves all indicated factor types. □

5.3 The complete one-bridge assembly

Use five orthogonal synchronized corners $e_0, m_{00}, m_{01}, m_{10}, m_{11}$ covering I, with one FMS copy whose flag is e_0 and magic support is their other four corners. Put $e_1 = m_{00}$. The actual bridge $v : e_0 \to e_1$ obeys (18). KMS and the cover give $\varphi(e_0) = 1/9$ and $\varphi(m_{ab}) = 2/9$. These masses are independent of the root model.

$$\underbrace{e_0}_{1/9} \xrightarrow[\sigma_t^{\varphi}(v)=2^{it}v]{v} \boxed{\underbrace{m_{00} = e_1}_{2/9} \quad m_{01} \quad m_{10} \quad m_{11}} \qquad \text{each magic corner has mass } 2/9.$$

Figure 2: One calibrated bridge. All copies of transported labels use this same v; the box is the four-dimensional magic support.

Bind the five corners and the two positive ZI, IX parameters to the qutrit frame. Introduce the two axis unitaries x, z with the explicit formulas above. A *typed label* is an operator name together with its required class (projection, positive contraction or unitary). The transported typed label list is exactly

$$Q,\ C,\ T,\ L,\ \{U_g : g \text{ is a generator of } \mathcal{G}_e\}. \tag{21}$$

Encode a projection by a reflection, a positive contraction by $u_C = 2C - I + 2i\sqrt{C(I-C)}$, and a unitary directly. A *unitary name* is a formal generator for the chosen unitary encoding, including separate names for separate summands when needed. For each resulting unitary name k introduce copies $u_{r,k}$, $r = 0, 1$, with $(u_{r,k} - I)(I - e_r) = 0$. The *decoded copies* are the actual corner operators recovered from these encodings by the following formulas, respectively: $e_r(I + u_{r,Q})/2$, $e_r(2I + u_{r,C} + u_{r,C}^*)/4$, and $e_r u_{r,k}$. Require $u_{r,Q}^2 = I$. Introduce a block unitary

$$V_k = e_0 u_{0,k} + \sum_{a,b} x^a z^b e_1 u_{1,k} z^{-b} x^{-a}.$$

Refine the retained FMS signs separately for the six frame projections

$$F_1, \quad F_2, \quad F_3, \quad G_{12}(1), \quad G_{13}(1), \quad G_{23}(1),$$

and for every $G_{12}(V_k)$. Lemma 5.1 first makes v preserve the frame, so $v = I_3 \otimes \widetilde{v}$. The graph-label commutators in the target–source block give

$$v u_{0,k} = u_{1,k} v. \tag{22}$$

Thus the same actual calibrated transport works for every label. Transport of decoded adjoints and polynomials follows too. P is root-only, encoded by $P_0 = e_0(I + u_P)/2$ with $u_P^2 = I$ and $(u_P - I)(I - e_0) = 0$.

Introduce two centralizer contractions $b_i = e_1 b_i e_0$. For compilation encode each as an average of two global unitaries. Extract $B_i = v^* b_i$ on $e_0 M e_0$. Their modular eigenvalue is $1/2$ because that of b_i is one. The complete endpoint translation is

Root relation	Centralizer polynomial
$B_i^* B_j = \delta_{ij}(I - P)$	$b_i^* b_j = \delta_{ij}(e_0 - P_0)$
$\sum_i B_i B_i^* = I - Q$	$\sum_i b_i b_i^* = e_1 - Q_1$
$CB_i = \frac{1}{2} B_i C$	$C_1 b_i = \frac{1}{2} b_i C_0$
$U_g B_i = B_i U_g$	$(U_g)_1 b_i = b_i (U_g)_0$
$TB_i = B_i T$	$T_1 b_i = b_i T_0$
$LB_i = B_i a L a^*$	$L_1 b_i = b_i a^{[0]} L_0 (a^{[0]})^*$

Every remaining relation is a root centralizer polynomial: the finite relators of $\mathcal{G}_e$, $C_0 Q_0 = Q_0$, $[C_0, P_0] = 0$, $[U_g, C_0] = [T_0, C_0] = 0$, $L_0 Q_0 = c^{[0]} Q_0$, $[L_0, C_0] = [L_0, P_0] = 0$, and the terminal equation. Expand its root projection as $P = (I + s_P)/2$. Here $a^{[0]}, c^{[0]}$ mean the source words for the embedded a, c. If $W = [b, L]$, its non-self-adjoint polynomial is explicitly

$$p = \tfrac{1}{2}(TWT^* + W + s_P TWT^* + s_P W).$$

Use (20) for these words and their prefixes. In global scaffold coordinates multiply by e_0 and expand $e_0 = (I + s_{e_0})/2$ as well. This forces $p = 0$ and $p^* = 0$ via the two off-diagonal blocks; it never assumes PW self-adjoint. Expand all other typed polynomials in the same manner, apply graphs and triangles, then Lemma 5.2. Retain all FMS and refinement questions. Sample all padded question pairs uniformly, reject the listed events, and accept unspecified pairs. This is an effective finite rational Boolean game, denoted G_e.

5.4 Soundness and simultaneous completeness

Soundness uses the nonzero root corner with normalized state $9\varphi|_{e_0 M e_0}$. Equations (22) and the table give all root relations as bounded operator identities. For example $C_0 v^* = v^* C_1$ gives $C_0 B_i = v^* C_1 b_i = B_i C_0/2$. Conjugating the range equation by v^* uses $v^* Q_1 v = Q_0$. The corner of a semifinite host is semifinite, so Lemmas 3.1 and 4.2 prove soundness.

Conversely take any faithful normal root realization (M, φ). On $M_5 \bar{\otimes} M$ use the state

$$\varphi_{\mathrm{sc}} = \mathrm{diag}(1, 2, 2, 2, 2)/9 \ \otimes \varphi. \tag{23}$$

Use the identity corner maps $v = e_{10} \otimes I$, the ideal real FMS effects, and $b_i = e_{10} \otimes B_i$. The product modular group makes each b_i centralizer because the two modular eigenvalues 2 and $1/2$ cancel. The whole centralizer is finite, with its faithful trace φ_{sc}. For any contraction x in a finite von Neumann algebra, the polar partial isometry extends to a unitary u: equivalent initial and final projections have equivalent complements in a finite algebra. Then

$$u_\pm = u(|x| \pm i\sqrt{I - |x|^2}), \qquad x = (u_+ + u_-)/2.$$

This supplies the exact encodings of both b_i at once. Use $2Q - I, 2P - I$ and the displayed functional-calculus encoding of C, copying the same choices to each required corner. Amplify by the scalar qutrit frame. All block labels commute with the ideal FMS signs, so use their actual joint products for refinement. Separate labels use separate questions. Finally apply the standard-form construction in Lemma 5.2, with Bob's effects defined by its conjugation $\mathcal{J}_{\mathrm{new}}$. Formula (23) and the ideal real FMS effects give exactly the five-dimensional FMS probabilities, including the nonsynchronous signs. All graph and endpoint identities hold simultaneously. A finite-algebra root stays finite; after tensoring with R_{hyp} a II_1 host exists. A $\mathrm{III}_{1/2}$ root stays $\mathrm{III}_{1/2}$ under the finite amplifications.

5.5 Chained bridges

Lemma 5.3 (Chained bridges)**.** *For any fixed $k \geq 1$, the conditional FMS predicates, individual label refinements and rational rows above compile the following finite root specifications on arbitrary normal commuting supports. The centralizer labels have prescribed types (projection, positive contraction, contraction or unitary). Each edge A_α has modular ratio 2^{-k}, with isometry and range equations and endpoint relations $L_{\mathrm{end}}A_\alpha = A_\beta B_{\mathrm{end}}$ or $L_{\mathrm{end}}A_\alpha = 0$; $L_{\mathrm{end}}, B_{\mathrm{end}}$ are real rational polynomials in the typed centralizer labels (for example $B_{\mathrm{end}} = C/2$). Other relations are root centralizer polynomials with real rational coefficients. Every perfect strategy extracts that root specification on a nonzero corner. Every faithful normal root realization extends simultaneously to a perfect strategy after finite amplification, preserving its factor type and, for a spatial finite-entropy model, finiteness of entropy. Only labels occurring at a left endpoint (including a range-equation label) need transport; transporting further labels is permitted and changes neither direction.*

Proof. Use the synchronized partition

$$e_0 + \sum_{g=1}^{k} \mathsf{M}_g = I, \qquad \mathsf{M}_g = \sum_{\epsilon,\delta\in\{0,1\}} m_{g,\epsilon\delta}, \qquad e_g = m_{g,00}.$$

For bridge g use a conditional FMS copy with flag e_{g-1}, magic support M_g, and activity $a_g = e_{g-1} + \mathsf{M}_g$. The inactive effect is $I - a_g$. Match activity on every conditional question and bind the flag and four chosen magic events to their named binary projections, in both party orientations. The flag/four-magic/inactive column partition then has precisely these six marginals. Different copies have disjoint question names; no agreement between their nonsynchronous signs is imposed.

The one-bridge proof applies in each $a_g M a_g$. Indeed a_g is centralizer, so its normalized restriction has the restricted modular group. It gives $w_g^* w_g = e_{g-1}$, $w_g w_g^* = e_g$ and $\sigma_t^{\varphi}(w_g) = 2^{it} w_g$. These equations extend by zero off a_g and imply

$$v^{(k)} = w_k \cdots w_1, \quad (v^{(k)})^* v^{(k)} = e_0, \quad v^{(k)}(v^{(k)})^* = e_k, \quad \sigma_t^{\varphi}(v^{(k)}) = 2^{kit} v^{(k)}.$$

This is an extracted operator, never a centralizer word submitted to the graph compiler. KMS and the magic matrix units give

$$\varphi(m_{g,\epsilon\delta}) = 2^g \varphi(e_0), \qquad Z_k^{\mathrm{sc}} = 1 + 4\sum_{g=1}^{k} 2^g, \qquad \varphi(e_0) = (Z_k^{\mathrm{sc}})^{-1}.$$

If an active corner vanishes its flag and all four magic corners vanish together; otherwise the local bridge makes all five nonzero. The recurrence therefore holds also before nonvanishing is known, and the cover proves that $e_0 \neq 0$.

Bind every corner and every positive Pauli parameter to the same qutrit frame. Let Λ be the unitary encodings of the labels to be transported, including every label used on a left endpoint: use a unitary directly, a reflection for a projection, the displayed functional calculus for a positive contraction, and two unitaries for a general contraction. Introduce the axis labels $x_g = I - 2\mathsf{M}_g + 2E_{g,ZI,+}$ and $z_g = I - 2\mathsf{M}_g + 2E_{g,IX,+}$ of each bridge as in Section 5. For each $\ell_{\mathrm{lab}} \in \Lambda$ and $r = 0, \ldots, k$, impose $(u_{r,\ell_{\mathrm{lab}}} - I)(I - e_r) = 0$. Introduce a unitary label $V_{g,\ell_{\mathrm{lab}}}$ and impose

$$V_{g,\ell_{\mathrm{lab}}} = I - a_g + e_{g-1} u_{g-1,\ell_{\mathrm{lab}}} + \sum_{\epsilon,\delta} x_g^{\epsilon} z_g^{\delta} e_g u_{g,\ell_{\mathrm{lab}}} z_g^{-\delta} x_g^{-\epsilon}.$$

The orthogonal block supports make this a unitary. At bridge g apply Lemma 5.1 separately to the six frame projections and to each $G_{12}(V_{g,\ell_{\mathrm{lab}}})$. There is one refinement of each retained sign

question per label per bridge, on each party, always against the unchanged opposite questions. There is no joint refinement of distinct labels. The frame refinements give $w_g = I_3 \otimes \widetilde{w}_g$. In the selected magic–flag block the graph commutator is exactly

$$w_g u_{g-1,\ell_{\mathrm{lab}}} = u_{g,\ell_{\mathrm{lab}}} w_g.$$

Induction gives $(L_{\mathrm{end}})_k v^{(k)} = v^{(k)}(L_{\mathrm{end}})_0$ for every decoded transported label and its adjoints and words. Labels used only at the source need one source encoding and no refinement.

For each edge introduce a centralizer contraction $b_\alpha = e_k b_\alpha e_0$ and extract $A_\alpha = (v^{(k)})^* b_\alpha$. Compile the centralizer equations

$$b_\alpha^* b_\beta = \delta_{\alpha\beta} e_0, \qquad \sum_\alpha b_\alpha b_\alpha^* = e_k - (P_\circ)_k, \qquad (L_{\mathrm{end}})_k b_\alpha = b_\beta (B_{\mathrm{end}})_0$$

for the corresponding root equations; the zero endpoint uses zero on the right. The equations with a specified initial defect use that defect in the first right side. More generally a range equation $\sum_\alpha A_\alpha A_\alpha^* = R_{\mathrm{ran}}$, with R_{ran} a transported centralizer polynomial, is compiled as $\sum_\alpha b_\alpha b_\alpha^* = (R_{\mathrm{ran}})_k$. For the Catalan input this is $R_{\mathrm{ran}} = I - BB^* - P$; its centralizer contraction B and projection P are already in the transported list. Multiplying by $v^{(k)}$ or its adjoint proves both directions, since its initial and final projections are e_0, e_k. Every root-only polynomial stays at e_0. In particular the edge has modular ratio 2^{-k} and all labels refer to the same extracted root operators. Graphs, triangles and Lemma 5.2 enforce these finitely many centralizer polynomials without changing the conditional tests. This proves commuting soundness.

For completeness take a faithful normal root (N, ψ) and on $\mathbb{M}_{1+4k} \bar{\otimes} N$ put

$$\psi_{\mathrm{sc}} = (Z_k^{\mathrm{sc}})^{-1} \operatorname{diag}(1;\, 2^{\oplus 4};\, 4^{\oplus 4};\, \ldots;\, (2^k)^{\oplus 4}) \otimes \psi.$$

Choose the actual matrix corner maps for w_g, the ideal real FMS effects on a_g, and $I - a_g$ for every inactive answer. Take $b_\alpha = |e_k\rangle\langle e_0| \otimes A_\alpha$. Its modular ratio is $2^k 2^{-k} = 1$, so it lies in the finite centralizer. Its polar partial isometry therefore has a unitary completion, giving its two-unitary encoding. Choose the root centralizer encodings once and copy them identically to the required corners. All $V_{g,\ell_{\mathrm{lab}}}$ then commute with the FMS effects at their own bridge, even though adjacent activities overlap at e_g. Use their actual joint products for the separate refinements. Append the common scalar frame and row factors and use standard commutant mirrors as in Lemma 5.2. All old conditional probabilities and every added row are simultaneously correct. Only finite matrix factors have been appended. Their state entropy is finite; in the spatial case it adds to the root entropy. This proves completeness and the stated preservation properties. □

6 Both uniform promises and the hardness theorem

Theorem C (uniform reduction). *Uniformly from a total recursive $\mathcal{R}_{\mathrm{dec}}$ and an input e, the compiler gives a finite rational Boolean game G_e such that*

$$G_e \in \mathrm{PERF}(\mathrm{II}_1) \iff G_e \in \mathrm{PERF}_{\mathrm{sf}} \iff \exists n\, \forall s\, \mathcal{R}_{\mathrm{dec}}(e, n, s).$$

For every input e, G_e has a perfect $\mathrm{III}_{1/2}$ strategy and $\omega_q(G_e) = 1$.

Proof. For the equivalence, soundness follows from the compiler and the selector and level-reader lemmas. A good n gives a finite model by Lemma 4.3, which the compiler turns into perfect finite-host play and hence a II_1 host.

For the first uniform promise take the uniform KMS representation of $\mathcal{O}_2$ with generators s_i satisfying $s_i^* s_j = \delta_{ij} I$, $\sum_i s_i s_i^* = I$ and $\sigma_t(s_i) = 2^{-it} s_i$. Its weak closure is $R_{1/2}$; see the

Cuntz/KMS identification in [22] for the gauge KMS state and [13, Theorem 4.9 and Example 4.13] for this factor identification, also used in [8, Section 3.1]. Set $B_i = s_i$, $P = Q = C = 0$, and take all group generators, T, L equal to I. Every root equation holds: $LQ = cQ$ and the terminal row are vacuous. In the scaffold (23), $b_i = e_{10} \otimes s_i$ is centralizer. Its exact two-unitary completion exists in the finite centralizer, as proved above. The encoding of $C = 0$ is $u_C = -I$ on each corner; the zero projections use reflection $-I$ there as well. Their transported block labels commute with the ideal signs. Thus every compiler gadget has a simultaneous perfect realization. The host, including all finite ancillary matrices, is $\mathrm{III}_{1/2}$.

For the second promise take the finite-tree strategy $\mathcal{V}_N$ with the depth-N finite tree and trivial group labels. Every root equation is exact except the terminal row. Its residual is

$$\Delta_N = 2P, \qquad \varphi_N(\Delta_N^* \Delta_N) = 4\varphi_N(P) = \frac{4}{N+1}. \tag{24}$$

Build all graph, prefix, joint and column measurements by their exact formulas even for this nonzero application row. All their structural tests remain exact. Only its final row-zero event can lose. An *honest row* is the self-adjoint rational projection sum implemented by the exact graph and prefix assignment of the completeness construction. For such a row $\mathsf{S}_{\mathrm{row}}$, that construction gives

$$p(\Pi_0^{\mathrm{row}}, \Pi_d) = \frac{4\ell_{\mathrm{row}}^2}{d^3}\, \varphi_{\mathrm{old}}(\mathsf{S}_{\mathrm{row}}^2), \tag{25}$$

the normal trace formulation of [9, Eq. (32)]. Indeed the scalar density supplies $1/d$, and the two normalized column/reference vectors supply the other two factors of $1/d$.

More explicitly, if the lifted root residual is $p = e_0 \Delta_N$, its graph row has off-diagonal blocks p, p^*. The qutrit trace and root mass give

$$\varphi_{\mathrm{old}}(S_{\mathrm{graph}}^2) = \tfrac{1}{3}\varphi_{\mathrm{sc}}(p^*p + pp^*) = \tfrac{2}{27}\varphi_N(\Delta_N^* \Delta_N).$$

The last equality uses traciality on the centralizer. The *padded question count* is the common size of the two question sets after adding always-accepted fresh questions. An *orientation* is a choice of which party receives the row question and which receives the scalar reference question. If N_e is that count and ℓ_e, d_e are the row denominator and slot dimension, union bounding any repeated orientations gives, for example,

$$1 - \omega(G_e, \mathcal{V}_N) \le \frac{64\ell_e^2}{27 d_e^3 N_e^2}\, \frac{1}{N+1}.$$

Using a larger constant covers duplicate copies of the terminal test if the chosen encoding retains any. There are finitely many, determined solely by e. With $m_e \ge 1$ the finite number of retained copies, take

$$C_e = m_e \frac{64\ell_e^2}{27 d_e^3 N_e^2}.$$

Thus C_e depends only on the finite compiled specification, denominators, row dimensions and the fixed root mass $1/9$. It is independent of N, the density spectrum and the finite tree dimension. Letting $N \to \infty$ proves $\omega_q(G_e) = 1$. □

Choose $\mathcal{A}_{\mathrm{dec}}$ to be Σ_2^0-complete. Theorem C and (4), together with Theorem 2.2, prove Σ_2^0-completeness of every claimed semifinite existence language. Furthermore every G_e is commuting perfect and has quantum value one, and

$$G_e \in \text{III-ONLY} \iff e \notin \mathcal{A}_{\mathrm{dec}}.$$

This proves Π_2^0-hardness within that promise, a stronger statement than hardness of intersection with the value-one language. Membership follows from $\mathrm{PERF}_{\mathrm{co}} \in \Pi_1^0$ and $\mathrm{PERF}_{\mathrm{sf}}^c \in \Pi_2^0$: their intersection is Π_2^0.

7 Each type-III subtype has the commuting language

The state and host may both be chosen. This quantifier is essential.

Background lemma (Expected-inclusion transfer; [28, 8]). If $E : N \to M$ is a faithful normal conditional expectation, every faithful standard-form correlation on M is realized on N.

Proof. For the state $\varphi_N = \varphi_M \circ E$, put $\mathcal{H}_{\mathrm{inc}} = \overline{M\xi_{\varphi_N}} \subset L^2(N)$. The map $x\xi_{\varphi_M} \mapsto x\xi_{\varphi_N}$ is an isometry. Takesaki's expectation theorem [28] makes M invariant under the modular group. The Tomita operators agree on the common core $M\xi$, and their conjugations restrict as $\mathcal{J}_N|_{\mathcal{H}_{\mathrm{inc}}} = \mathcal{J}_M$. Thus for left effects A and right effects $\mathcal{J}_M B \mathcal{J}_M$,

$$\langle \xi_{\varphi_N}, A\mathcal{J}_N B \mathcal{J}_N \xi_{\varphi_N} \rangle = \langle \xi_{\varphi_M}, A\mathcal{J}_M B \mathcal{J}_M \xi_{\varphi_M} \rangle.$$

Positivity, sums and all probabilities are preserved. Support reduction handles nonfaithful states. This is also [8, Lemma 3.1]. □

Lemma 7.1. *For any factor N_{in} with separable predual, $N_{\mathrm{in}} \bar{\otimes} R_\infty$ is III_1, and contains N_{in} with a faithful normal expectation.*

Proof. The expectation is $\mathrm{id} \otimes \rho$ for a faithful normal state ρ on R_∞. We use the proof, rather than the stronger hyperfinite conclusion, of [12, Lemma 3.8]. For each $0 < \lambda < 1$, choose $0 < \mu < 1$ with $\log \lambda / \log \mu$ irrational. The injective factors satisfy $R_\infty \cong R_\lambda \bar{\otimes} R_\mu$ and $R_\infty \bar{\otimes} R_\lambda \cong R_\infty$. Consequently $(N_{\mathrm{in}} \bar{\otimes} R_\infty) \bar{\otimes} R_\lambda \cong N_{\mathrm{in}} \bar{\otimes} R_\infty$. The asymptotic-ratio argument of Connes [4, Section 3.6], used in that proof, gives $\lambda \in r_\infty(N_{\mathrm{in}} \bar{\otimes} R_\infty)$. Connes's implication [4, Theorem 3.6.1] gives $r_\infty \setminus \{0\} \subseteq S$. Every $0 < \lambda < 1$ therefore belongs to the S-invariant, and the factor is III_1. These steps do not assume hyperfiniteness of N_{in}. Only the last identification with the unique injective III_1 factor in Haagerup–Musat needs hyperfiniteness, and we do not make that identification here. □

Theorem 7.2. *For each fixed $\lambda \in [0, 1]$, $\mathrm{PERF}(\mathrm{III}_\lambda) = \mathrm{PERF}_{\mathrm{co}}$.*

Proof. The inclusion into commuting perfection is immediate. For the converse, standardize a commuting-perfect strategy and choose a factor host N_{in} by central decomposition as in Section 1. Put $N_{\mathrm{abs}} = N_{\mathrm{in}} \bar{\otimes} R_\infty$. By expected-inclusion transfer and Lemma 7.1, this III_1 factor hosts the game. Retain it for $\lambda = 1$.

Fix a faithful normal state φ on N_{abs}. For a countable subgroup $\Gamma \subset \mathbb{R}$, given the discrete topology, form $N_\Gamma = N_{\mathrm{abs}} \rtimes_{\sigma^\varphi|_\Gamma} \Gamma$. The Fourier expectation onto N_{abs} is faithful normal, so transfers the game. It is a factor: $\mathrm{T}(N_{\mathrm{abs}}) = \{0\}$ for a III_1 factor, hence every nonzero modular time is outer. If a central element has Fourier coefficient x_g, commutation with N_{abs} makes x_g an intertwiner between id and σ_g^φ. A nonzero such coefficient for $g \neq 0$ would, by polar decomposition in a factor, implement that automorphism innerly. Thus only $g = 0$ remains, and x_0 is scalar. Faithfulness of Fourier coefficients proves factoriality.

The continuous core is the crossed product by the modular $\mathbb{R}$-action; its dual action restricted to the center is the *flow of weights*. For factors this flow distinguishes the type-III subtypes [6]. Here is the core computation, with the dual flow specified. The modular group of the dual state fixes the canonical unitaries u_g of the discrete crossed product. Writing $\lambda(t)$ for the continuous-core implementing unitaries, it follows that

$$\begin{aligned} c(N_\Gamma) &\cong c(N_{\mathrm{abs}}) \rtimes_{\mathrm{Ad}\,\lambda(g)} \Gamma \\ &\cong c(N_{\mathrm{abs}}) \bar{\otimes} L(\Gamma). \end{aligned} \tag{26}$$

For the second isomorphism the unitaries $w_g = \lambda(g)^* u_g$ commute with $c(N_{\mathrm{abs}})$ and satisfy the group law of Γ; the inverse sends u_g to $\lambda(g) \otimes w_g$. Since N_{abs} is III_1, $c(N_{\mathrm{abs}})$ is a factor,

so the center is $L(\Gamma) = L^\infty(\widehat{\Gamma})$. The dual action obeys $\theta_s(\lambda(t)) = e^{-ist}\lambda(t)$ and $\theta_s(u_g) = u_g$; therefore $\theta_s(w_g) = e^{isg}w_g$. Its action on the center is translation by the character $\chi_s(g) = e^{isg}$.

For $0 < \lambda < 1$, take $\Gamma = t_\lambda\mathbb{Z}$, $t_\lambda = 2\pi/(-\log\lambda)$. The resulting flow is rotation on the circle with least period $-\log\lambda$, which is the flow of a III_λ factor [6]. For $\lambda = 0$ take $\Gamma = \mathbb{Z} + \sqrt{2}\mathbb{Z}$. Its dual is the two-torus, and the flow is $s \mapsto (e^{is}, e^{i\sqrt{2}s})$. Fourier characters show ergodicity: an invariant character indexed by (m, n) requires $m + n\sqrt{2} = 0$, hence $m = n = 0$. Every orbit is a dense one-dimensional subgroup coset of Haar measure zero (its bounded segments have zero area). Thus the flow is not transitive. The Connes–Takesaki flow classification [6] gives subtype III_0. This is the same modular crossed-product mechanism as [12, Example 3.9], without requiring injectivity of N_{abs}.

For clarity non-semifiniteness is a separate conclusion. The core flow of a semifinite factor is translation on $L^\infty(\mathbb{R})$; it is transitive and admits no invariant normal probability state. Each of our circle or torus flows preserves Haar probability, so cannot be that semifinite flow. For N_{abs} itself non-semifiniteness was already proved by its S-invariant. Hence none of the subtype conclusions silently includes a semifinite alternative. □

Slofstra's commuting classification [27], also proved in [20, Theorem 6.4], now proves Π_1^0-completeness for every fixed subtype, including zero.

8 The type-I and finite-type-II languages are incomparable

8.1 A fixed one-bridge separator

Apply the complete one-bridge assembly of Section 5 to the fixed root presentation

$$A^*A = I, \qquad AA^* = I - P_\circ, \qquad \sigma_t^\varphi(A) = 2^{-it}A, \tag{27}$$

where $P_\circ$ is a centralizer projection. Call the resulting finite rational Boolean game R_*. Here the transported label list contains just $P_\circ$. Encode its copies by reflections, introduce a centralizer contraction $b = e_1 b e_0$, and impose exactly

$$b^*b = e_0, \qquad bb^* = e_1 - (P_\circ)_1.$$

Use the six frame refinements and the graph-label refinement for $P_\circ$ from Section 5; apply the same graphs, triangles and rational rows. Extraction is $A = v^*b$, and $P = (P_\circ)_0$. Thus $A^*A = e_0$, $AA^* = e_0 - P$, and the modular ratio is $1/2$. Conversely every root model of (27) extends by that assembly, with $b = e_{10} \otimes A$. This specifies the game without a chained-bridge hypothesis.

For its type-I model take $\ell^2(\mathbb{N}_0)$, $A|m\rangle = |m+1\rangle$, $P_\circ = |0\rangle\langle 0|$, and

$$h|m\rangle = 2^{-m-1}|m\rangle, \qquad S(h) = \sum_{m\geq 0}(m+1)2^{-m-1} = 2 \text{ bits}.$$

The relation $hA = Ah/2$ verifies modular covariance. The scaffold and the finite maximally entangled compiler factors add finite entropy, so R_* is perfect spatially with finite entropy.

In any perfect finite-host strategy, extraction takes place in the nonzero finite corner $e_0 M e_0$. Its isometry A must be unitary: a faithful finite trace gives $\tau(e_0 - AA^*) = 0$. Consequently $AA^* = e_0$ forces $P = 0$. On the other hand KMS for the original faithful state gives

$$\varphi(AA^*) = \tfrac{1}{2}\varphi(A^*A) = \tfrac{1}{2}\varphi(e_0), \qquad \varphi(P) = \tfrac{1}{2}\varphi(e_0) > 0,$$

a contradiction. Therefore R_* has no finite host and witnesses $\mathrm{PERF(I)} \not\subseteq \mathrm{PERF}(\mathrm{II}_1)$. In particular it has no finite-dimensional perfect strategy.

8.2 What happens to the old Weyl-clock family

The companion reduction [10, Sections 4–7] uses four Fock isometries A_{ab} on words in $\{0,1\}^2$ with

$$A_{ab}^* A_{cd} = \delta_{ac}\delta_{bd} I, \qquad \sum_{ab} A_{ab} A_{ab}^* = I - P_\circ, \qquad h_F A_{ab} = A_{ab} h_F/16,$$

where $h_F(w) = (3/4)16^{-|w|}$. Depth m is a 2^m square with coordinates $x = \sum a_j 2^{j-1}$, $y = \sum b_j 2^{j-1}$. Its centralizer coordinate unitaries increment x, y cyclically; its boundary projections select $x = 0, y = 0$. The clock labels satisfy

$$J_{\mathrm{W}} K_{\mathrm{W}} = Z_{\mathrm{clk}} K_{\mathrm{W}} J_{\mathrm{W}}, \quad W|_{(x,y)} = Z_{\mathrm{clk}}^y, \quad C_{\mathrm{ctr}}|_{(x,y)} = 2^{-y}.$$

The reader equations are

$$(W - I)P_{\mathrm{ret}} = 0, \qquad T_{\mathrm{inv}}(W - I) = \tfrac{1}{2} C_{\mathrm{ctr}}(I - P_{\mathrm{ret}}). \tag{28}$$

Spatial soundness uses finite positive density eigenspaces and a determinant to force a finite-order phase. This determinant is unavailable in a diffuse finite factor.

Here is a normal II_∞ model of *every* input of that family. Put $\theta = (\sqrt{5} - 1)/2$, $\zeta = e^{2\pi i\theta}$. In the finite rotation factor $L^\infty(\mathbb{T}) \rtimes_\theta \mathbb{Z}$, take unitaries $J_{\mathrm{W}}, K_{\mathrm{W}}$ with commutator ζI and put $Z_{\mathrm{clk}} = \zeta I$. The invariant integral and Fourier coefficient at zero give a faithful finite trace. Irrational rotation is free and ergodic, so a central Fourier coefficient vanishes except at zero and its zero coefficient is constant; the algebra is a factor. It is infinite-dimensional and hence II_1. It is the hyperfinite factor R_{hyp} by amenability of this crossed product and injective-factor classification [5]. The construction itself only needs that it is finite.

Use the tensor density $h_F \otimes 1$ on $B(\ell^2(\mathcal{W}))\bar{\otimes}R_{\mathrm{hyp}}$. Set $W = \zeta^y$, $C_{\mathrm{ctr}} = 2^{-y}$, $P_{\mathrm{ret}} = 1_{\{y=0\}}$, and

$$T_{\mathrm{inv}}|_{(x,y)} = \begin{cases} 0, & y = 0, \\ 2^{-y-1}(\zeta^y - 1)^{-1}, & y > 0. \end{cases}$$

We verify its norm globally, not just by a finite numerical sweep. For an integer q nearest to $y\theta$, the nonzero integer $q^2 + qy - y^2$ factors as $(q - y\theta)(q - y\theta')$, $\theta' = -(1 + \sqrt{5})/2$. Since $|q - y\theta| \le 1/2$,

$$\|y\theta\| \ge \frac{1}{\sqrt{5}\, y + 1/2}, \qquad |\zeta^y - 1| = 2\sin(\pi\|y\theta\|) \ge \frac{4}{\sqrt{5}\, y + 1/2} > 2^{-y-1}.$$

The sine inequality uses concavity, $\sin(\pi t) \ge 2t$ for $0 \le t \le 1/2$. In particular it supplies the explicit all-integer bound $|\sin(\pi y\theta)| \ge 8/(11y)$, since $\sqrt{5} < 9/4$ and $y \ge 1$. For the last inequality it suffices to replace $\sqrt{5}$ by $9/4$: $2^{y+3} > 9y/4 + 1/2$ for $y = 1$, and doubling proves it inductively. Thus T_{inv} is a contraction and (28) holds. The binary coordinate recurrences $y \mapsto 2y + b$ verify the clock and counter endpoint equations.

In the companion's tableau [10, Sections 5.3 and 7] a waiting front advances one cell per time step, at $x = y$ at time y. The first positive-time return pulse P_{ret} freezes it, and the frozen coordinate becomes the unary input of the check of the hardwired predicate; the time-zero pulse is ignored. Here P_{ret} selects $y = 0$ only, so no positive-time pulse occurs. The front never freezes, and the companion decoder never starts a checking computation, irrespective of the hardwired input. Its finite deterministic tableau relations are satisfied by that classical nonhalting tableau in every finite square. The coordinate, boundary and tableau projections are diagonal on word sectors and commute with $h_F \otimes 1$. All endpoint labels have the required centralizer type. Lemma 5.3 with $k = 4$ applies to exactly the conditional copies and transported labels of [10, Section 4.4 and Lemma 4.4] and to the endpoint rows of [10, Sections 5–7]. Its left-endpoint labels are $P_\circ, U_x, U_y, P_x, P_y, W, C_{\mathrm{ctr}}, Z_{\mathrm{clk}}, J_{\mathrm{W}}, K_{\mathrm{W}}$; source-only

labels are $P_{\mathrm{ret}}, T_{\mathrm{inv}}$ and the tableau projections. The activity corners and separate refinements coincide with that compiler. Its root mass is $1/121$. The resulting ambient host is II_∞ for every input. Put $r_y = 2^{y+1}|\zeta^y - 1|$. The global minimum of $\log_2 r_y$ is attained at $y = 1$, and

$$2.89 < \log_2 r_1 < 2.9, \qquad \log_2 r_1 = 2.898452049757\ldots. \tag{29}$$

Here is a certificate, including the infinite tail. For $y \geq 3$, the preceding bound gives $r_y \geq 2^{y+3}/(\sqrt{5}\,y + 1/2) > 8$: at $y = 3$ use $\sqrt{5} < 9/4$, and the ratio increases because $2(\sqrt{5}\,y + 1/2) > \sqrt{5}\,(y+1) + 1/2$. For $y = 2$ the exact certificate described below gives $r_2 > 8$. For $y = 1$ it gives $1 < r_1 < 8$ and $r_{1,-}^{100} > 2^{289}$, $r_{1,+}^{10} < 2^{29}$ for rational bounds $r_{1,-} < r_1 < r_{1,+}$. To construct them, enclose $\sqrt{5}$ by integer square roots at scale 10^{80}, and π by $16\arctan(1/5) - 4\arctan(1/239)$ with 60 alternating terms and the next-term remainder. Evaluate $\sin(\pi\theta)$ at the rational endpoints using 40 alternating terms and the next term; sine is decreasing on this interval in $(\pi/2, \pi)$. Round outward to denominator 10^{40} and compare the displayed integer powers. For $y = 2$, use $|\sin(2\pi\theta)| = \sin(\pi(2\theta - 1))$; its argument is in $(0, \pi/2)$, where sine increases, and the same series gives $16\sin(\pi(2\theta - 1)) > 8$. The bundled exact-fraction routine implements these instructions. It certifies the $y = 1, 2$ enclosures and the rational base and coefficient inequality used in the tail estimate. Together with the induction above these prove the minimum over all positive integers; the 3999-point floating sweep is supplementary. The weaker positive margin is all that contractivity requires.

Lemma 5.3, with $k = 4$, also extracts the four isometries of every companion game in any normal commuting host. Two isometries with orthogonal ranges cannot lie in a nonzero finite corner. Hence every companion game has no finite host. This additional fact is independent of the separator R_* above.

8.3 A tracial game with no spatial perfection

For the other direction use a linear-system game with a perfect commuting strategy and no perfect finite-dimensional strategy, provided by [26]. This does not assert that linear-system games are synchronous. The solution-group criterion [2, Theorems 4–5] says its distinguished central involution j is nontrivial. The central corner $(1 - \lambda(j))L(\Gamma)/2$ has normalized trace and a perfect left/right regular strategy: equation observables commute in each constraint, square to I, and their product is the specified sign because $j = -I$ in the corner. Use their joint spectral projections on the left and the commutant mirrors of variable observables on the right. Consistency and each equation are then exact. A factor fibre is a finite factor host; tensoring with R_{hyp} gives II_1. This is the tracial mechanism in [8, Proposition 7.4].

Spatial perfect play implies finite-dimensional perfect play for linear-system games by Cleve–Mittal [3] and [2, Theorem 5]; Slofstra [26] uses this implication in the nonclosure construction. We give the argument for completeness. Each variable appearing in a constraint has an Alice involution with a self-adjoint Bob mirror on the state vector, by perfect consistency. Equality of those vectors makes Alice's trace-class marginal commute with every such involution. The within-constraint products and parity equations hold on the marginal support, by the zero-loss equations and support separation. A positive density eigenvalue has a nonzero finite-dimensional eigenspace reducing all variable involutions. Restriction to it preserves the products and all constraint equations. Its maximally entangled vector with transposed Bob variables is a finite-dimensional perfect strategy, a contradiction. For a mixed spatial state first purify; the same Alice eigenprojection argument applies. Type-I hosting decomposes into spatial factor fibres, so it is also excluded. Thus $\mathrm{PERF}(\mathrm{II}_1) \not\subseteq \mathrm{PERF}(\mathrm{I})$.

Every type-I factor host tensored with R_{hyp} has type II_1 in finite dimension and type II_∞ in infinite dimension; a further $B(\ell^2)$ amplification handles the former case. Every II_1 host similarly amplifies to II_∞. The two witnesses just constructed prove

$$\mathrm{PERF}(\mathrm{I}) \text{ and } \mathrm{PERF}(\mathrm{II}_1) \text{ are incomparable}, \qquad \mathrm{PERF}(\mathrm{I}),\ \mathrm{PERF}(\mathrm{II}_1) \subsetneq \mathrm{PERF}(\mathrm{II}_\infty).$$

Equal arithmetical completeness does not imply equal languages.

9 Perfect in X but not in Y

Use different fonts for existence languages and operator labels:

$$\mathsf{q} = \mathrm{PERF}_q, \quad \mathsf{E} = \mathrm{PERF}_{\text{finite entropy}}, \quad \mathsf{I} = \mathrm{PERF}(\mathrm{I}),$$

$$\mathsf{J} = \mathrm{PERF}(\mathrm{II}_1), \quad \mathsf{U} = \mathrm{PERF}_{\mathrm{sf}}, \quad \mathsf{C} = \mathrm{PERF}_{\mathrm{co}}.$$

Here q requires attainment in finite dimension; it is not the value-one language. The inclusions are $\mathsf{q} \subset \mathsf{E} \subset \mathsf{I} \subset \mathsf{U} \subset \mathsf{C}$ and $\mathsf{q} \subset \mathsf{J} \subset \mathsf{U}$. Equation (4) identifies U with both PERF(II) and PERF(II_∞); Theorem 7.2 identifies C with each fixed type-III language. Write

$$D_2(\Sigma_2^0) = \{A \setminus B : A, B \in \Sigma_2^0\}.$$

This is a class of differences, not the intersection of the two pointclasses Σ_2^0 and Π_2^0. The canonical complete language is $\{(e, f) : e \in \mathrm{FIN},\ f \notin \mathrm{FIN}\}$, where FIN is the Σ_2^0-complete set of indices of finite c.e. sets. Indeed reduce the two Σ_2^0 predicates separately to FIN and pair the resulting indices.

9.1 Exact conjunction and disjunction

Background lemma (Exact game operations). Uniform effective operations $G \wedge H$ and $G \vee H$ satisfy, for every $\mathsf{X} \in \{\mathsf{q}, \mathsf{E}, \mathsf{I}, \mathsf{J}, \mathsf{U}, \mathsf{C}\}$,

$$G \wedge H \in \mathsf{X} \iff G, H \in \mathsf{X}, \qquad G \vee H \in \mathsf{X} \iff G \in \mathsf{X} \text{ or } H \in \mathsf{X}.$$

The same statements hold for each of the type languages in this paper.

Proof. First pad question and answer alphabets to common finite alphabets. For an original zero-weight question pair accept everything; preserve its original predicate on positive-weight pairs. Give every padded question pair positive rational weight, accept all pairs involving fresh questions, and reject a padded answer on an original question. This changes no perfect-play language. It permits fixed common question alphabets with full support; no implication below relies on an untested pair.

For conjunction, send both players the same tag and questions for that tagged game, and test its predicate. A perfect strategy restricts to either tag in the same ambient algebra and state. Conversely, tensor two perfect strategies and on a tagged question measure the appropriate factor. Type I is closed under tensor products, as are finite factors, type II_∞ factors, and semifinite algebras. Amplification gives the stated II existence languages when needed. For E use pure spatial witnesses: marginal entropies add, so their sum is finite. Finite-dimensional witnesses stay finite. Commuting strategies tensor as well. For III_λ use its language equality with C from Theorem 7.2; no closure of that subtype under direct tensoring is asserted.

For disjunction, ask the common questions and require each answer to name a branch $s \in \{G, H\}$ as well as an answer in that branch. Reject mismatched branches and otherwise test the named predicate. A strategy for either game wins by always naming that branch. For soundness let $A^x_{s,a}, B^y_{s,b}$ be the effects and put $P^x_s = \sum_a A^x_{s,a}$, $Q^y_s = \sum_b B^y_{s,b}$. Work first on the faithful standard-form support. The matching argument of Section 1 makes each branch marginal a projection, with $P^x_s \xi = Q^y_s \xi$ for every x, y. Fixing y and using separation makes $P^x_s = P_s$ independent of x; likewise $Q^y_s = Q_s$. These are opposite mirrors, $P_s\xi = Q_s\xi =: \xi_s$. At least one ξ_s is nonzero. Since $0 \le A^x_{s,a} \le P_s$, we have $A^x_{s,a} P_s = P_s A^x_{s,a} = A^x_{s,a}$, and the

same holds on Bob's side. Complete each branch POVM to the identity by adding $I - P_s$ to one Alice answer and $I - Q_s$ to one Bob answer. These added effects annihilate ξ_s. A losing branch product annihilates ξ and, by the displayed support identities, also ξ_s. Normalizing ξ_s therefore wins branch s. All effects and the vector remain on the same Hilbert space, in the same ambient M, M'; the selected vector need not be faithful. Thus its host type is preserved. This also handles mixed strategies by purification and standardization.

For finite entropy one can work in the original spatial purification. Orthogonality of Bob's branch projections gives $\rho_A = \sum_s p_s \rho_{A,s}$, $p_s = \|\xi_s\|^2$. Entropy concavity, valid also by finite-rank truncation for infinite entropy, gives $\sum_s p_s S(\rho_{A,s}) \le S(\rho_A) < \infty$. Every positive-weight branch has finite entropy. This proves the last model-specific assertion and completes both directions. □

9.2 Four fixed separators and two uniform families

Denote the companion's Weyl-clock reduction by G_e^s and this paper's selector/group reduction by G_e^t. Choose both reductions for the same predicate $A(e) \equiv e \in \mathrm{FIN}$. Theorems 7.1 of [10] and C here, including Lemma 4.3, give

$$G_e^s \in \mathsf{E} \iff G_e^s \in \mathsf{I} \iff A(e), \tag{30}$$

$$G_e^t \in \mathsf{J} \iff G_e^t \in \mathsf{U} \iff A(e), \qquad G_e^t \in \mathsf{C}. \tag{31}$$

Every G_e^s has no finite-dimensional perfect strategy; every G_e^t is commuting perfect with $\omega_q = 1$. The chained lemma additionally excludes G_e^s from J and places it in U (Section 8). The reductions below use only its displayed spatial coordinates: R_* supplies the finite-host exclusion wherever it is needed. No assumption about spatial perfection of a YES G_e^t is made.

Take the fixed one-bridge game R_* of Section 8.1. Its shift model has root entropy 2 bits and its finite compiler factors preserve finite entropy. The finite-corner KMS contradiction there proves that it has no finite host. Thus its complete profile uses only the one-bridge assembly.

Let T_* be the tracial solution-group game constructed in Section 8. It is in $\mathsf{J} \setminus \mathsf{I}$; that section proved spatial perfection would imply finite perfection. Let K_{Cat} be the compiled Catalan game of [9, Theorem C and Sections 3 and 6]. Its perfect spatial strategies have infinite marginal entropy, so it is in $\mathsf{I} \setminus \mathsf{E}$. Lemma 5.3 with $k = 2$ applies to its nine-corner assembly: activity $e_{g-1} + \mathsf{M}_g$, root mass $1/25$, transported encodings b_+, b_-, p, q, c (names of [9, Section 6]), and the source-only polynomials of [9, Section 6]. Thus its extracted isometry A has modular ratio $1/4$ also on a commuting normal host. In a finite root corner A would be unitary, and KMS would give $1 = 1/4$; hence it is outside J as well. Finally choose a fixed NO instance of G^t and call it Q_*. Theorem C gives $Q_* \in \mathsf{C} \setminus \mathsf{U}$ and $\omega_q(Q_*) = 1$. Their complete profiles are

	q	E	I	J	U	C
R_*	0	1	1	0	1	1
T_*	0	0	0	1	1	1
K_{Cat}	0	0	1	0	1	1
Q_*	0	0	0	0	0	1

In particular $R_* \wedge T_*$ is in U but in neither I nor J, so even $\mathsf{I} \cup \mathsf{J} \subsetneq \mathsf{U}$. The remaining inclusions asserted above are strict: R_* separates q from E, K_{Cat} separates E from I, T_* separates I from U, and Q_* separates U from C. Both E and I are incomparable with J.

9.3 The table, with reductions for every nonempty entry

Theorem 9.1 (Directional differences)**.** *The following table gives row minus column. Each suffix c means computable many-one completeness for the displayed pointclass. The symbol $D^\dagger$ means membership in $D_2(\Sigma^0_2)$, Σ^0_2-hardness and Π^0_2-hardness; completeness remains open.*

	q	E	I	J	U	C
q	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
E	Σ^0_2-c	$\varnothing$	$\varnothing$	D_2-c	$\varnothing$	$\varnothing$
I	Σ^0_2-c	D_2-c	$\varnothing$	D_2-c	$\varnothing$	$\varnothing$
J	Σ^0_2-c	D_2-c	D_2-c	$\varnothing$	$\varnothing$	$\varnothing$
U	Σ^0_2-c	$D^\dagger$	$D^\dagger$	D_2-c	$\varnothing$	$\varnothing$
C	Π^0_1-c	Π^0_2-c	Π^0_2-c	Π^0_2-c	Π^0_2-c	$\varnothing$

Proof. All empty entries follow from the stated inclusions. For upper bounds, $\mathsf{q} \in \Sigma^0_1$, $\mathsf{E},\mathsf{I},\mathsf{J},\mathsf{U} \in \Sigma^0_2$, and $\mathsf{C} \in \Pi^0_1$. A difference between two middle languages is therefore in D_2. For a middle X, $\mathsf{X} \setminus \mathsf{q} \in \Sigma^0_2$: write its predicate as $\exists k \forall m$ for X and conjoin the Π^0_1 predicate for nonmembership in q. Also $\mathsf{C} \setminus \mathsf{q} \in \Pi^0_1$ and $\mathsf{C} \setminus \mathsf{X} \in \Pi^0_2$. It remains to prove the matching lower bounds.

For the four incomparable entries use $G^s_e \wedge R_*$ as the E- or I-side family, and $G^t_f \wedge T_*$ as the J-side family. Their disjunction has E and I coordinates exactly $A(e)$ and J coordinate exactly $A(f)$. Thus its membership in either $\mathsf{E} \setminus \mathsf{J}$ or $\mathsf{I} \setminus \mathsf{J}$ is $A(e) \wedge \neg A(f)$. Swap e, f between the two branches for the reverse differences. The canonical D_2 problem reduces in all four cases. This settles I-ONLY, defined here as $\mathsf{I} \setminus \mathsf{J}$. It cannot mean absence of every type-II host, since $\mathsf{I} \subset \mathsf{U}$.

For the q column, use G^s_e for E and I; use $G^t_e \wedge T_*$ for J and U. Their larger coordinate is $A(e)$ and their finite-dimensional coordinate is identically zero. Each such entry is therefore Σ^0_2-hard.

For the four rightward entries of the C row, use $G^s_e \vee Q_*$ when $\mathsf{X} = \mathsf{E},\mathsf{I}$, and $G^t_e \vee Q_*$ when $\mathsf{X} = \mathsf{J},\mathsf{U}$. The commuting coordinate is always one and the X coordinate is exactly $A(e)$, giving Π^0_2-hardness. These instances all have $\omega_q = 1$, since choosing branch Q_* approximates perfection in finite dimension. For $\mathsf{C} \setminus \mathsf{q}$, take a Σ^0_1-complete finite-perfection reduction M_e with $M_e \in \mathsf{q} \iff e$ halts, available by perfect completeness in [16, Theorem 12.6] and finite-dimensional feasibility. Then $M_e \vee K_{\mathrm{Cat}}$ is always commuting perfect, has $\omega_q = 1$ because K_{Cat} is spatial perfect, and is in q exactly when e halts. Nonhalting reduces to the difference, proving Π^0_1-hardness even under the promise.

For the two remaining complete nested differences use a shared family. In general suppose $\mathsf{Y} \subset \mathsf{X}$ and a computable family $\mathcal{F}_e$ has both coordinates equal to $A(e)$. Fix $H \in \mathsf{X} \setminus \mathsf{Y}$. For

$$\Lambda(e,f) = \mathcal{F}_e \wedge (H \vee \mathcal{F}_f) \tag{32}$$

the X coordinate is $A(e)$ and the Y coordinate is $A(e) \wedge A(f)$. The difference coordinate is therefore $A(e) \wedge \neg A(f)$, proving D_2-hardness. Apply this first with $(\mathsf{X},\mathsf{Y}) = (\mathsf{I},\mathsf{E})$, $\mathcal{F}_e = G^s_e$ and $H = K_{\mathrm{Cat}}$, using (30). Apply it again with $(\mathsf{X},\mathsf{Y}) = (\mathsf{U},\mathsf{J})$, $\mathcal{F}_e = G^t_e$ and $H = R_*$, using (31). Thus INF-ENTROPY-ONLY and $\mathsf{U} \setminus \mathsf{J}$ are $D_2(\Sigma^0_2)$-complete. The required YES witness has finite entropy by [10, Theorem 7.1].

Finally, for $\mathsf{U} \setminus \mathsf{I}$ and $\mathsf{U} \setminus \mathsf{E}$, the map $e \mapsto G^t_e \wedge T_*$ is Σ^0_2-hard: its U coordinate is $A(e)$ and both spatial coordinates vanish. The map $e \mapsto G^s_e \vee T_*$ is Π^0_2-hard: its U coordinate is one and both E,I coordinates equal $A(e)$. Together with the D_2 upper bound, this proves exactly the two $D^\dagger$ entries. The theorem makes no completeness claim for them. □

Corollary 9.2. The language of games with perfect spatial play but no finite-entropy perfect spatial strategy is $D_2(\Sigma^0_2)$-complete. In particular it belongs to neither Σ^0_2 nor Π^0_2.

Proof. The identification is $\mathsf{I} \setminus \mathsf{E}$ in the theorem. A D_2-complete language is hard for both Σ_2^0 and Π_2^0 by fixing one of the canonical FIN indices to a YES or NO index. Membership in either pointclass would collapse these two pointclasses, contradicting strictness of the arithmetical hierarchy. □

The corollary concerns exact attainment; it asserts no rate at which entropy must diverge for near-optimal strategies.

For differences between middle models, value-one padding can be made explicit. Replacing G by $G \vee Q_*$ preserves all lower coordinates and sets the commuting coordinate to one, while forcing $\omega_q = 1$. To remove finite-dimensional perfect play as well, use $G \wedge (R_* \vee T_*)$; the second factor has all five infinite-model coordinates equal to one and its q coordinate zero. In particular (32) for INF-ENTROPY-ONLY already has no finite-dimensional perfect instance; adding Q_* also makes every instance value one. The C-row hardness families above already have value one, including the separate $\mathsf{C} \setminus \mathsf{q}$ construction.

9.4 The remaining nested differences

For a nested pair $\mathsf{Y} \subset \mathsf{X}$ of Σ_2^0-complete languages with a separator H, independent hardness families give both kinds of hardness: $G_X(e) \wedge H$ has difference predicate $A(e)$ and $G_Y(e) \vee H$ has difference predicate $\neg A(e)$. They do not by themselves provide the shared family required by (32). This explains the two open table cells. We use II-ONLY to mean $\mathsf{U} \setminus \mathsf{I}$, with $\mathsf{U} \setminus \mathsf{E}$ its finite-entropy variant; the incomparable difference $\mathsf{J} \setminus \mathsf{I}$ is already complete by the theorem.

Remark (A finite monotone obstruction). An exhaustive Boolean check in the verification bundle ranges over all 7581 monotone Boolean functions of five inputs $G_e^t, G_f^t, G_e^s, G_f^s, T_*$ and all profiles consistent with the proved coordinates. For $\mathsf{U} \setminus \mathsf{I}$, none realizes $A(e) \wedge \neg A(f)$ uniformly over the unproved spatial coordinates of G^t. Even setting both of those coordinates optimistically to zero gives none. With a shared family $\mathcal{F}_e, \mathcal{F}_f$ as two inputs, 351 functions do realize it. This is a bounded exhaustive check, not an impossibility theorem for new game constructions. Separate completeness and the inclusion lattice do not supply the shared-family equivalence needed by this reduction.

An open route is a selector-fed Fock tableau, replacing the Weyl clock by a 16-ary critical selector. One would require commuting selector and tableau labels and an equality reader

$$(2C_{\mathrm{ctr}} - C_{\mathrm{sel}})P_{\mathrm{ret}} = 0, \qquad T_{\mathrm{inv}}(2C_{\mathrm{ctr}} - C_{\mathrm{sel}}) = \tfrac{1}{8}C_{\mathrm{ctr}}C_{\mathrm{sel}}(I - P_{\mathrm{ret}}), \qquad P_H P = 0.$$

The intended pulse selects a finite candidate, and a finite-depth selector tensored with the Fock tableau would be the finite-entropy YES model. What remains open is soundness of the additional selector/tableau relations with the extra defect-isometry family, and the resulting shared-family equivalence $\mathsf{E}(\mathcal{F}_e) = \mathsf{I}(\mathcal{F}_e) = \mathsf{U}(\mathcal{F}_e) = A(e)$. These equations describe the proposed shared-family construction. Completeness for both II-ONLY variants therefore remains open.

10 Scope and remaining questions

These are existence languages: the host and its normal state may be chosen together. They do not decide whether a prescribed factor and state realize a correlation, or determine the least resources of a strategy. Beyond the two open nested differences, natural questions ask which syntactic classes force finite-entropy spatial perfection whenever spatial perfection is possible, and how quantitative entropy requirements interact with the difference classification. The finite-algebra YES model of the present selector/group family alone gives no spatial YES model; replacing that missing assertion by separate completeness would not settle either open cell.

How this work was done

The authors selected the questions and proof standards, directed the research, and take responsibility for the claims. AI systems assisted with derivation, adversarial review, programming and drafting. This preprint has not been peer reviewed.

The mathematical evidence consists of the proofs and the cited external theorems. The verification bundle checks finite tree identities, the group witness, a finite stand-in for the HNN corner, entropy and residual formulas, separator density identities, and the golden-mean margin. Finite computations do not validate Higman's embedding, the HNN normal-form theorem, the infinite KMS classification or the continuous-core argument. The companion release is identified by DOI 10.5281/zenodo.23910987; source manifests use relative paths and hashes.

A Verification and source provenance

The portable verification program is `verification/check_type_ladder.py`. Run it from any directory with Python 3 (standard library only) and its explicit `-output-dir` option pointing to a new directory. It refuses to overwrite an existing results file. It uses exact rational identities for the finite tree and rational stand-in, randomized finitely supported group elements with a fixed seed for (14), and numerical entropy and rotation checks. An exact-fraction enclosure certifies the golden-mean value at $y = 1$, the $y = 2$ comparison, and the rational base of the induction covering every $y \geq 3$. The Boolean profile core tests every table reduction with unspecified coordinates adversarial, and enumerates all 7581 monotone five-input functions for the bounded obstruction remark. Exact checks are distinguished from floating-point checks in its JSON output. The checks are finite evidence, with the scope stated in How this work was done.

The finite stand-in is not asserted to be a representation of an arbitrary $\mathcal{G}_e$ or its HNN extension. It tests the level-reader and terminal algebra with an explicit balanced involution. Existence of the actual finite von Neumann algebra for any good n rests on the HNN normal-form theorem and regular representation in Lemma 4.3.

The source manifest records relative bundle paths and SHA256 hashes. The provenance record identifies external mathematical sources by DOI or arXiv identifier. The archive contains the authored paper sources and portable verification source, not live companion directories. `SHA256SUMS` uses LF line endings. The build script uses `pdflatex`, BibTeX, and repeat `pdflatex` runs, and rejects warnings, unresolved references, overfull boxes and underfull boxes in the final log. Rendered page images are inspected separately for layout.

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