Preparation relativity in noisy local systems

Certified common readouts, selected limits, and the scope of memory

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Abstract

For rational finite-range probabilistic cellular automata, effective compactness recovers the equivalence between agreement of all stationary laws on a local indicator and a common almost-sure empirical limit with uniform statistical certificates. Corresponding common-readout statements hold for positive rational Gibbs specifications on $\mathbb{Z}^2$ and finite-depth finite-range local quantum channels, with different experimental meanings. This forgetting half recovers existing computability and certification principles. The contribution is exact preparation-selected spectra in specified noisy and equilibrium experiments, paired with recovered common-readout certification. The standard limit-computable upper bound is matched on specified intervals (the persistence interval of the routed experiment; $\Delta_2^0 \cap [1/3, 2/3]$ in the fixed line, square-lattice Gibbs and three-dimensional quantum experiments). These fixed experiments include an ordinary origin average on the line, a square-lattice Gibbs boundary readout, and a three-dimensional noisy quantum record. The joint statement retains explicit model, noise, preparation and observation quantifiers. The general dimension-threshold statement remains Conjecture. General realization of exponentially mixing temperature sets, its dynamics counterpart, and a physical prediction remain open; specific exact interval magnets are distinguished from the general open problem.

Claude Opus 5.5, GPT-6.1 Sol, GPT-6.2 Sol and GPT-6.2 Astra assisted with this work; the authors take responsibility for every claim.

Reading convention. Appendix A records the evidential scope of the cited results and distinguishes archival preprints, work in preparation, conjectures and open problems.

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0 The organizing claim and its status

The organizing contribution is exact preparation-selected spectra in specified noisy and equilibrium experiments, paired with recovered common-readout certification. In the effective compact-state settings of Theorem 2.1, a singleton compatible-state readout interval is equivalent to sound universal enclosures of arbitrarily small width. This is a recovery and reorganization of effective compactness, with the prior computability, strong-law and certified-equilibrium results of Marcovici–Sablik–Taati, Galatolo–Hoyrup–Rojas, Breiman, Jenkinson and Fawzi–Fawzi–Scalet credited in Section 2 [29, 30, 31, 32, 33]. The new half identifies the exact spectrum of computably selected definite limits, rather than only an upper bound on their complexity. In the named PCA, square-lattice Gibbs and quantum-channel experiments, all limit-computable values in the specified interval occur; the PCA realizations include ordinary origin averages for one fixed line rule from computable deterministic starts. Theorems 2.2 and 2.3 and the equilibrium and quantum realizations state the distinct quantifiers [10, 27, 12, 13, 14]. Their conjunction is the organizing contribution: a certified common-readout theorem beside exact selected spectra, without identifying scalar computability with preparation independence.

A useful question is therefore: which part of a long-run measurement is fixed by the local rule, and which part is selected by preparation? We use "derivable" in the operational sense of a rule-based approximation with a sound error certificate that is uniform over the admissible preparations. This meaning is stronger than the bare existence of an algorithm computing a particular selected real. A preparation-dependent readout can be rational. Its rationality does not supply a certificate that the same readout occurs from the other preparations.

The distinction is also relative to the observation. A rule with several stationary states can have an indicator on which all of them agree. That indicator falls under the common-readout theorem even while another indicator distinguishes the phases. Conversely, a failure of convergence at a date-by-date observation is not automatically a failure of a Cesaro average. The theorems below specify which experiment supplies the value.

Other results sharpen the choice of output task. A fixed cubic rule transports a limit-lemma sequence in one computable preparation to a single-site limit of halting degree at rate $1/100$ [39]. An application of the companion recovery theorems to a distinct operational task separates decidable bounded-depth exact recovery in self-similar codes, robust approximate-recovery hardness and exact-threshold hardness with a computable infimum. In the stated code classes, unsteered single-excitation propagation has a computable value, while worst-case fresh inputs steering a dissipative memory along one ray give a robust undecidable gap without broadcast [38]. Their scopes are stated below.

Conjecture 0.1 (Preparation-relativity extension)**.** *For effectively presented positive-rate local systems beyond the classes established here, an appropriately specified compact compatible-state set and local affine readout admit sound complete finite enclosures; moreover, a suitable*

nondegenerate preparation-selection experiment realizes all limit-computable values in its persistence interval. The model class, allowed preparations and experiment must be stated before either conclusion can be tested.

The conjecture separates an enclosure assertion from a realization assertion. The latter is not inferred from the presence of two phases. A model satisfying only the enclosure assertion is not thereby a counterexample to the proved selection constructions. An extension claiming native, fixed-site ordinary averages would need an additional theorem for that observation protocol.

What is contributed by the synthesis

The contribution is exact preparation-selected spectra in specified noisy and equilibrium experiments, paired with recovered common-readout certification: the named PCA, $\mathbb{Z}^2$ Gibbs and quantum-channel realizations retain their distinct preparation and observation quantifiers. Published theorems are cited rather than re-proved; no new glue lemma is needed here. Undecidability of PCA ergodicity is already known from Toom, who credits Kurdyumov, and from the deterministic ergodicity–nilpotency comparison of Bušic, Mairesse and Marcovici [63, 64, 65]. The results below refine that diagnosis to fixed positive whole-symbol replacement or a fixed rule with halting-indexed rate windows. A universal precision search can coexist with an undecidable agreement promise. Xu and collaborators give a nearby zero-temperature one-site certificate-versus-diagnosis obstruction [46, Supplementary Corollary 3.6]. For one fixed routed square-lattice PCA, on computable windows of positive replacement rates, capacity zero cannot be uniformly distinguished from capacity at least $1 - h_2(1/4) \approx 0.18872$ bits per observation [14]. For one fixed finite-energy square-lattice interaction, all-DLR uniqueness and a fixed odd-observable phase gap encode halting on designated inverse-temperature windows [2, version 2.1.0, Theorem B]. These are scoped obstructions to deciding the promise, not obstructions to the soundness of a certificate once found. Sections 8 and 9 give the limitations and refutation conditions.

The term "only observable" will be used, when needed, as shorthand for the absence of a computable precision horizon for a specified noncomputable selected limit. It does not assert that all preparation-dependent values are noncomputable, that a finite observation determines an infinite real, or that the local rule alone specifies a preferred preparation. Those would be different statements and are not consequences of either cited theorem.

1 Setting and definitions

We use the effective-state formulation of Douglas and Mghirbi (in preparation) [10]. A local description R specifies finite rule data, range and lattice $\mathbb{Z}^d$, with computable coefficients. The compatible states form a nonempty compact convex set K: all stationary laws for a PCA, all DLR laws for a Gibbs specification, or all stationary states for a local quantum channel. Spatial translation invariance is imposed only when expressly stated. A computable local affine readout f takes values in $[0, 1]$. Write $f(K) = [a, b]$.

A preparation ρ is independent of future innovations. It is computable when finite marginals have a uniform fast approximation algorithm. A deterministic configuration is computable when its symbols are computable uniformly in position. Gibbs selection instead uses computable exterior collars and a computable cofinal sequence of volumes. A selected experiment has a computable description and uniformly computable finite readouts. A real is limit-computable if it is the limit of a total computable rational sequence, without a specified convergence modulus.

Write $V(R, \rho, f)$ only when the stated experiment has a definite value. For classical trajectories this may mean the deterministic almost-sure limit of $A_T = T^{-1} \sum_{t<T} f(X_t)$; for Gibbs systems it may mean a finite-volume expectation limit; for quantum channels it may mean an unmeasured Cesaro expectation. These are different conventions. Preparation invariance means equality over the preparations named in the assertion. A *derivable common value* additionally has a finite, sound error certificate uniform over that class.

We use explicit descriptive memory notions. *Compatible-readout memory* means $a < b$ for the chosen K, f. *Failure of all-state attraction* means that no single stationary law attracts every preparation locally. *Finite-time readout persistence* means that the extrema of $P^t f$ retain a positive limiting gap. *Scalable quantum storage* means a size-growing storage horizon with the fidelity and encoder/decoder quantifiers of the cited theorem. Agreement on one readout is weaker than agreement of states. These notions are not silently identified.

For a fixed experiment and readout, the *computably selected spectrum* is the set of deterministic definite limits obtained from computable admissible preparations. A spectrum theorem restricted to an interval I characterizes these limits in I; it makes no assertion about limits outside I. For the line trajectory theorem the preparations are deterministic computable configurations, and the observation is the ordinary average at one fixed site [27].

A universal certificate is a finite rational enclosure that is sound even when agreement fails; termination at arbitrarily fine precision is promised only under agreement. A confidence certificate additionally bounds the probability of a deviation at *any* later date. "Positive rate" refers to strictly positive transition probabilities, or to the explicitly specified independent replacement/depolarizing noise; it is not a claim about a laboratory rate. Entropies in Section 5 are in nats unless information rate is explicitly in bits. Definitions do not assert realizability of an arbitrary abstract K by a finite rule.

2 The principle as two scoped theorems

2.1 A universal interval and a certificate

Theorem 2.1 (Common-readout boundary law; recovery)**.** *Assume the effective interval premise: computable nested rational closed intervals J_n contain $f(K)$, and every point of $\bigcap_n J_n$ is realized by a compatible state. Then $f(K) = \bigcap_n J_n = [a, b]$, with attained endpoints, a left-c.e. and b right-c.e. The following are equivalent: all compatible states agree on f; the enclosure widths tend to zero; and finite sound universal enclosures of arbitrarily small width can be found by search. On that promise the common value is computable uniformly from the finite presentation and precision [10].*

Memory notion: compatible-readout agreement versus compatible-readout memory.

This theorem is a recovery and reorganization of effective compactness, not a priority claim for the forgetting half. Marcovici, Sablik and Taati proved that a computable PCA with a unique invariant measure has a computable invariant measure [29, Proposition 2.2]. Galatolo, Hoyrup and Rojas give the related deterministic effective-compactness result under their computability and measure hypotheses [30, Corollary 4.2.1]. Breiman's every-start Feller-chain strong law is a classical precedent for the trajectory implication [31]. The finite-cone endpoint argument is the Markov counterpart of the deterministic ergodic-optimization formula [32, Proposition 2.2]. For local quantum Hamiltonians, Fawzi, Fawzi and Scalet already give ground-state and equilibrium expectation intervals and certified algorithms [33, arXiv version 3, Theorems 1.1, 1.2, 3.6 and 3.7]. The model-specific selection realizations below are a separate contribution [10, 12, 27, 13].

The certificate is not a decision procedure for the singleton promise. A search that produces a sufficiently narrow interval has produced a finite checkable object; it does not provide an algorithm deciding whether a narrower object will eventually exist on an arbitrary input. This distinction is the relevant computational content of the theorem. "With a certificate" refers to sound precision bounds, with no polynomial-time or practical sample-size assertion.

For a rational finite-range PCA on the full shift over $\mathbb{Z}^d$, with finite alphabet and fixed finite d, finite-cone Cesaro functions give complete stationary-readout intervals. For a local indicator, singleton agreement over all stationary laws is equivalent to a common in-probability empirical limit from every preparation, and yields almost-sure convergence and all-later confidence certificates [10]. The deterministic-table-plus-uniform-replacement specialization is also established by Douglas and Mghirbi (in preparation) [11]. Thus the preparation quantifier includes noncomputable stationary laws. Restricting it to computable preparations alone is not justified by this equivalence.

For orientation, the finite quantities used by the cited certificate are

$$F_m = \frac{1}{m} \sum_{j<m} P^j f, \qquad L_m = \min_x F_m(x), \quad H_m = \max_x F_m(x).$$

They are rational for the rational PCA presentation, and dyadic m gives nested intervals. This is a description of the certificate being cited, not a new proof. The model-specific statement in the schema gives, with $q_m = (L_m + H_m)/2$, positive η and integer $B \geq 1$,

$$\mathbb{P}\Big(\exists T \geq mB : \ |A_T - q_m| > \frac{H_m - L_m}{2} + \eta + \frac{1}{B}\Big) \leq \frac{4}{B\eta^2}. \tag{1}$$

A reported closed exceptional inequality requires the strict slack specified there [10]. The bound is uniform in the initial law. Its output has an experimental interpretation, but the computation needed to obtain m may be enormous.

2.2 What changes between equilibrium and quantum dynamics

For a finite-alphabet finite-range Gibbs specification on $\mathbb{Z}^2$ given by strictly positive rational weight tables, with rational local f, the analogous enclosures optimize finite-volume expectations over all finite exterior collars. Their intersection is the range over all DLR laws. A singleton is equivalent to uniform boundary-independent convergence of these expectations, and the finite partition sums certify the common value. The certificate applies to later containing volumes, so it also bounds a cofinal exhaustion [12]. Canel proves effective closedness of the Gibbs class for the stated finite-range computable Hamiltonian at positive computable temperature [43, Example 5.10 and Theorem 5.11]; the present formulation supplies explicit collar enclosures and their finite-volume meaning. Binder and collaborators and Gayral–Hoyrup give related equilibrium-computability and optimization results under their respective hypotheses [44, 45]. The memory notion here is DLR readout multiplicity, not a Glauber mixing time. Positivity of the finite partition functions is part of the presentation.

The difference between uniqueness and attraction is concrete: Chassaing–Mairesse give a line PCA with one invariant law and nonconverging instantaneous laws; Jahnel–Külske give an exponentially localized three-dimensional example with a quasiperiodic orbit [47, 48]. Neither uniqueness alone nor a scalar enclosure premise is substituted for all-state attraction.

An equilibrium boundary is not an initial row evolving in time. An assertion about the common expectation of a local spin in all DLR laws does not by itself identify a convergence rate for a stochastic dynamics sampling those laws. In particular, the enclosure horizon is a volume horizon. A paper using it as a time horizon would need an additional sampling theorem with the relevant boundary and update conventions.

For a computable finite-depth finite-range quantum channel, compatible states are the stationary states and finite-cone Cesaro observables have certified extremal eigenvalue enclosures. The common-readout consequence concerns uniform convergence of *unmeasured Cesaro expectations* [10][11]. Its memory notion is stationary local-expectation multiplicity. A continuing invasive instrument is a different channel, so a measurement-record assertion requires the instrument-specific theorem. Brambilla, Tosini and Perinotti develop finite local instruments for causal reversible quantum cellular automata, a nearby measurement framework rather than an unmeasured-expectation theorem [49]. The circuit realization cited later

supplies a classical invariant subalgebra for a more specialized record interpretation; it does not convert arbitrary measurements into passive observations.

The Heisenberg adjoint of a trace-preserving quantum channel is unital as an operator map. That formal property is unrelated to the separate resource question of whether the Schroedinger dynamics preserves the maximally mixed reference. We keep those uses distinct when discussing entropy. Otherwise a routine duality identity could be mistaken for a restriction excluding noisy quantum memory.

2.3 Selected limits: the upper bound and the extra realization premise

Theorem 2.2 (Computable selection and exact spectra; in preparation)**.** *A definite limit of uniformly computable finite-experiment readouts is limit-computable. The same upper inclusion holds for uniformly bounded empirical observations converging in probability to a deterministic constant when their expectations are uniformly computable. Equality with all limit-computable values in an interval additionally requires a realization theorem [10].*

Such equality holds for the routed experiment built from a rational finite-range classical PCA or a computable quantum channel with an effectively known finite backward cone: on one extra lattice coordinate, independently evolving layers pass a fixed binary detector. If $[a_t, b_t]$ *is the readout range after* t *rounds of the original channel and* $[a, b] = \bigcap_t [a_t, b_t]$*, the deterministic almost-sure frequency limits from computable preparations of the routed system are exactly*

$$[a, b] \cap \Delta^0_2.$$

The classical preparation can be a deterministic computable configuration. The quantum preparation can be a computable product of pure layer states. Endpoints are included [14].

Memory notion: finite-time readout persistence of the original channel; stationary detector range and selected frequencies of its routed extension.

The upper inclusion is standard. The noiseless frequency benchmark is the Downey–Jockusch–Schupp density classification, recorded by Downey, Jockusch, McNicholl and Schupp: densities of computable subsets of $\mathbb{N}$ are exactly $\Delta^0_2 \cap [0, 1]$ [50, Theorem 2.1]. Avigad, Gerhardy and Towsner give finite-window stability bounds for ergodic averages even when a full computable convergence rate is unavailable [51]. The realization claim here adds its specified positive noise and passive observation.

Here Δ^0_2 denotes limit-computable reals in the representation of Section 1. The positive-width interval is a property of a specified detector. The theorem does not require a decision procedure for positive width. It also does not claim the same selection theorem for the original un-routed dynamics. The added coordinate makes separately prepared layers available at successive dates; that is a resource in the experiment.

Two other precise realizations broaden the evidence without changing these quantifiers. For one fixed nearest-neighbour zero-field square-lattice Ising interaction, with equal-spin weight 16 and unequal-spin weight 1, computable deterministic boundaries along one fixed computable cofinal sequence of rectangles realize exactly the limit-computable expectations in $[1/3, 2/3]$; the corresponding finite-volume laws converge weakly [12]. For one fixed eight-layer three-dimensional circuit with seven qubits per cell, independent single-qubit depolarization at $e = 3^{-30}$ after every layer, and one fixed one-qubit projector, computable diagonal preparations realize exactly $\Delta_2^0 \cap [1/3, 2/3]$ as deterministic almost-sure ordinary single-record frequency limits in that interval [13]. The latter statement uses the circuit's particular invariant classical subsystem and routing convention. Coquille–Velenik supply the static finite-volume plus/minus-mixture approximation; van Enter–Medveď–Netočný and van Enter–Netočný–Schaap study random-boundary phase selection and pure-phase subsequences [52, 53, 54]. The equilibrium realization above instead selects a prescribed limit along one fixed cofinal sequence using deterministic computable collars. Kretschmann–Werner supply causal memory-channel representations, and Drechsler–Oreg–Aharonov relate CSS memories under Pauli noise to classical stochastic processes [55, 56]. These precedents do not supply the particular fixed-circuit diagonal-preparation record theorem.

Theorem 2.3 (Line-native exact selected spectrum; in preparation)**.** *At the fixed positive rational whole-symbol replacement rate of [3, version 2.0.0, equation (3.1)], there are one fixed finite-alphabet finite-range rule Q on $\mathbb{Z}$ and one fixed Boolean one-site event f for which the computably selected spectrum $\Sigma^{\mathrm{comp}}(Q, f)$ of deterministic almost-sure origin-average limits satisfies*

$$\Sigma^{\mathrm{comp}}(Q, f) \cap [1/3, 2/3] = \Delta_2^0 \cap [1/3, 2/3].$$

Here selection ranges over computable deterministic initial configurations. For every $r \in \Delta_2^0 \cap [1/3, 2/3]$, a preparation is obtained uniformly from a total convergent rational approximation to r, without a convergence modulus. The observation is the ordinary all-date average at the fixed origin [27].

Memory notion: preparation-selected ordinary origin averages; no all-state-attraction promise.

This realization uses the line itself and needs no added routing coordinate. It does not assert ergodicity of Q, instantaneous-density convergence or full weak convergence of row laws. A separate theorem on the line fixes a strictly one-way rule and characterizes computably selected stationary weak limits; its numerical density example has interval $[1/4, 3/4]$ [11]. On $\mathbb{Z}^2$, one fixed computable inhomogeneous preparation and a local reporter, with the rule compiled from a limit index at the same replacement rate, realize exactly $\Delta_2^0 \cap [1/3, 2/3]$ for ensemble-density limits and almost-sure ordinary frequencies; a deterministic computable preparation supplies the frequency version [11]. These are additional realizations with different fixed and varying inputs.

2.4 The exact point of the organizing claim

In every realization of the dependent half, the computable preparation carries the target: a limit-lemma sequence is laid out in space, in initial data or exterior collars, and the memory transports it to the readout. Preparation dependence means precisely that this spatially encoded target, together with the specified observation protocol, selects the long-run value. The local rule supplies the persistent medium and the transport; it need not generate the target independently of its preparation [14, 27, 12, 13, 39, 127, 128].

The common-value theorem and the selection theorem quantify different things. The common-value theorem ranges over all compatible states or all initial laws. The selection theorem ranges over computable descriptions of selected experiments, and asserts a definite deterministic limit for the output. Allowing an arbitrary noncomputable mixture weight would permit values outside Δ^0_2 and would invalidate its upper inclusion. Conversely, a stationary mixture of phases with different temporal means need not have its mixture expectation as an almost-sure temporal mean.

Within the routed setting both statements are available for one experiment: interval collapse gives the common certified value, while positive width supplies noncomputable selected frequencies as well as computable ones. That is the exact scoped form of the proposed divide. Elsewhere we cite the matching parts separately. The slogan "dependent implies uncomputable" is not used as a theorem: even a system realizing a noncomputable frequency also realizes rational frequencies in the interior of its interval.

2.5 A preparation-encoded single-site limit at a specified rate

Theorem 2.4 (Selected single-spin halting degree; in preparation)**.** *There exist one fixed finite deterministic rule on the cubic lattice $\mathbb{Z}^3$, one computable initial law independent of the rate, and one binary spin at the origin such that, under independent uniform whole-site replacement at rate $p \in (0, 1]$, its ordinary single-site probability has a limit $\ell(p)$. At $p = 1/100$, the time-iterated laws converge weakly to a stationary law and*

$$\deg_T \ell(1/100) = \mathbf{0}'.$$

At every other positive computable $p \in (0, 1]$, $\ell(p)$ is computable. Here $\mathbf{0}'$ is the Turing degree of the halting problem [39].

Memory notion: a preparation-selected ordinary one-site probability limit; no temporal averaging or optimization.

This is an instance of the dependent half of the organizing claim, rather than a criticality result. The computable, spatially inhomogeneous preparation lays out a limit-lemma sequence on incoming planes; the dynamics transports that sequence to the observed spin.

In the construction,

$$\theta_n = \sum_{\substack{i \le n \\ i \text{ halts within } n \text{ steps}}} 4^{-(i+1)}, \qquad \ell(1/100) = \frac{1-m}{2} + m \lim_n \theta_n,$$

with known computable $m > 0$. Localization to $1/100$ comes from the observed autonomous component's own coexistence point. That component attracts all starts off this point; the full rule still coexists below it [39]. The noncomputability is carried by the preparation's limit, rather than generated by criticality.

Hellouin de Menibus and Sablik already characterize noiseless cellular-automaton limit measures from computable initial measures as exactly the limit-computable ones: their realization of shift-invariant targets permits an enlarged alphabet and a target-dependent automaton [40, arXiv version 3, Corollary 1]. Marsan–Sablik characterize stationary-law accumulation sets as the noise tends to zero [57]; Delacourt–Hellouin de Menibus give a higher-dimensional deterministic limit-measure counterpart [58], while Coronel and collaborators realize semicomputable invariant-measure simplices in computable systems [59]. The theorem above is a preparation-selected one-site probability of halting degree at a specified positive rate: it fixes the noisy rule and the preparation across rates. Gacs and Toom supply positive-error information survival, and Marsan–Sablik–Törmä give reentrant ergodicity of a fixed noisy line rule [60, 20, 124, 61]. Xu and collaborators use a halting-indexed Hamiltonian family and an extremum over KMS ground states; Shiraishi–Matsumoto use time and spatial averages in isolated quantum systems [46, 41]. These are different parameter and observation conventions.

2.6 Two Ising trajectory realizations

Theorem 2.5 (Ordinary programming for a moving-frame readout of an interacting reversible medium; in preparation)**.**

Fix the binary PCA on $\mathbb{Z}^2$ with neighbourhood $\{0, e_1, e_2, e_1 + e_2\}$. If k of the four input bits are one, the output is one with probability

$$q(k) = \frac{1000^{\,2k-4}}{1 + 1000^{\,2k-4}},$$

independently across sites and dates. For every limit-computable $r \in [1/3, 2/3]$, a deterministic computable initial configuration, obtained uniformly from a total convergent rational approximation to r, gives

$$\mathbb{P}\left(\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} X_t(0) = r \right) = 1.$$

The rule and origin-bit detector are fixed [127].

Memory notion: preparation-selected ordinary averages at the native PCA origin, corresponding to a moving Ising readout.

Under the identification of native site (i, j) at date t with physical Ising site $(i+j+t, i-j)$, this rule is the checkerboard heat bath of the nearest-neighbour zero-field Ising model at $\beta = \log(1000)/2$. The frame moves one physical site per date; the physical Ising detector is therefore not at rest. The reversible-medium background is due to Kozlov–Vasilyev and Dai Pra–Louis–Roelly [130, 129]. The interface input is the aligned Dobrushin-strip estimate of Lubetzky–Martinelli–Sly–Toninelli [131, Theorem 5.3, equation (5.3)]; it is used with no slits and endpoints at the same normal height. The ordinary-time preparation compiler is due to Douglas and Mghirbi (in preparation) [127].

Theorem 2.6 (Zero-temperature fixed-site programming: diffusion of a single interface delivers a scale-encoded preparation to the origin; in preparation)**.**

Consider continuous-time zero-temperature Glauber dynamics on $\mathbb{Z}^2$, with rate-one site clocks and fair ties. There is a uniform algorithm taking an index of any total rational sequence converging to $r \in [-1, 1]$ and producing a deterministic computable spin configuration such that

$$\mathbb{P}\left(\lim_{T\to\infty} \frac{1}{T} \int_0^T \sigma_{(0,0)}(t)\, dt = r \right) = 1.$$

No convergence modulus is an input. The measured spin stays at the fixed physical origin [128].

Memory notion: preparation-selected ordinary time averages at a site at rest, carried by one unbounded directed interface.

The exclusion-process identification of the corner-flip interface has the classical precedent of Spohn; symmetric-exclusion duality and its heat-equation description are standard in Liggett [132, 133]. The preparation programs the initial interface on increasing spatial scales, and diffusion delivers it to the fixed column. Long-time behaviour encoded in initial data at infinity has PDE precedents in Vázquez–Zuazua and Cazenave–Dickstein–Weissler [134, 135]. The Repnikov–Eidelman criterion relates stabilization of bounded heat-equation data to their large-scale spatial means [136]. These precedents concern diffusion and asymptotics; the uniform computable preparation with the stated almost-sure fixed-spin ordinary average is the result cited here [128].

3 Forgetting, preparation, and undecidable classification

3.1 A computable common value need not have a decidable promise

Theorem 3.1 (Fixed-noise forgetting is undecidable; cited)**.** *For the computable family of noisy local rules in [1, version 3.0.0, Theorem A and Corollary 3.1], ergodicity is equivalent to halting at one fixed positive rational whole-symbol replacement rate. In its six-dimensional natural Wang construction, the equivalence holds throughout* $0 < \varepsilon \leq 10^{-24}$*. A tiling input gives two distinct translation-invariant stationary laws. The same release gives a four-dimensional natural-rule result at* $0 < \varepsilon \leq 10^{-30}$ *and a three-dimensional result for north/east deterministic tile inputs, with their stated restrictions [1, version 3.0.0, Theorems 12.5 and 12.11].*

Memory notion: all-state attraction in the ergodic branch; stationary-state multiplicity in the memory branch.

This is a fixed positive replacement-noise refinement of PCA ergodicity undecidability. Toom already proves the diagnosis for translation-invariant binary line PCA with transitions in $\{0, 1/2, 1\}$, and for one-dimensional continuous-time local processes with rates in $\{0, 1\}$; Bušic–Mairesse–Marcovici give the deterministic ergodicity–nilpotency comparison [63, 64, 65]. The reference for the present refinement is DOI `10.5281/zenodo.24265098`. The theorem is not a lower bound for evaluating a fixed finite cone. Each such cone is a finite computation. It rules out a total classifier for the asymptotic promise across this effectively presented family. A partial search for a certificate on the forgetting branch is therefore consistent with undecidability of the branch.

Theorem 3.2 (One rule, undecidable noise windows; cited)**.** *There is one finite-alphabet finite-range rule on* $\mathbb{Z}$ *and a computable sequence of disjoint nonempty rational rate windows* I_n*. Throughout* I_n*, uniform replacement noise gives uniform exponential ergodicity if the indexed program halts; if it does not halt, the laws from a specified preparation fail to converge weakly. A computable radius-one family has the same dichotomy at one fixed positive rational rate [2, version 2.1.0, Theorems A and C].*

Memory notion: all-state attraction versus failure of instantaneous-law convergence; stationary multiplicity is not substituted for the latter.

General reentrance under positive uniform noise is already established on the line by Marsan, Sablik and Törmä [61, Theorem 3.1], using Gacs's positive-rate memory [60, 20]; their three regimes are not halting-indexed windows. Marsan–Sablik's zero-noise stability test is another undecidable classification problem [62, Theorem 6.1]. The DOI of the designated-window theorem is `10.5281/zenodo.24408026`. The nonhalting witness can be

temporal rather than a pair of stationary phases distinguished by the original readout. This is why the persistence interval of the routed theorem was defined through extrema of $P^t f$, and not through a stationary phase count of the un-routed system alone. It is also why an empirical-average claim needs its own argument: a nonconverging sequence of row laws may still have convergent Cesaro averages.

3.2 Which stationary questions belong to the Rice class

Theorem 3.3 (Stationary-state Rice theorem on the line; cited)**.** *For the classical line category of deterministic finite-range tables followed by independent uniform replacement at the fixed rational rate of equation (3.1), every predicate nonconstant on the realized data (N, E, L) is undecidable. Here N counts extreme translation-invariant stationary laws, E is all-law ergodicity, and L records a translation-invariant stationary law with nondecaying local covariance. The analogous line quantum theorem uses finite repeating circuits with algebraic local maps and independent single-qubit depolarization after every layer. The refinement retains counts of ordered and Bragg-ordered extremes [3, version 2.0.0, Theorems 3.1–3.3].*

Memory notion: the expressly defined phase-count, all-state-attraction and covariance data; no equivalence of these coordinates is assumed.

This is a Rice theorem for specified stationary data under fixed noise. Kari's deterministic limit-set Rice result, as recorded in his nilpotency paper and Guillon–Richard, and the typical-configuration and measure-limit results of Boyer and collaborators and Hellouin de Menibus–Sablik are prior Rice theorems [66, 67, 68, 40]. The DOI of the stationary-data theorem is `10.5281/zenodo.24353745`. The qualifier "nonconstant on realized data" is part of the result. It cannot be replaced by "any nontrivial sentence about stationary states." The source itself gives a decidable entropy-floor property outside that data class [3, version 2.0.0, Theorem 13.1]. Its finite phase-count cuts on the line have stated arithmetic upper bounds and hardness results; the conditional completeness results in dimension seven are a different assertion. We do not transfer them to the line.

3.3 Storage and computation use different output tasks

Theorem 3.4 (Undecidable scalable storage on rings; cited)**.** *One fixed periodic local quantum circuit on rings, with finitely many qubits per site and independent single-qubit depolarization after every elementary layer, has computable disjoint rational noise windows. On a nonhalting window, explicit noiseless encoders and time-dependent decoders, computable from window index, size and decoding date and independent of the rate, preserve entanglement fidelity at least $15/16$ up to $\lfloor c_1 L \rfloor$ elementary-layer dates for all sufficiently large L. On a halting window, for every $\eta > 0$ a finite time bound uniform in size and rate makes optimal fidelity at most $1/2 + \eta$ for every encoding. The scalable-memory labels are Π_1^0-complete [4, version 3.1.0, Theorem D].*

Memory notion: scalable finite-ring quantum storage, with size taken as a resource; not multiple stationary states on a fixed ring.

Dünnweber, Styliaris and Trivedi prove scalable memory and fault-tolerant computation below a positive noise threshold for one dissipative translation-invariant local rule in two dimensions, with circuits encoded in initial states [69, version 3]. Drechsler, Oreg and Aharonov give a CSS/Pauli correspondence with classical stochastic processes and a four-dimensional Toom-based memory [56]. Shiraishi–Matsumoto's fixed-product-start thermalization test concerns a closed Hamiltonian system [41]. The result here is undecidable scalable quantum storage on a fixed ring family with separated noise windows. The DOI is `10.5281/zenodo.24431439`. Positive final depolarization makes a fixed finite ring a finite-dimensional noisy problem; the storage theorem asks about a family of sizes. Its linear horizon in L must not be written as indefinite storage at fixed L. The all-encoding upper branch also includes controller encodings, so a description of the intended encoder alone understates the upper theorem's quantifier.

One fixed finite-range translation-invariant periodic circuit on rings also supports fault-tolerant computation from product preparations. Given a Toffoli/Hadamard circuit, a precision request, and independent single-qubit depolarization below the theorem's fixed threshold after every elementary layer, classical basis preparations encode the task and a prescribed classical decoder recovers its computational-basis output distribution with polynomial resources [5, version 1.0.0, Theorem A], DOI `10.5281/zenodo.24357735`. The memory notion is task-dependent preservation of a computation through its execution. This is one-dimensional fixed-rule fault-tolerant computation. Aharonov–Ben-Or already prove a threshold theorem for controlled circuits, including nearest-neighbour architectures [70]; the autonomous two-dimensional and CSS-memory constructions above are additional capability comparisons [69, 56]. Bravyi–Terhal's low-dimensional energy-barrier obstruction concerns local stabilizer Hamiltonians [71], a different storage model. This theorem establishes an operational capability; it is not another proof of undecidability and does not identify stationary local expectations with quantum circuit outputs.

3.4 A precise forgetting benchmark

For a positive-rate classical PCA on $\mathbb{Z}^d$ admitting a stationary Bernoulli measure, Marcovici and Taati's Theorem 1 gives exponential local convergence from every initial law to that measure [19, Theorem 1, arXiv version 1]. The memory notion is all-state attraction. The same work proves an undecidable stationary-Bernoulli admission test in dimension at least two [19]. This benchmark covers its product-reference hypothesis, not an arbitrary meaning of "unital" for all classical and quantum models.

For deterministic finite-range classical rules followed by uniform replacement, the entropy-price result uses two translation-invariant stationary branches distinguished by a one-site event. It bounds the entropy-density erasure required to sustain their separation [15]. The

result and its nonuniform-bath extensions distinguish rule erasure from the bath's supply of order. In particular, a reversible rule coupled to a strictly positive nonuniform bath can remember in the stated three-dimensional construction while its own rule erasure is zero [15]. We therefore restrict "memory needs nonunital total dynamics" to the relevant reference-preservation setting and do not use it as an unqualified theorem about a local rule in isolation.

3.5 Computable preparations do not supply computable horizons

The selected realizations in Section 2 begin with computable preparations. Their noncomputable limits are not inserted as arbitrary oracle weights. A finite description can generate a rational sequence whose eventual stabilization time is unbounded by a computable modulus. The selection theorem transfers that distinction into its specified experiment. This is the sense in which the organizing claim locates the obstruction in persistent preparation selection, rather than in an assumption of noncomputable input data.

For an actually computable selected value, the obstruction can disappear for that value without disappearing for the universal experiment. For a noncomputable selected value, an effective precision horizon strong enough to approximate its deterministic limit from finite expectations would compute the value and is excluded by the cited upper-bound reasoning [11]. The distinction concerns certified inference about an asymptotic value. It is not a prohibition on collecting finite observations or on estimating ordinary finite-time probabilities.

3.6 Self-similar codes: exact recovery and depth

Bulk recovery in self-similar tensor-network codes is an application of the companion recovery theorems to a distinct operational task, with logical recoverability as its memory notion. The output task is reconstruction from a retained boundary region, after losing any one of five child intervals; it differs from attraction of temporal laws [38]. Steinberg, Feld and Jahn supply a nearby state-dependent holographic recovery setting [72].

Theorem 3.5 (Exact recovery with bounded and growing depth)**.** *For one fixed bounded-depth local isometric circuit, or finite-state address-dependent gate placement, repeated at every fivefold refinement, with fixed local dimension and no seam register, exact all-scale recovery is decidable. Entries lie in a computable field with decidable equality and effective conjugation; algebraic entries suffice, whereas arbitrary computable real entries are not covered.*

Beyond bounded depth, exact all-scale recovery is Π_1^0*-complete* on explicit templates *with fixed gate alphabet and dimension and nearest-neighbour depth* $O_w(\log(k+2))$ *at refinement level* k*. For each computable unbounded nondecreasing nonnegative integer function* g*, an explicit template with depth* $O_w(1+g(k))$ *suffices, where* w *is the finite program specifying the instance [38].*

Memory notion: exact all-scale recovery of bulk subsystems from the specified boundary regions.

The repeated-circuit and explicit-template restrictions matter: arbitrary level-dependent gate choices already permit hardness at depth one. These results do not classify all codes of unbounded depth [38]. The bounded-depth theorem applies a finite endpoint compiler and standard weighted-automaton zeroness. Complementary privacy for subsystem correction is due to Kretschmann–Kribs–Spekkens [73, Theorem 5]; bounded causal-cone cancellation is standard in Vidal and Evenbly–Vidal [74, 75], and finite linear representations and zeroness have the Schützenberger, Tzeng and Kiefer–Murawski–Ouaknine–Wachter–Worrell precedents [76, 77, 78].

Chuklanov compiles parameterized ladder generators into width-independent weighted automata and tests circuit equivalence by difference-series zeroness; Abdulla and collaborators verify represented all-size circuit families with transducers and synchronized weighted tree automata [79, 80]. Ross–Wesley instead obtain parameter-sampling cutoffs at fixed circuit arity [81]. Miller-Bakewell gives finite verification of diagram families under a connectivity condition, and Huang and collaborators verify quantum correction programs for specified errors [82, 83]. Ferris–Poulin and Farrelly and collaborators give tensor-contraction decoding algorithms on finite codes [84, 85]. The companion theorem's extra scope is the joint all-scale erasure query with arbitrary spectators; all-word positivity has a different undecidability precedent in Kliesch–Gross–Eisert [86].

The growing-depth templates use standard reversible history storage [87]. Shiraishi–Matsumoto also use reversible computation to select asymptotic readouts in isolated line dynamics, while Cubitt and collaborators give arbitrarily delayed positive-coherent-information witnesses [41, 88]. The recovery contribution is the explicit scheduled template, with fixed gates and dimension and program-dependent depth constants [38].

3.7 Approximate recovery: a robust recorded gap

Let f_* be the infimum over finite cutoffs, bulk targets and specified child-interval losses of the optimal worst-case subsystem recovery fidelity, using squared Uhlmann fidelity and allowing a reference system. For the recorded construction the same formula holds for standard entanglement fidelity at a maximally mixed target with product spectator inputs [38]. Bény–Oreshkov give the complementary-channel characterization of optimized worst-case recovery, including subsystem correction [89].

Theorem 3.6 (Recorded approximate-recovery gap)**.** *On the explicit homogeneous depth-one rational isometric fivefold tree-code template with bulk dimension* 7*, radial dimension* $681{,}408 = 448 \cdot 39^2$*, and retained transition records, deciding* $f_* \geq 1 - \varepsilon$ *is* Π_1^0*-complete for each fixed rational* $0 < \varepsilon < 48/49$*. The explicit template has a robust undecidable promise gap*

$$f_* \geq 25/49 \qquad \textit{versus} \qquad f_* = 1/49.$$

A second promise on the same fixed-dimensional template separates $f_* \geq 33/49$ *from* $f_* = 1/49$*, and excludes a uniform algorithm with absolute error at most* $12/49$*. Exact recovery in the same bounded-depth class is decidable [38].*

Memory notion: approximate worst-case all-scale subsystem recovery, with transition records retained in the encoding.

This is uniform inapproximability across finite instances; it does not assert noncomputability of every individual fidelity value. This is a fixed-dimensional recovery-fidelity realization of a known probabilistic-automaton gap. Rote's Theorem 1(c) supplies binary 18-state strict emptiness at 1/18; his Theorem 7 and Appendix A apply Gimbert–Oualhadj gap amplification to produce the 39-state four-letter machine, with one letter duplicated at the fifth port [90, 91]. Condon–Lipton and Nasu–Honda are earlier probabilistic-automaton hardness sources; Blondel–Tsitsiklis transfer emptiness to matrix-product boundedness [92, 93, 94].

Elkouss–Pérez-García already encode a probabilistic-automaton value in finite-state channel capacity, obtaining an undecidable separated comparison and a uniform approximation obstruction [95]. Here the optimized recovery fidelity satisfies $f_* = 1 - (48/49)\sup_w p(w)$ on the recorded template, and exact recovery remains decidable [38]. Eisert–Müller–Gogolin give measurement-word occurrence undecidability, and Wolf–Cubitt–Pérez-García discuss an asymptotic singlet-fidelity decision problem [96, 97]. Orthogonal transition histories belong to this particular encoding; Theorems 3.7 and 3.8 rule out treating their presence as a general derivability criterion.

3.8 Single-excitation propagation and exact equality

Theorem 3.7 (Computable infimum and undecidable root threshold)**.** *On an explicit rational single-excitation depth-one tree-code template with one amplitude matrix per port and radial dimension* 575,960*,* f_* *is uniformly computable from the finite instance. Deciding whether* f_* *equals its root value* $4/5$ *is nevertheless* Π_1^0*-complete. Every other rational threshold is decidable, uniformly when the threshold is an input promised different from* $4/5$ *[38].*

Memory notion: the same all-scale recovery fidelity, compared at an exact root value.

This template establishes no positive-gap hardness. Its two-dimensional no-arrival state is rotated at every cell, forwarding a fixed nonorthogonal quantum record of the logical sector. It is therefore not described as record-free [38]. This is a single-excitation recovery realization of a known cutpoint obstruction. Turakainen relates generalized linear automata to stochastic languages, and Yakaryilmaz–Say show that measured one-way Kondacs–Watrous automata recognize the stochastic languages with unbounded error [98, 99, 100]. The lower-bound chain uses Neary's five-pair Post correspondence construction and Rote's strict-emptiness theorem [101, 90]. Blondel–Jeandel–Koiran–Portier separate decidable strict from

undecidable non-strict cutpoint tests for algebraic unitary measure-once automata; Derksen–Jeandel–Koiran provide effective algebraic-group methods for the strict case [102, 103]. The port matrices here are contractions although the full scatterer is isometric.

3.9 Input steering and dissipative memory

Theorem 3.8 (Computable unsteered value and an input-steered gap)**.** *For fivefold self-similar single-excitation tree codes with an inert no-arrival state in a fixed rational presentation, where letters come from addresses alone and no input can steer the excitation, the all-scale optimal recovery fidelity is an exactly computable algebraic number and every rational threshold is decidable. With state-independent unitary continuation letters, the result extends to the stated two-sector polarized readout [38].*

There is also a homogeneous rational depth-one fivefold tree-code template with radial dimension $7 \cdot 158 \cdot 7^{80}$*, about* $5 \cdot 10^{70}$*, for which worst-case fresh bulk inputs steer a dissipative memory along one ray. On this template it is undecidable to distinguish*

$$f_* \geq 1/6 \qquad \textit{from} \qquad f_* = 1/49, \qquad f_* \leq 9/49 \ \textit{in every instance.}$$

Thus the high side lies in $[1/6, 9/49]$*. The fresh-input quantifier is essential: this reduction does not give hardness when those inputs are frozen. No fixed source channel reaches more than* $2/\Delta + O(1)$ *disjoint recovery footprints at distinguishing gap* $\Delta > 0$ *[38].*

Memory notion: all-scale subsystem recoverability; in the hard class, a dissipative memory steered by worst-case fresh inputs.

In the unsteered class, the summable arrival tail follows from the Gram probability budget, with an inert vacuum; absence of records is not the operative hypothesis. The algebraic optimum and threshold procedures use standard finite-dimensional methods, with quantum reachability and rational-semigroup closure precedents in Ying–Feng–Yu–Ying and Hrushovski–Ouaknine–Pouly–Worrell [104, 105]; Derksen–Jeandel–Koiran, Blondel and collaborators and Nosan and collaborators supply algebraic-group and unitary-cutpoint comparisons [103, 102, 106].

For the input-steered code, the per-pair probability budget permits at most $2/\Delta$ nonexceptional receivers. The exceptions are at most six early slices, one footprint containing the spine leaf, and one containing all children of an off-spine source site. The bound concerns disjoint recovery footprints and each pair of source inputs; these codes also serve pooled disconnected receivers, and the corresponding general receiver bound remains open [38]. Classical-state quantum automata and quantum secret sharing are relevant memory and receiver comparisons [125, 126]. The gap mechanism is the known probabilistic-automaton memory mechanism of Elkouss–Pérez-García, realized here in subsystem recovery [95, 38].

Theorem 3.8 is an application of the companion recovery theorems to a distinct operational task: fresh inputs steer a dissipative memory that retains full weight on one ray. The

analogy with the memory/forgetting criterion is that a retained distinction matters to the specified readout; this is an analogy, not a further instance of the common-readout theorem. Broadcast is not needed. The 25/49 high value belongs to the earlier recorded construction, not this one [38].

Open question. Is dissipation necessary for a robust undecidable gap under this fresh-input recovery task? State-independent unitary continuation in the stated class has a computable measure-once quantum-automaton value by the algebraic methods above; no general necessity theorem for dissipation is established [38, 103, 102].

4 Where undecidability appears: a threshold conjecture

4.1 Instances with explicit dimension and category

The line theorem of Section 3 gives undecidable all-state attraction under positive replacement noise in one spatial dimension [2, version 2.1.0, Theorems A and C]. Toom already establishes line ergodicity undecidability in a broader stochastic class [63]. Gacs's reliable-medium constructions establish that persistent information under positive local errors is possible on the line [60, 20]. These are compatible facts in the same broad dimension, not a theorem saying that every rule capable of memory has an undecidable parameter problem. The categories and presentation resources of the two constructions remain part of their respective results.

For finite-range equilibrium specifications with strictly positive finite weights, the square-lattice example in [2, version 2.1.0, Theorem B] has undecidable designated windows of all-DLR uniqueness and exponential weak mixing versus two symmetry-related translation-invariant DLR states. In a nonhalting window an odd one-site observable has means at least $1/4$ and at most $-1/4$. The memory notion is equilibrium readout multiplicity. No behavior is asserted outside the designated windows. The result is at finite temperature and finite energy; the additional spatial coordinate is the equilibrium space–time coordinate, not an extra independent time variable in the Gibbs state.

The familiar one-dimensional finite-state, finite-range, strictly positive equilibrium setting has a unique Gibbs law at finite temperature [26, Corollary 6.41]; it is the comparison class in this dimensional discussion. This observation refers to equilibrium Gibbs states, not to positive-rate one-dimensional PCA, whose temporal evolution is a different problem. We do not use an equilibrium no-phase statement to deny dynamical memory on the line.

For quantum storage, the ring theorem already gives undecidable scalable storage windows with independent single-qubit depolarization after every layer [4, version 3.1.0, Theorem D]. For computation, one fixed local ring circuit has the product-preparation capability stated

in [5, version 1.0.0, Theorem A]. Dünnweber–Styliaris–Trivedi and Drechsler–Oreg–Aharonov supply nearby two- and four-dimensional autonomous memory comparisons [69, 56]. The memory notions here are scalable storage and task-dependent fault tolerance, respectively. A finite noisy ring with a unique stationary state is compatible with both results because the operational resources include growing size and a prescribed execution horizon.

A further low-dimensional incumbent is the zero-temperature result of Bausch, Cubitt, Lucia and Perez-Garcia: spectral-gap undecidability already holds for their one-dimensional translation-invariant nearest-neighbour quantum Hamiltonians [36, Theorem 1]. This supports comparison across named low-dimensional categories; it does not prove the positive-noise threshold conjecture or identify a spectral gap with storage fidelity.

4.2 What the general statement would have to say

Conjecture 4.1 (Threshold principle)**.** *Fix a precisely specified effectively presented category of local systems, its noise or positive-temperature convention, its admissible readout tasks, and its resource quantifiers. At the least spatial dimension in which that category admits the stated persistent long-run distinction, the corresponding unrestricted family of long-run classification problems includes an undecidable problem.*

This formulation makes a family-level prediction. It does not make the contradiction "a simple decidable rule refutes the conjecture." A family containing a constant rule can still contain an undecidable classification problem. Nor does the conjecture assert that every scalar selected by every memory is uncomputable. The exact selected spectrum is a separate issue from the existence of an undecidable classification task.

A useful test of the conjecture needs an independently defined category. Defining a category by the same simulation construction that supplies the undecidability reduction would merely restate that construction. In particular, choosing the readout task after inspecting the reduction can conceal the substantive question. A genuine extension should state the allowed alphabets, local dimensions, range bounds, noise kernels and preparations before giving the reduction or an obstruction.

4.3 A conditional reading of "lowest dimension"

The evidence supports lowest-dimension instances in the stated categories: classical dynamics, equilibrium and quantum operational tasks. It does not supply a universal dimension theorem. Zero-dimensional finite-state positive Markov chains and fixed finite-dimensional depolarized quantum systems have finite linear-algebra descriptions, while infinite spatial media permit a different asymptotic problem. That elementary distinction motivates the comparison; it does not prove that every infinite category inherits a halting simulation.

The threshold conjecture is intentionally separate from the effective enclosure theorem.

Effective enclosure concerns the complexity of a common compatible-state readout. A dimension claim concerns which families can sustain distinctions and encode computations. One can establish a certificate theorem in a category without establishing its sharp dimension for memory, and one can exhibit memory without proving the full selected-limit spectrum. Keeping these obligations separate makes the conjecture assessable rather than a generalization by terminology.

5 What memory costs and what an observation reveals

5.1 Stationary distinction and entropy density

Fix a deterministic translation-equivariant finite-range classical rule Φ on $\mathbb{Z}^d$, followed by independent uniform whole-symbol replacement at rate $0 < \varepsilon < 1$. Write $a = 1 - \varepsilon$. For a translation-invariant stationary law ν, the rule's logical erasure density is

$$e(\nu) = h(\nu) - h(\nu\Phi),$$

where h is spatial entropy per site in nats. Consider two stationary branches whose post-bath probabilities of one fixed one-site event differ by at least δ. The optimized cost $M(\varepsilon, \delta)$ is the infimum of the larger branch erasure density over the allowed rules and branches. Range may grow along the approximating sequence. This is a stationary density currency; it is not the mutual information about a message in a finite observation record [15].

Theorem 5.1 (Sharp stationary entropy-density price under uniform reset; in preparation)**.** *In the preceding setting, put $H_2(x) = -x \log x - (1 - x) \log(1 - x)$ with natural logarithms. For each fixed $d \geq 2$ and even alphabet size $q \geq 2$, the exact infimum is*

$$M(\varepsilon, \delta) = H_2\left(\frac{1 - \delta}{2}\right) - H_2\left(\frac{1 - \delta/a}{2}\right), \qquad 0 < \delta < a. \tag{2}$$

For $a \leq \delta \leq 1$, no such finite-range memory is admissible and the cost is $+\infty$. The finite expression is a lower bound in all dimensions and alphabets. Fixed odd alphabets require the stated event-size optimization, while a matching general one-dimensional upper construction is not asserted [15].

Memory notion: two translation-invariant stationary branches with a specified one-site probability gap after the bath.

Taati and Marcovici–Taati use entropy injection, boundary leakage and entropy balance to quantify forgetting [107, 19]. Aharonov–Ben-Or–Impagliazzo–Nisan give the noisy reversible-circuit antecedent with inputs supplied initially, while Nelson–Rajakumar–Gullans bound local relative entropy and correlations for geometrically local depolarizing unitary circuits [108, 109]. The optimization here is the sharp stationary entropy-density price for a

one-site distinction under uniform reset, in the stated dimensions and alphabets. It is an unpublished checked result, cited as in preparation. The equality is an infimum over implementations, not a claim that a particular finite rule attains the formula exactly. A prescribed margin strictly inside the feasible interval can be approached by increasing the interaction range. The theorem does not fix that range while varying all tolerances, and it does not give a compact laboratory device attaining the infimum.

At fixed $0 < \delta < 1$, the formula gives

$$M(\varepsilon, \delta) \sim \varepsilon\delta \operatorname{atanh}(\delta) \quad (\varepsilon \downarrow 0).$$

Thus a fixed nonzero margin costs order ε in this optimization. A high-fidelity request that approaches saturation can instead cost order $\varepsilon \log(1/\varepsilon)$, for example $\delta = 1 - k\varepsilon$ with fixed $k > 1$ [15]. These different regimes should not be combined into a universal logarithmic price. The endpoint $\delta = a$ is excluded by strict feasibility, even though the finite-expression limit from below exists.

5.2 Where the environment enters the account

For a full-support reference distribution π preserved by a positive product bath θ, the entropy-price analysis instead uses relative entropy density and the bath's energy term. Its lower bound includes

$$D(p\|\pi) - D(p\theta\|\pi)$$

for a pre-bath one-site marginal p, together with the separate nonnegative rule-erasure constraint. A nonuniform reference changes the currency from Shannon entropy alone to a free-energy-type deficit [15]. The reversible three-dimensional construction with a nonuniform bath therefore does not contradict the uniform-reset price: it obtains order from a different resource.

The reference is not a cosmetic normalization. If a bath has a preferred distribution, comparing input and output against the uniform law can assign the bath's ordering action to the rule. The cited accounts explicitly retain that action. The same caution applies to prepared auxiliary inputs: a stream of pure or otherwise ordered ancillas cannot be treated as a free reference-preserving wire in an erasure claim.

The logical units also matter. Formula (2) is in nats per site per round. The capacity below is in bits per detector observation. Multiplying one by an observation time does not yield the other unless an additional theorem identifies the charged region, branch and encoding. A local stationary marginal gap is not itself the information content of a message.

5.3 Persistent readout as a routed binary channel

The routed theorem identifies the communication capacity of its routed detector with the capacity of a binary-input channel whose two click probabilities are the persistence endpoints

a, b. With h_2 denoting binary entropy in bits, this is

$$C(a,b) = \max_{0\le\lambda\le 1} \left[h_2((1-\lambda)a + \lambda b) - (1-\lambda)h_2(a) - \lambda h_2(b)\right]. \tag{3}$$

This is Shannon's binary-input discrete memoryless channel capacity formula [110]. It is positive exactly when $a < b$. Arbitrarily correlated classical inputs or entangled quantum input layers obey the upper bound; product-layer constructions attain the operational capacity [14]. The memory notion is finite-time detector persistence made available through the routing coordinate. The receiver obtains one binary observation per date and has no access to an entire row.

Elkouss–Pérez-García supply the prior use of operational channel capacity as a computability witness in a finite-state memory channel [95]. The application here is the routed detector's reduction to its endpoint channel, including correlated or entangled preparations in the converse [14]. The coordinate used for routing is charged as part of the model. The source's quantum operation is a causal translation of whole layers; a noisy nearest-neighbour gate implementation of that translation is an additional engineering assumption. The mathematical capacity result does not silently grant that implementation, just as it does not grant free copies of a quantum input.

5.4 The unrestricted in-place information rate

Theorem 5.2 (Stationary variational rate and coding; in preparation)**.** *For a fixed finite-alphabet finite-range rational positive-rate classical PCA on $\mathbb{Z}^d$, a fixed finite passive detector, and a receiver restricted to its finite record, let I_T be the maximum information about a message encoded in the initial microscopic configuration obtainable from T observations. If $Q_k(x)$ is the next-word output law from row x, put*

$$J_k(\mu) = H(\mu Q_k) - \int H(Q_k(x))\,\mu(dx)$$

in bits. Then

$$\rho = \lim_{T\to\infty} \frac{I_T}{T} = \max_{\mu P=\mu} \inf_{k\geq 1} \frac{J_k(\mu)}{k}.$$

A maximizing stationary law can be chosen time-ergodic. Rates strictly below a positive ρ admit all-large-horizon codes with vanishing maximum error; above ρ the average success decays exponentially [16].

Memory notion: retained information about a microscopic initial message, optimized over preparations; not merely selection of a stationary phase label.

Verdú–Han give the general operational information-spectrum capacity formula; Nguyen–Armand study stationary and ergodic capacity-achieving inputs for consistent stationary ergodic channels [111, 112]. The result here is an in-place passive-detector coding theorem with the entire microscopic message supplied in the initial row. The distinction between a stationary optimizing law and a restricted entrance-law input is binding. The source proves the operational rate in the displayed experiment, while equality with its proposed unrestricted entrance-law capacity remains open. A stationary law can randomize microscopic configurations that are not themselves entrance laws. Replacing that microscopic-message resource by a message selecting only an entrance law changes the channel input constraint.

An ordinary north-east-center (NEC) majority-rule experiment, with uniform replacement rate $e = 2 \cdot 3^{-24}$, realizes all limit-computable frequencies in $[1/3, 2/3]$ for logarithmically weighted averages and separately for geometrically sampled averages. Its full ordinary record has $I_T = \Omega(\log T)$. It does *not* establish ordinary all-date convergence to all these targets in that native medium [16]. We retain the logarithmic lower bound and the explicitly changed averaging conventions. No fitted numerical power is asserted as a theorem. Toom's stability theorem and the quantitative contour treatment by Swart–Szabó–Toninelli are the relevant native-NEC stability precedents [124, 113].

5.5 Revelation rent and the order of quantifiers

For the uniform-reset classical model with passive observations and the full physical-site cone costs defined in the revelation-price result, a positive universal finite-horizon dynamical price is false. When the medium and preparation resources may vary with the horizon, initial

spatial redundancy makes the infimum dynamical cost zero. For one medium fixed before the delay tends to infinity, a reliable delayed bit instead has a sharp quadratic full-cone cost in time. At positive fixed reliable rate the sharp growth orders are quadratic in place and cubic in the routed geometry [17].

These are full-cone resource bounds in the specified experiments. The charged cone includes background computation and full site states. These powers are not an intrinsic heat attributable solely to the message. The source distinguishes a record-adapted union deficit, separately routed budgets, and gate erasure under an explicit complete port/resource account. Its gate bound requires that account. A smaller circuit producing only the detector output is not automatically charged the cost of computing the complete medium.

The finite inventory keeps the initial deficit bank and observation credit. For gate implementations satisfying its assumptions, gate erasure dominates the relevant created-deficit budget. Landauer supplies the physical logical-reset precedent [114]. A Landauer conversion then requires a bath temperature, accessible-side-information convention and physically local gate operation [17]. This is a conditional heat implication, not an attained heat measurement. Section 9 reserves the physical-prediction open problem until an apparatus and resource account are specified.

These accounts explain why no single "rent per fact" follows from the selected-limit theorem. A spectrum classifies possible asymptotic values. A coding theorem classifies reliable bits per observation. A cost theorem charges a particular dynamics, bath and geometry. The cited results connect particular pairs of these quantities under their hypotheses; the synthesis keeps those hypotheses attached to the connection.

6 Exact phase structure: proved cases and open characterization

Open problem. Problem P remains open as a two-sided characterization. Specific exact interval magnets are proved below; the general interval case and unrestricted realization remain open. The exact dynamics counterpart also remains open.

6.1 The statement under investigation

Fix rational $0 < a < b$. A magnet has a finite alphabet and a translation-invariant finite-range interaction Φ on $\mathbb{Z}^2$ with finite rational energies. Write exponential weak mixing as EWM and define

$$\mathcal{E}_\Phi = \{\beta \in [a, b] : \Phi \text{ has EWM at } \beta\}.$$

The convention is an all-volume boundary bound: there are $C, m > 0$ such that for every finite Λ, every pair of exterior conditions, and each $|f| \leq 1$ supported in $V \subset \Lambda$, the expectation difference is at most

$$C|V| \exp[-m \operatorname{dist}(V, \Lambda^c)].$$

The proposed Problem P asks whether the sets $\mathcal{E}_\Phi$ are exactly the relatively effectively open subsets of $[a, b]$, with uniform compilation from an enumerating program and magnetic order throughout the complement [18]. The memory notion on the complement would be two translation-invariant DLR states distinguished by a fixed odd local observable, with a uniform rational gap.

Theorem 6.1 (The upper direction of Problem P; in preparation)**.** *For the preceding two-dimensional finite-rational-energy magnet class, $\mathcal{E}_\Phi$ is effectively open relative to $[a, b]$, uniformly from the interaction and rational endpoints. This uses the stated all-volume exponential weak-mixing convention and finite block certificates [18].*

Memory notion: the certified absence of equilibrium boundary dependence with an exponential bound; not uniqueness without a rate.

The proof uses the two-dimensional weak-to-strong mixing theorem of Martinelli, Olivieri and Schonmann [34, Theorem 1.1]. The finite-range, factor-$|V|$ formulation and the restriction of the resulting strong-mixing estimate to squares are as explained by Spinka [35, Section 1.2 and Remark 2]. No arbitrary-volume strong-mixing implication is substituted for that import. Ott gives a further weak-to-strong spatial-mixing treatment with the geometric and finite-size context explicit [115]. The contribution here is effective finite-size certification of the exponential weak-mixing region, due to Douglas and Mghirbi (in preparation) [18].

Effective openness is not a decision algorithm for membership at every inverse temperature. A finite certificate recognizes a positive instance and covers it by a rational interval; a

boundary point need not be decided. Uniqueness without an exponential rate is a different predicate.

6.2 Specific exact positive-temperature intervals with finite rational interactions

Theorem 6.2 (Finite rational interval magnets; in preparation)**.** *Put* $t(\beta) = 1/(1+\exp\beta)$*. For rational* $0 < a < b$ *and rational* t_c *with* $t(b) < t_c < t(a)$*, a finite-alphabet homogeneous finite-range magnet with rational energies on* $\mathbb{Z}^2$ *can be constructed so that*

$$\mathcal{E}_\Phi = [a, \beta_c), \qquad \beta_c = \log\frac{1-t_c}{t_c}.$$

At every $\beta \in [\beta_c, b]$*, including the endpoint, it has two translation-invariant DLR states with opposite expectations at least* $1/4$ *in magnitude for a fixed rational odd local observable and a fixed-point-free interaction symmetry. The EWM assertion concerns the full spin system and arbitrary exterior conditions [28].*

Memory notion: equilibrium readout multiplicity on the ordered interval, including its endpoint.

Finite products give further exact intervals in the same finite rational magnet class. One example has

$$[a, b] = [1/2, 2], \qquad \beta_c = -\log\frac{3+\sqrt{2}}{10}.$$

Here it is the exponential coordinate $\exp(-\beta_c)$ that is algebraic. More generally, the stated finite-product criterion uses an integer scale $d \geq 1$, an algebraic $z = \exp(-\beta_c/d)$, and the exclusion of other real conjugates in the warmer segment $(z, \exp(-a/d)]$ [28]. This is a criterion for that product construction, not a classification of the transition points of all finite rational magnets.

6.3 The remaining realization problem

The exact interval constructions do not establish realization of an arbitrary relatively effectively open set. A general construction must control the full ordinary-noise system at unbounded ranks, provide sound and sufficiently frequent sampling, recover from erroneous local alarms, and establish warm-side mixing from every initial state. Scheduled all-rank sampling results and finite-rank homogeneous implementations do not by themselves supply such an autonomous infinite-rank Gibbs construction [18, 28].

Open problem. General realization. Construct a finite-rational-energy magnet for every relatively effectively open set, with EWM on that set and uniform-gap magnetic states

throughout its complement. The general interval case remains open beyond the exact classes stated above.

A transition at Chaitin's Ω, the halting probability of a fixed universal prefix-free Turing machine, and order on a large-measure set containing no computable temperature remain conditional consequences of a general realization theorem. Effective-open-cover and computability reductions do not themselves construct the required finite interaction [18, 28]. The exact interval examples above have computable endpoints and do not settle those existence questions. Purcell, Li and Cubitt already realize an Omega transition between gapped and gapless zero-temperature phases of a two-dimensional quantum Hamiltonian, with an interaction parameter as control [116]. The open target here concerns positive-temperature classical finite rational interactions and a uniform odd-readout gap.

6.4 The dynamics counterpart

Open problem. Dynamics counterpart. An exact characterization of the replacement-noise sets on which one fixed local rule forgets, with the attraction and certificate convention stated, remains open. The published line noise diagram gives undecidable behavior on designated disjoint windows [2, version 2.1.0, Theorems A and C]. The infinite-alternation theorem gives interleaving memory and attraction windows and hence infinitely many boundary points of the ergodicity set, accumulating at zero; it does not prescribe the transition rates between those windows [7, version 1.0.0, Theorem A]. Neither theorem supplies the missing exact-set characterization. In a broader category, Marsan–Sablik realize every G_δ containing 1 as the common ergodicity and uniqueness set for uniform-replacement perturbations of continuous Cantor maps; their effective version realizes computable Π^0_2 G_δ sets [117, Theorem 3.19 and computability remarks]. Those maps may be nonlocal and need not commute with shifts, leaving the local cellular-automaton question open.

7 Checkable comparisons with existing frameworks

7.1 Effective compactness and prior certified readouts

Marcovici, Sablik and Taati are a direct incumbent for the forgetting half: their Proposition 2.2 states that a computable PCA with a unique invariant measure has a computable invariant measure [29]. Theorem 2.1 is a recovery in an observable-level interval framework, and a common scalar readout need not require uniqueness of the entire state. Galatolo, Hoyrup and Rojas provide related effective-compactness results in their deterministic setting [30, Corollary 4.2.1]; Breiman and Jenkinson supply the strong-law and endpoint precedents credited above [31, 32]. Fawzi, Fawzi and Scalet already obtain certified local equilibrium and ground-state expectation intervals [33, arXiv version 3, Theorems 1.1, 1.2, 3.6 and 3.7]. Canel's effective closedness of the positive-temperature Gibbs class and the equilibrium-

computability and optimization work of Binder and collaborators and Gayral–Hoyrup are additional nearby precedents [43, Example 5.10 and Theorem 5.11][44, 45]. Xu and collaborators' Supplementary Corollary 3.6 gives a particularly close certificate-versus-diagnosis contrast: a halting-indexed one-dimensional KMS ground-state family has one-site expectation set either $\{0\}$ or $[0, 1]$, and no uniform computable convergence rate for certified upper bounds on its maximum [46]. This is a zero-temperature extremum over all ground states. None of these precedents is claimed as a new uniqueness-implies-computability theorem here.

The additional contribution is exact preparation-selected spectra in specified noisy and equilibrium experiments, paired with recovered common-readout certification. The realizations keep their stated fixed and varying inputs: ordinary line PCA trajectories, fixed square-lattice Gibbs boundary selection, and a fixed noisy three-dimensional channel [27, 12, 13]. The joint statement puts these lower realizations beside the recovered common-readout certification theorem. It does not claim that scalar computability characterizes preparation independence.

7.2 Landau and Wilson: what question is being answered?

Landau's phase-transition work organizes transitions through thermodynamic and symmetry considerations [21]. Wilson and Kogut develop renormalization-group descriptions of critical phenomena and the epsilon expansion [22]. The checkable contrast here is a question of scope: neither reference is used as a theorem giving a uniform algorithm that classifies every finite local rule. The positive-temperature window construction of [2, version 2.1.0, Theorem B] supplies a family of inverse-temperature queries for one finite-energy square-lattice interaction whose outcomes encode halting. Its conclusion concerns algorithmic classification, rather than a failure of symmetry or scaling methods in models where those methods apply.

The common-readout theorem supplies an approximation method when all compatible states agree on a specified affine observable. It does not prove that a renormalization flow has a computable fixed-point representation, nor that computable fixed points occur exactly in systems with all-state attraction. Those would require a chosen space of renormalization maps and an effective representation of its fixed points. They are not consequences of compactness of stationary-state marginals. Watson, Onorati and Cubitt already construct a renormalization-group map with uncomputably complex flows and undecidable fixed-point behavior for their quantum Hamiltonian family [118]. That result supplies a concrete RG computability comparison with its own map and predicates. The present comparison therefore avoids equating "derivable" with a fixed-point ansatz.

The operational distinction is concrete. Given a local indicator and a singleton stationary interval in the classical setting, the certificate theorem supplies finite sound numerical bounds. Given the rule alone with no agreement promise, the cited undecidability result prevents a total universal classifier of forgetting. Symmetry data may be useful inputs to a model-specific argument, but the synthesis does not claim that they decide that promise or

that the theorem removes the need for them.

7.3 Pirogov–Sinai: hypotheses and effective use

In the finite-state finite-range low-temperature version of Pirogov–Sinai theory, one assumes finitely many periodic ground states and a Peierls condition controlling the energy cost of defects; the phase analysis applies at sufficiently low temperature [23][37, Chapter 7, Sections 7.2 and 7.4]. These hypotheses supply a controlled contour regime, not a total algorithm for every positive-temperature interaction. The finite-energy window theorem supplies instances outside any claim of a total diagram-classification algorithm [2, version 2.1.0, Theorem B]. These two statements can both hold. This paper proves no sharp characterization of the boundary of Pirogov–Sinai applicability, and no equivalence between its hypotheses and the enclosure or selection premises.

For a particular model, an effective contour bound may itself provide a usable finite certificate. A certificate can then be incorporated in a derivation with its constants and domain stated. That is a way to compare methods on the same input, rather than interpreting the existence of an undecidable family as an obstruction to all contour calculations. The unresolved exact-set target in Section 6 is substantially stronger than the existence of disjoint undecidable windows.

7.4 Quantum ground-state undecidability

For two-dimensional translation-invariant nearest-neighbour quantum Hamiltonians, Cubitt, Perez-Garcia and Wolf prove that the spectral-gap decision problem is undecidable [24]. Bausch, Cubitt, Lucia and Perez-Garcia establish spectral-gap undecidability already for one-dimensional translation-invariant nearest-neighbour quantum spin chains [36, Theorem 1]. Bausch, Cubitt and Watson construct a continuous one-parameter family with an uncomputable phase diagram [25]. The comparison concerns different predicates: their cited ground-state and spectral questions concern the thermodynamic low-energy problem; [4, version 3.1.0, Theorem D] concerns finite noisy circuit storage with growing ring size, and [2, version 2.1.0, Theorem B] concerns all-DLR equilibrium uniqueness versus magnetic states at positive temperature. None of these observables is identified with the others by this synthesis.

The distinction between constrained and finite-energy equilibrium results is visible even within the cited work. The earlier positive-temperature phase-diagram theorem has nearest-neighbour energies in $\{0, 1, +\infty\}$ and an undecidable *translation-invariant* Gibbs uniqueness diagram. At every positive temperature it also has non-translation-invariant Gibbs states [8, version 1.2.0, Theorem A / Theorem 7.3], DOI `10.5281/zenodo.24144192`. It is therefore not an all-DLR uniqueness example for positive finite weights. The later square-lattice theorem with $\{0, 1\}$ energies supplies the stated all-DLR finite-energy windows [2, version

2.1.0, Theorem B]. Its behavior outside those windows is left uncontrolled. Gayral–Sablik–Taati realize computability classes in zero-temperature limits of two-dimensional finite-range rational interactions; Barbieri–Bissacot–Dalle Vedove–Thieullen give two-dimensional finite-range zero-temperature chaos, extending the higher-dimensional finite-range construction of Chazottes–Hochman [119, 120, 121]. Gayral–Sablik's noisy-tiling stability and undecidability are related finite-defect precedents [122, 123]. These temperature-limit and static-defect questions differ from an all-DLR envelope at each designated positive temperature.

The comparison is adjudicable by inspecting the input representation, dimension, temperature convention and quantified state set in each theorem. Merely replacing the term "ground state" by "stationary state" in a summary would lose the predicate being decided. A quantitative device-level comparison would need additional resource estimates; no such performance ranking is claimed here.

7.5 Gacs and the distinction between existence and diagnosis

Gacs constructs reliable one-dimensional media under positive local errors [60, 20]. Toom already proves general stochastic ergodicity undecidability and credits Kurdyumov; Bušic–Mairesse–Marcovici give the deterministic comparison [63, 64, 65]. Marsan–Sablik–Törmä give reentrance under positive uniform noise, while Marsan–Sablik's stability diagnosis concerns noise tending to zero [61, 62]. The line noise-diagram theorems show that diagnosing all-state attraction is undecidable on their fixed windows or fixed-rate family [2, version 2.1.0, Theorems A and C]. Ordinary selected frequencies also occur in one dimension: Theorem 2.3 gives one fixed line rule and one fixed one-site event with exact limit-computable spectrum in $[1/3, 2/3]$ from computable deterministic starts [27]. This construction needs no added routing coordinate and makes no ergodicity or instantaneous-law-convergence claim. The routed theorem and the line weak-limit theorem remain distinct additional results [14, 11]. The noiseless density classification of Downey–Jockusch–Schupp, recorded by Downey–Jockusch–McNicholl–Schupp, and the deterministic CA limit-measure theorem of Hellouin de Menibus–Sablik predate these positive-noise realizations [50, Theorem 2.1][40, version 3, Corollary 1]. Their restricted-class agreement and target-dependent rule should not be identified with agreement over all compatible states. Marsan–Sablik's noisy realization selects stationary accumulation sets as the rate vanishes [57].

7.6 Values, selected limits and classification

The cited theorems answer three computational questions. A common-readout certificate computes a value uniformly on its agreement promise. An exact selected-spectrum theorem classifies possible definite limits of computable experiments. A reduction proving undecidability concerns classification across an effectively presented family [10, 27, 14][2, version 2.1.0, Theorems A–C]. A theorem answering one question does not answer the others. Sec-

tion 9 preserves these distinctions in its refutation conditions.

7.7 Thermalization and frustration-freeness

Shiraishi and Matsumoto's undecidable thermalization is a close closed-system incumbent for the forgetting question: it concerns isolated one-dimensional shift-invariant nearest-neighbour quantum systems, a fixed product preparation, and the long-time average of a spatially averaged single-site observable, under their stated observable and promise hypotheses [41, Theorems 1 and 2]. Our all-state-attraction questions quantify over every initial law under continuing positive noise. These are distinct relaxation predicates.

Shiraishi proves frustration-freeness decidable at each fixed local dimension for the stated translation-invariant nearest-neighbour square-lattice Hamiltonians with algebraic entries [42, version 1, Theorem 1 and Corollary 2]. This is nonuniform in dimension: the finite cutoff is not computable as a function of dimension. Assuming consistency of ZFC, a frustration-free Hamiltonian exists whose frustration-freeness is unprovable in ZFC [42, version 1, Theorems 3 and 4]. Thus computability of a value, decidability of a predicate and provability in a chosen formal theory require separate statements. No identification of these notions is made by the recovery theorems.

8 Scope limits

8.1 Existence constructions and physical rates

The fixed-noise and ring-storage theorems are existence constructions with large finite alphabets, local registers, ranges or periods, and very small sufficient noise scales [1, version 3.0.0, Theorems 2.1, 12.5, 12.11][4, version 3.1.0, Theorem D and Section 1.5]. Their formal finiteness is not a practical resource estimate. In particular, the line theorem's rational noise is specified from fixed construction constants rather than advertised as a useful decimal threshold [3, version 2.0.0, equation (3.1)]. A local representation can be finite and still be far beyond accessible experimental size.

No measured prediction is claimed. The existence of a mathematical memory or a noise-window reduction does not establish that an apparatus realizes its rule, bath independence, detector convention or preparation. The physical-prediction open problem in Section 9 remains open. The preparation and cone-accounting assumptions in the cost results must be established for a proposed implementation before a heat comparison can be made.

8.2 Noise can maintain a distinction

One radius-one rule on $\mathbb{Z}^3$ is nilpotent without noise but has two translation-invariant stationary laws at a positive uniform-replacement rate; the same rule is exponentially ergodic at sufficiently small and sufficiently large positive rates [6, version 1.2.0, Theorem A; Theorems 4.2 and 5.1], DOI `10.5281/zenodo.24120279`. The memory notion is stationary multiplicity after perturbation. Nilpotence of the deterministic rule therefore does not classify the noisy stationary problem in this example.

A separate fixed rule on $\mathbb{Z}^4$ has infinitely many alternating designated noise windows accumulating at zero: two translation-invariant stationary laws on one sequence, all-state attraction on the interleaving sequence. Its behavior between the windows is not determined [7, version 1.0.0, Theorem A / Theorem 7.7], DOI `10.5281/zenodo.24122609`. This statement supplies repeated transitions, not an exact classification of every noise rate. The associated five-dimensional Gibbs result concerns uniqueness among translation-invariant Gibbs measures, and does not assert all-DLR uniqueness [7, version 1.0.0, Theorem B / Theorem 8.3].

These examples prevent a monotone "more noise means less memory" reading of the synthesis. The bath is part of the total dynamics. They also prevent using deterministic simplicity as a proxy for the computability of the noisy long-run problem. The exact noise model in each cited example is uniform whole-symbol replacement; robustness to arbitrary perturbations is an additional question.

8.3 Preparation, stationarity and convergence are separate

The lower selection results allow spatially inhomogeneous computable preparations or computable boundary sequences. Homogeneous product starts are not substituted for these preparations. The routed quantum result permits a causal translation of layers, and its selected states belong to a specified invariant classical subsystem. The general stationary quantum enclosure theorem concerns unmeasured expectations [10, 13, 14]. None of those conventions is dispensable when interpreting a claimed observation record.

The random-boundary Ising result illustrates another distinction. For the planar Ising model at every $K > K_c$, with fair independent boundary spins in its stated volume convention, the almost-sure weak cluster set of finite-volume Gibbs laws is the whole interval of plus/minus mixtures. The empirical distribution across sizes nevertheless converges to equal endpoint masses [9, version 2.0.0, Theorems A and C], DOI `10.5281/zenodo.24401253`. The memory notion is equilibrium boundary selection and subsequential state multiplicity. This is not an ordinary time-average realization theorem for a fixed native heat-bath trajectory.

A cluster set, an empirical distribution across volume sizes, a definite ensemble-density limit and a deterministic almost-sure frequency are thus four distinct output tasks. The cited results concern different tasks. This paper does not infer an ordinary native Ising PCA frequency compiler from the random-boundary Gibbs theorem, or from the existence of plus and minus stationary phases.

8.4 Which extrapolations remain unavailable

Theorem 2.5 gives ordinary programming for the moving-frame Ising readout, and Theorem 2.6 gives ordinary fixed-site programming at zero temperature. At positive temperature, ordinary interior time-average programming at a site at rest remains open. For rate-one Glauber dynamics at each computable finite $\beta > \beta_c$, almost-sure programming of a prescribed computable phase-sign sequence is proved, through cumulative and epoch averages approaching the corresponding signed spontaneous magnetization at computable observation times; this does not give an ordinary interior limit. That limit still requires Lifshitz-type control of interface motion [128]. The ordinary NEC convergence target, unrestricted entrance comparison, general Problem P realization, and exact dynamics characterization remain open [16, 18, 28].

The general threshold principle remains Conjecture 4.1. The exact entropy formula concerns an optimized stationary one-site distinction with the stated dimension and alphabet restrictions, rather than all kinds of memory. The revelation cost concerns explicitly charged cones and complete gate accounting. A broader mathematical or physical principle would require its own hypotheses and evidence; it is not added as an interpretation of those scoped results.

DOI-bearing records are archival releases, often preprints; no peer-review status is in-

ferred. Boundary, selection and cost components remain in preparation. The evidence table distinguishes these publication statuses and the remaining mathematical questions.

8.5 Standard models and measurement assumptions

The boundary results stated here are theorems inside standard stochastic, equilibrium and quantum models [10, 12, 13, 38, 39]. They force no deviation from standard physics. A measurable divergence from standard quantum predictions would require an additional axiom about measurement; these results do not support such an axiom. They establish limits on selected computations and asymptotic classification within the models. They supply neither an observed departure from standard predictions nor evidence choosing an alternative measurement rule. The physical prediction in Section 9 therefore remains an open problem.

9 Refutation conditions and an open physical problem

9.1 A mathematical challenge must preserve the quantifiers

A proposed counterexample to Theorem 2.1 must supply the model's effective presentation, the compact compatible-state set, the same fixed computable affine readout, and the sound complete interval premise. It must then show singleton agreement while ruling out the asserted universal precision search or computability. A noncomputable coefficient, a noncomputable readout, or incomplete finite relaxations would instead remove a hypothesis. In the classical trajectory version, a limit common only to a selected computable subclass of preparations does not meet the all-preparation premise [10][11].

For the conjectural extension, a system whose natural finite constraints fail completeness would be informative about the proposed extension even if it is no counterexample to the proved interval theorem. A clear report should identify which of those two claims is challenged. The distinction matters because a conditional compactness theorem cannot be tested by assuming its missing model-specific premise without justification.

A computably presented rational PCA with one stationary law attracting every preparation locally, but with a noncomputable stationary cylinder probability, would contradict the cited common-readout result in its scope. The state set here is the full stationary set and the readout is the same finite cylinder. A claim about a fixed finite-ring storage horizon, or about an invasive quantum record with additional memory, is not that counterexample. For quantum channels the corresponding statement uses the common unmeasured local stationary expectation [10, 11].

A slow convergence rate alone would not refute the theorem. The certificate search makes no efficiency assertion, and a bound that is not known a priori can still be found on the agreement promise. To challenge the stated conclusion one needs failure of computability or of the sound universal certification assertion, rather than a large experimental relaxation time.

9.2 Upper inclusions and realization claims have different tests

A bounded experiment with uniformly computable finite expectations, a computable preparation description, and convergence in probability to a deterministic real outside Δ_2^0 would contradict the selected-limit upper inclusion. All three conditions are needed. A random limiting variable whose expectation lies in a phase interval is not a deterministic limiting frequency; an unrestricted noncomputable mixture weight is not a computable preparation [10][14].

The realization direction has another test. To refute the exact routed spectrum, one would need a limit-computable target in its specified persistence interval for which the cited computable preparation construction fails under its actual wiring and detector assumptions.

Failure of a native, un-routed fixed-site experiment to realize that target would not refute it. Likewise, failure of a homogeneous-start protocol does not refute a theorem permitting inhomogeneous computable starts.

These distinctions also constrain positive evidence. A noncomputable selected value in one example proves existence of such a value there; it does not establish the full interval spectrum for another medium. A theorem about geometric-date averages does not settle the ordinary all-date average. A stationary expectation in a convex mixture does not establish a deterministic time average on a sample drawn from that mixture.

9.3 Cost and information-rate tests

A challenge to formula (2) must keep the post-bath one-site gap, two translation-invariant stationary branches, deterministic finite-range rule, uniform replacement and erasure-density optimization. Charging only the easier branch, changing the bath reference, or treating pre-bath separation as post-bath separation changes the optimization [15]. A challenge to the finite-cone rent statements must keep the same initial-bank and observation credits and the same complete cone or gate cost [17].

For Theorem 5.2, the message is encoded in a microscopic initial configuration, the detector is fixed and passive, and the receiver has the record only. Restricting messages to a smaller entrance-law family does not challenge its optimized value. Supplying a free continuing quantum instrument with its own nonunital reset would change the resource account in a heat assertion. The corresponding cost extensions explicitly charge these resources [16, 17].

9.4 The threshold conjecture can be tested at the family level

A refutation of Conjecture 4.1 would identify a predeclared category at its least dimension for the specified persistent distinction, and give a decision procedure for the whole corresponding family of long-run tasks claimed to include an undecidable one. A single solvable model is insufficient. A decision theorem for an alphabet-bounded or attractive subcategory is relevant only if that restriction is part of the category to which the conjecture was applied. This family-level criterion is deliberately stronger than finding a tractable memory example.

The same discipline applies to a supportive example: a new reduction in a category chosen independently of the existing simulation constructions would extend the evidence. Rephrasing an old reduction in the language of that construction would not establish the general conjecture. The question concerns the relation between persistence resources and algorithmic classification in a specified category, not a ranking of how complicated individual local tables look.

9.5 Physical prediction: an open problem

Open problem. Physical prediction. No device-specific calibrated prediction or measured outcome is claimed. Specify an apparatus implementing a particular local medium, its preparation and independent-noise calibration, a detector and continuing-measurement convention, a finite-time information target, and a complete erasure/heat resource account before inserting a numerical prediction. The intended comparison would involve the information revealed about a specified prepared message and the erasure charged to a specified implementation. The present stationary-density price and asymptotic rate theorem alone do not supply a universal finite-time inequality of "capacity times time" without transient terms, and they do not identify stationary entropy-density erasure with measured device heat. Initial order and auxiliary supplies must be included under the chosen account [15, 16, 17].

An experimental discrepancy would then require its own interpretation: it could challenge the physical realization assumptions, the bath calibration, the measurement model, or a stated quantitative mathematical bound. With no apparatus model yet specified, calling such a slot a falsifiable device prediction would overstate the evidence. The physical problem remains open, alongside the general phase-set realization problem and its dynamics counterpart.

10 Further directions

The general phase-set realization problem goes beyond the exact rational-parameter and finite-product intervals of Theorem 6.2. An arbitrary relatively effectively open warm set requires control at accumulating interval endpoints and an autonomous unbounded-rank implementation with sound sampling, recovery and all-start mixing. The Omega transition and large-measure order without a computable temperature remain conditional until this problem is resolved [18, 28]. The exact replacement-noise-set dynamics counterpart is a separate question.

The observation protocol is another substantive direction. Theorem 2.3 settles the ordinary origin-average realization for its fixed line rule. Theorems 2.5 and 2.6 add the moving-frame positive-temperature Ising readout and the fixed-site zero-temperature ordinary average. What remains open for positive-temperature Glauber dynamics is ordinary interior time-average programming at a site at rest: the almost-sure phase-sign sequence programming proved at each computable finite $\beta > \beta_c$ does not supply that limit, which still needs Lifshitz-type control of interface motion [128]. Ordinary NEC realization also remains open. Boundary-selected Gibbs expectations, convergent instantaneous laws, ordinary time averages and weighted or sampled averages cannot be interchanged without a theorem [27, 12, 16].

A physical test requires a specified apparatus and resource account. Its finite observation record, initial order, accessible side information, continuing measurement and gate erasure must be matched to the hypotheses of a quantitative theorem. Abstract capacity, stationary entropy density and full-cone cost are different quantities; converting units alone does not identify them [15, 16, 17]. The physical-prediction problem therefore remains open.

The joint statement established at the present scope combines a recovery of certified common readouts with new exact selected spectra in the named PCA, square-lattice Gibbs and quantum-channel experiments. Extending that statement requires model-specific enclosures and realization theorems, with the preparation and observation quantifiers fixed in advance. The threshold extension remains Conjecture 4.1.

A Evidence register

Archival preprints are cited by version and theorem identifier, with version-specific DOIs in the bibliography. Unpublished results are by Douglas and Mghirbi, in preparation. The table records proved scopes, conjectures and open problems; definitions and comparisons in the main text do not assert additional theorems.

Claim and citation	Status
Singleton readout and arbitrarily fine universal certificates; recovery with prior credit [10, 29, 30, 31, 32, 33].	proved, in preparation (recovery of cited prior results)
Rational PCA confidence certificates; all-DLR $\mathbb{Z}^2$ Gibbs collars; unmeasured quantum Cesaro expectations [10, 12, 13].	proved, in preparation
Exact selected spectra and the rate-localized single-spin limit: routed detector, native line trajectories, Ising boundaries and specified quantum channel [14, 27, 12, 13, 39].	proved, in preparation
Moving-frame Ising ordinary programming and zero-temperature fixed-site programming, Theorems 2.5 and 2.6 [127, 128].	proved, in preparation
Fixed-noise forgetting; v3.0.0, A/3.1 and 12.5, 12.11 [1].	proved in [1]
Line noise and finite-energy Gibbs windows; v2.1.0, A–C [2].	proved in [2]
Stationary Rice class; v2.0.0, 3.1–3.3, 13.1 [3].	proved in [3]
Ring storage, v3.1.0, D; computation, v1.0.0, A; one-dimensional spectral-gap incumbent [4, 5, 36].	proved in [4, 5, 36] (archival preprints for the Zenodo releases)
Bernoulli-stationary PCA forgetting [19]; stationary entropy export [15].	proved in [19]; proved, in preparation [15]
Exact-depth recovery decision and hardness; approximate gaps; computable root infimum; unsteered versus input-steered recovery [38].	proved, archival preprint
Necessity of dissipation for the fresh-input recovery gap; general bounds for pooled disconnected receivers [38].	open
General preparation extension and dimension-threshold principle.	conjecture
Exact optimized stationary entropy price, feasibility and noise regimes [15].	proved, in preparation
Routed binary capacity and microscopic-message variational rate with coding [14, 16].	proved, in preparation
NEC logarithmic information bound and weighted/geometric-date realizations [16].	proved, in preparation
Ordinary all-date NEC realization [16].	open
Horizon-dependent zero infimum; fixed-medium cone costs and credited inventory [17].	proved, in preparation
Effective openness of two-dimensional exponential weak-mixing temperature sets [18]; mixing import [34, 35].	proved, in preparation [18]; import proved in [34, 35]

Claim and citation	Status
Exact rational-t_c and specified finite-product interval magnets, including ordered endpoints [28].	proved, in preparation
Problem P realization, general interval case, Chaitin-Ω magnet and exact dynamics set.	open
Nilpotent noisy memory and alternating windows [6, 7].	proved in [6, 7]
Planar Ising boundary-mixture recurrence for every $K > K_c$; v2.0.0, A/C [9].	proved in [9]
Fixed-site phase-sign programming at computable finite $\beta > \beta_c$ [128].	proved, in preparation
Positive-temperature ordinary interior limit at a site at rest; unrestricted entrance-law comparison.	open
Device implementation and a calibrated physical prediction.	open

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