A trace costs one quantifier: the exact complexity of perfect play in nonlocal games

Seth Douglas Nidhal Mghirbi

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Abstract

Given a finite nonlocal game, can the players win with certainty? We find the exact level of the arithmetical hierarchy at which this is decided, for each standard model of quantum strategy. With normal states on tensor products of Hilbert spaces, perfect play is Σ_2^0-complete, also when the strategy must have finite entanglement entropy, so it cannot be decided even with an oracle for the halting problem. This completes a table in which finite-dimensional perfection is Σ_1^0-complete, commuting perfection is Π_1^0-complete and value one is Π_2^0-complete. Inside the commuting model the answer depends on the von Neumann type of the players' algebra: perfect play in a type II_1 algebra, or in any semifinite algebra, is Σ_2^0-complete, while in a type III_λ factor, for each fixed $\lambda \in [0,1]$, it is equivalent to commuting perfection. In this sense a trace costs one existential quantifier. Commuting perfect play without semifinite perfect play is Π_2^0-complete; the remaining semifinite-without-spatial differences are $D_2(\Sigma_2^0)$-complete. In the hardness proofs the existential witness is a finite object that every perfect strategy must choose: the order of a clock phase in the spatial model, or the depth of a critical binary tree in semifinite algebras. The same level governs communication: deciding whether a finite classical channel can send a given number of messages in one use with zero error, helped by normal entanglement, is Σ_2^0-complete, and some channel's zero-error codes all need infinite entanglement entropy. Positive computable moment orders below one form a strict Σ_2^0-complete resource ladder; the infimal critical indices of spatially perfect games (attained by the realizing games) are exactly the right-c.e.-in-$0'$ reals in $[0,1]$.

AI systems assisted with this work. The authors take responsibility for its claims; see "How this work was done." This preprint has not been peer reviewed.

Programs and finite checks: https://github.com/Apsiape/perfect-play-arithmetical-hierarchy. This paper consolidates and supersedes the authors' preprints doi:10.5281/zenodo.23910986, doi:10.5281/zenodo.23925091 and doi:10.5281/zenodo.23924868 [24, 22, 21].

Contents

1 Introduction **5**
1.1 Why a trace costs a quantifier . . . 6
1.2 Further results . . . 7
1.3 Prior work and attribution . . . 7
1.4 Organization . . . 9

2 Preliminaries **9**
2.1 Games, strategies and models . . . 9
2.2 Arithmetical classes . . . 10
2.3 Conventions . . . 10
2.4 Supports and synchronization . . . 11
2.5 Notation . . . 12

3 Upper bounds: one existential quantifier **14**
3.1 Normalize one Schmidt coefficient . . . 15
3.2 An effective finite predicate . . . 15
3.3 No effective choice of stratum; vanishing largest weight . . . 16
3.4 Entropy budgets . . . 16
3.5 A finite tracial corner . . . 20

4 A finite compiler for normal strategies **22**
From a presentation to a game . . . 22
Normal states, supports and exact cancellation . . . 23
4.1 The normal flagged magic-square bridge . . . 26
4.2 Rational projection equations that certify products . . . 33
4.3 Rational rows as Boolean winning conditions . . . 34
4.4 Admissible presentations . . . 37
4.5 The n-bridge compiler . . . 38
4.6 Chained bridges on normal commuting hosts . . . 43
4.7 A dimension-independent completeness estimate . . . 44

5 The rooted tree and computation squares **45**
5.1 Normality removes every rootless sector . . . 45
5.2 Each depth is a finite square . . . 46
5.3 A finite deterministic tableau . . . 46
5.4 A worked front and mask example . . . 48

6 A finite-order clock and its period reader **48**
6.1 A determinant supplies a finite witness . . . 48
6.2 The clock and a counter without a positive lower bound . . . 49
6.3 A bounded inverse on every prime clock . . . 49

7 Spatial hardness: the reduction and value one **49**
7.1 Every constructed game has value one . . . 52

8 Exact Bell attainment **53**
8.1 A specified global bound . . . 53
8.2 No bound supplied . . . 54
8.3 A specified scalar value . . . 54

9 The entanglement spectrum and its decision problems **54**
9.1 Moment budgets are compact and effective . . . 55
9.2 Tree spectra separate the rungs . . . 55
9.3 Complexity, differences, and critical indices . . . 56
9.4 Approximate play: a dimension upper bound . . . 57

10 A uniform family and programmable critical indices **57**
10.1 The quarter-plane register and its finite presentation . . . 57
10.2 Normal classification, the clock, and all positive moments . . . 58
10.3 The comb and the independent checker . . . 59
10.4 Programming and classifying the minimum index . . . 61

11 Semifinite hosts: a critical tree selects a finite depth **62**
11.1 The critical-tree selector . . . 62
11.2 Finite models and a small example . . . 63
11.3 A finite presentation verifies the depth . . . 63
11.4 One bridge compiles the selector presentation . . . 65
11.5 Both uniform promises and the hardness theorem . . . 66

12 Each type-III subtype has the commuting language **68**

13 One selector for spatial and semifinite perfection **69**
13.1 The coupled root presentation . . . 70
13.2 The terminal sector meets the root . . . 70

14 Comparing the type languages **72**
14.1 A fixed one-bridge separator . . . 72
14.2 What happens to the Weyl-clock family . . . 72
14.3 A tracial game with no spatial perfection . . . 73
14.4 Exact conjunction and disjunction . . . 74
14.5 Four fixed separators and two uniform families . . . 75
14.6 The table, with reductions for every nonempty entry . . . 76

15 Zero-error codes and perfect play **77**
15.1 Channels and trace-class assemblages . . . 78
15.2 Saturation of a clique partition . . . 78
15.3 The normal correspondence . . . 79
15.4 The resource correspondence at every order . . . 80

16 Five bridges and the normal row **80**
16.1 An exact spectrum identity . . . 81
16.2 The tree at a general ratio . . . 82
16.3 Summability and its critical point . . . 82
16.4 Upper membership . . . 83
16.5 Five bridges for normal hardness . . . 83
16.6 Finite entropy already at five bridges . . . 84
16.7 The finite-dimensional row, as an attributed corollary . . . 84

17 The commuting row **84**
17.1 Normal functionals from a commuting code . . . 85
17.2 Synchrony makes the fine supports tracial . . . 86

18 The boundary between one-shot codes and capacity **86**
18.1 Theta bounds normal and commuting codes . . . 87
18.2 Saturation cancels a tensor power . . . 87
18.3 Arithmetical upper bounds for rates . . . 88
18.4 The exact rate identity and its limitations . . . 88
18.5 Exact common sums from approximate ones . . . 89
18.6 A fourteen-vertex test of the theta conjecture . . . 90
18.7 What a capacity gap would have to prove . . . 91

19 A channel whose codes require infinite entropy **92**
19.1 The squared Catalan density . . . 92
19.2 A four-bridge Catalan assembly . . . 94
19.3 Every normal code has infinite entropy . . . 94

20 Discussion and open problems **95**
20.1 Entropy budgets . . . 95
20.2 Prescribed tables and escaping mass . . . 95
20.3 The Fu–Miller–Slofstra remark . . . 96
20.4 Zero-error capacities . . . 96
20.5 Other questions . . . 96

How this work was done **97**

A The nested assembly **97**
A.1 Input, output, and the scope of the calibration . . . 97
A.2 Corners, parameters, and activities . . . 98
A.3 The complete transport list and the whole-activity block . . . 99
A.4 All quadratic translations and arbitrary coefficients . . . 100
A.5 Soundness and the Boolean game . . . 101
A.6 One simultaneous completeness model . . . 102
A.7 The loss constant, including the normalization factors . . . 103

B The bridge-count and branching variants **104**

C Finite checks and reproduction **105**
C.1 Supplementary exact rotation margin . . . 108

D Compiler validation instances **108**
D.1 Parametric presentation input and exact size formulas . . . 108
D.2 The concrete programs and both decision branches . . . 109
D.3 Exact symbolic checks of the infinite models . . . 110
D.4 Finite defects, the worst rows, and the loss bound . . . 111
D.5 Random algebra checks and the general regression . . . 112

E Results used from other papers **112**

1 Introduction

In a nonlocal game a referee sends each of two separated players a question from a finite list. Each player answers from a finite list, and a fixed rule decides whether they win. Shared entanglement can raise the winning probability. When can the players win with certainty?

For a fixed game this is a yes-or-no question. It has a precise answer only once one says what a quantum strategy is, and only once one separates approaching certainty from attaining it. The supremum of the winning probability can be one while no strategy reaches it. The standard models also differ: finite-dimensional tensor products, normal states on infinite-dimensional tensor products (the spatial model), and commuting operator algebras on one Hilbert space. Inside the commuting model one can further ask which von Neumann algebra hosts the players. This paper finds, for each of these models, the exact level of the arithmetical hierarchy at which perfect play is decided. Table 1 collects the answers.

Decision problem	Complete class	Source
Perfect finite-dimensional strategy	Σ_1^0	[32]
Perfect commuting strategy	Π_1^0	[53, 42]
Finite-dimensional value equals one	Π_2^0	[42]
Perfect spatial strategy; also with finite entropy	Σ_2^0	Theorem A
Perfect strategy with finite α-moment, fixed computable $0 < \alpha < 1$	Σ_2^0	Theorem D
Perfect play in type II_1, II_∞ or any semifinite algebra	Σ_2^0	Theorem B
Perfect play in a type III_λ factor, any fixed $\lambda \in [0,1]$	Π_1^0	Theorem B
Commuting perfect play but no semifinite perfect play	Π_2^0	Theorem B
Semifinite perfect play without spatial (or finite-entropy) perfection	$D_2(\Sigma_2^0)$	Theorem B
One-shot zero-error code, normal entanglement; also finite entropy	Σ_2^0	Theorem C
One-shot zero-error code, commuting operators	Π_1^0	Theorem 17.2

Table 1: Finite rational Boolean games and channels; computable many-one reductions. A value is a supremum. A perfect strategy attains one. The first three rows are known; the others are proved here.

Four results fill the table and its resource refinements. The first completes the standard row of models: spatial perfection sits one level above finite-dimensional perfection.

Theorem A (spatial perfection). *Deciding whether a finite rational* $0/1$ *game has a perfect normal spatial strategy is* Σ_2^0*-complete. The same holds when a perfect strategy with finite marginal entropy is required. Hardness holds under the promise* $\omega_q(G) = 1$*, for a family in which no instance has a perfect finite-dimensional strategy.*

The second sorts the commuting model by the type of the players' algebra. Here $\mathrm{PERF}(X)$ is the set of games with a perfect strategy whose state is normal on an ambient von Neumann algebra of class X (Section 2).

Theorem B (perfect play by von Neumann type). *Perfect normal play in a finite algebra, a type* II_1 *algebra, or a semifinite algebra is* Σ_2^0*-complete. The same holds for the type* II_∞ *and type* II *existence languages. For every fixed* $\lambda \in [0,1]$*, the type* III_λ *language equals commuting perfection and is* Π_1^0*-complete. Commuting perfection without semifinite perfection is* Π_2^0*-complete, with hardness on a family of commuting-perfect games all having* $\omega_q = 1$*. The two languages* semifinite but not spatial *and* semifinite but not finite-entropy

spatial *are $D_2(\Sigma_2^0)$-complete, also with no finite-host perfection. Every nonempty directional difference among the six models in Theorem 14.1 is complete for its displayed class.*

The communication result shows that the same level governs a communication task. A sender uses a finite classical channel once and must convey one of m messages with zero error, helped by shared entanglement. Write $\mathrm{ZE}_t(N, m)$ for the existence of such a code for the channel N in model t (Section 15).

Theorem C (zero-error communication). *For finite rational classical channels, $\mathrm{ZE}_{\mathrm{qs}}$ and $\mathrm{ZE}_{\mathrm{finS}}$ are Σ_2^0-complete. Moreover, there are a finite rational classical channel N and an integer m for which $\mathrm{ZE}_{\mathrm{qs}}(N, m)$ holds, and every normal spatial m-message zero-error resource has infinite marginal entropy on both sides.*

Theorem D (the entanglement spectrum). *For every fixed computable $0 < \alpha < 1$, perfect play with $\operatorname{Tr} h^\alpha < \infty$, denoted $\mathcal{P}_\alpha$, is Σ_2^0-complete. With $\mathcal{P}_0$ denoting finite rank, $\mathcal{P}_1$ finite entropy and $\mathcal{P}_{\mathrm{normal}}$ normal perfection, they form the strict ladder $\mathcal{P}_0 \subsetneq \mathcal{P}_a \subsetneq \mathcal{P}_b \subsetneq \mathcal{P}_1 \subsetneq \mathcal{P}_{\mathrm{normal}}$ for $0 < a < b < 1$; all Rényi orders above one collapse to normal perfection. One computable hardness family has a single perfect witness with all positive moments finite on every YES input, value one on every input, and no finite-dimensional perfect instance. Every nonempty difference between distinct positive computable rungs is $D_2(\Sigma_2^0)$-complete; a positive rung minus finite rank is Σ_2^0-complete. For a perfect witness with marginal h, its critical index is $\alpha_c(h) = \inf\{\alpha > 0 : \operatorname{Tr} h^\alpha < \infty\}$. Put $c(G) = \inf\{\alpha_c(h) : h$ is the marginal of a perfect spatial witness for $G\}$. The possible values of $c(G)$ (an infimum, attained by the realizing games) are exactly the right-c.e.-in-$0'$ reals in $[0, 1]$, already among games with finite-entropy perfect strategies.*

A running example. Consider the rooted tree of finite words w over four letters; $|w|$ is the word length and appending a letter selects one of four branches. Give the basis vector $|w\rangle$ density weight $(3/4)16^{-|w|}$. A density ratio $1/16$ means that each child has one sixteenth of its parent's weight. The weights sum to one and define a normal state of finite entropy. Its half-moment, the sum of the square roots of all weights, diverges: the four child square roots sum to the parent square root at each level. A bridge is a finite game gadget that doubles a density weight between two selected subspaces; four bridges give the factor 16. A fifth changes the ratio to $1/32$ and makes the half-moment converge, because $4/\sqrt{32} = 1/\sqrt{2}$. This example will explain the compiler, the strict resource ladder and the zero-error threshold. A clock of order three means a phase whose third power first equals one. Table 4 shows the depth-3 square for a clock of order three.

1.1 Why a trace costs a quantifier

Commuting perfection is Π_1^0 because the commuting state space is compact. Perfection then equals commuting value one, which a convergent hierarchy of relaxations refutes in finitely many steps whenever it fails. Spatial play has no such compactness: Schmidt mass can escape to infinitely many small coefficients. Our upper bound keeps one Schmidt coefficient from disappearing. After that coefficient is scaled to one, the amplitude operator lies in a norm-compact family, and losing probabilities become homogeneous zero equations that survive limits. Each lower bound on the largest Schmidt probability therefore gives a Π_1^0 condition, and their union over an integer is Σ_2^0.

The same integer appears for every semifinite host. A trace gives a finite corner whose overlap with the state can be bounded below. The overlap plays the role of the largest Schmidt amplitude; in type I a rank-one corner recovers it (Section 3.5). A type-III factor has no finite

corner, and Theorem B shows that its perfect-play language collapses to the commuting one. In this sense a trace costs exactly one existential quantifier.

The hardness proofs show that the quantifier is needed. In both cases the existential witness is a finite object that a perfect strategy is forced to choose. In the spatial model it is the order of a clock phase: a Weyl relation commutes with a trace-class density, each positive density eigenspace is finite-dimensional, and a determinant forces a root of unity. In a diffuse semifinite algebra no such determinant exists, but a trace forces a critical binary tree to end at a finite depth. A finite game then checks an infinite sequence of decidable conditions about the chosen object, through a deterministic computation or a group word. No computable function of the game bounds a sufficient clock order (Section 7) or a sufficient Schmidt stratum (Theorem 3.1).

Theorem C comes from a correspondence between games and channels due to Mančinska [37], which we extend from finite dimension to trace-class operators. A perfect strategy gives a code exactly when its Schmidt amplitudes are summable. The compiled games of Theorem A sit exactly at this threshold: with four bridges, the amplitude sum diverges for every perfect strategy. Five bridges restore summability. A Catalan version of the construction gives a channel whose game has finite-entropy perfect strategies, while every code of the required size has infinite entropy.

1.2 Further results

The body also proves several refinements of Theorems A–D. Theorem 3.1 and Section 3.4 show that the largest-Schmidt strata are Π^0_1 and that no total $0'$-computable function bounds a sufficient stratum. Section 3.4 proves compactness for full correlation tables under a supplied entropy budget; the entropy busy beaver has Turing degree $0''$ and eventually dominates every total $0'$-computable function. Theorem 8.1 shows that exact attainment of a rational Bell bound, of an unspecified supremum, or of a specified value is Σ^0_2-complete. In Section 14, the type-I and type-II_1 languages are incomparable, and both are proper subsets of the type-II_∞ language. Theorem 14.1 classifies every difference "perfect in X but not in Y" among six models. Perfect play in type I but not in type II_1, and perfect spatial play attainable only with infinite entropy, are $D_2(\Sigma^0_2)$-complete; the two semifinite-but-not-spatial cells are complete for the same difference class by the shared-family construction of Section 13. Section 16.7 records that one-shot zero-error feasibility with finite-dimensional entanglement is Σ^0_1-complete, a corollary of [37] and [32].

1.3 Prior work and attribution

The finite-dimensional row of Table 1 uses perfect completeness in Theorem 12.6 of Ji, Natarajan, Vidick, Wright and Yuen [32], not just a gap statement about a supremum. Upper membership follows by a dimension search and real quantifier elimination. Slofstra's embedding theorem [53] and the account of Mousavi, Nezhadi and Yuen [42, Theorem 6.4] give the commuting row. Their value-one theorem [42, Theorem 6.9] gives the third row. The commuting upper bound uses the convergent moment hierarchy of Navascués, Pironio and Acín [43]. The operator-solution framework for binary constraint system games originates with Cleve and Mittal [11]; Cleve, Liu and Slofstra gave the commuting-operator extension [10]. Slofstra also established nonclosure of finite-dimensional quantum correlations [52]. Lin's $\mathsf{MIP}^{co} = \mathsf{coRE}$ theorem concerns gap approximation in the commuting model [35]; it does not classify spatial attainment.

Infinite-dimensional spatial correlations were constructed by Coladangelo and Stark [12]. Kim, Paulsen and Schafhauser prove that perfect spatial play collapses to perfect finite-dimensional play for synchronous games [33, Corollary 3.11]. Lupini and coauthors establish

the corresponding equivalence for imitation games [36, Corollary 5.4]. Our games retain nonsynchronous bridge questions. No YES instance of the spatial reduction is synchronous or an imitation game, since either condition would imply finite-dimensional perfect play.

Mančinska's flagged magic-square construction supplies the bridge geometry [37]; the magic square itself is due to Mermin and Peres [40, 49]. The normal bridge, shared-label refinement, graph products and Boolean rational-row tests were developed in our entropy-game paper [23, Lemmas 4.1 and 4.2, Theorem B, Eq. (32)]; its Catalan game [23, Theorem C] is used here as a separator, and a four-bridge version of its Catalan assembly gives the infinite-entropy channel. The versions needed here are restated and proved in Section 4, which extends them to n bridges, arbitrary finite lists of endpoint labels and normal commuting hosts. The standard-form reduction, the commuting bridge and expected-inclusion transfer come from our earlier paper [20, Lemmas 1.6, 2.2 and 3.1]; the type rows of Table 1 complete the question left open in [20, Theorem 7.6]. The separation $\mathrm{PERF}(\mathrm{II}_1) \not\subseteq \mathrm{PERF}(\mathrm{I})$ uses the tracial mechanism of [20, Proposition 7.4] and Slofstra's nonclosure construction; the other direction uses a new one-bridge separator.

Group embeddings originate with Higman [30]; the effective form follows Chiodo [9, Theorem 2.2], with Craig's conversion [16] and the HNN normal form [31]. Subtype transfer uses Connes–Takesaki flow-of-weights theory [15], Takesaki's expectation theorem [54], Connes's invariants [13], and the argument inside Haagerup–Musat [27, Lemma 3.8]. The nearest operational work on host type is the commuting self-test for exact embezzlement of Paddock and Schmidt [46], and Harris's Cuntz-algebra self-testing and factor classification [29]. These concern exact embezzlement rather than the arithmetical classification of host-existence languages.

For communication, Cubitt, Leung, Matthews and Winter established the operator characterization of entanglement-assisted zero-error codes and the pseudo-telepathy construction [18]. Mančinska proved finite-dimensional clique saturation in Lemma 5, the converse and square-root state map in Theorem 6, and the entanglement-assisted independence-number characterization in Corollary 7 of [37]. Her tracial game value admits infinite-dimensional tracial strategies, but her zero-error results assume a finite-dimensional shared state. Her Corollary 8 gives the fifty-input flagged-magic-square channel with ten entangled messages and at most nine messages from finite-dimensional maximally entangled resources; our finite numerical checks reproduce that example as a regression check. Mančinska, Roberson and Varvitsiotis studied perfect strategies and independent-set game reductions [39]; Mančinska and Roberson developed quantum graph homomorphisms [38]. Harris proved universality results for synchronous graph homomorphism games [28]. Those game parameters should be distinguished from codes with arbitrary resources. Bhattacharyya, Mehta and Zhao proved undecidability for a restricted entanglement-assisted one-shot parameter of classical-to-quantum channels [7]; their formal one-shot parameter uses maximal entanglement and projective measurements [7, Definition 2.3]. The finite-dimensional row here removes these restrictions for classical channels and for a one-shot threshold. It does not resolve the unrestricted asymptotic capacity question.

Fu, Miller and Slofstra study fixed-scenario correlation membership [26]. In the discussion after their Theorem 1.1 they state that fixed-scenario membership in C_q and C_{qs} is in RE; the C_q case follows by dimension search, and we found no argument there for the C_{qs} case. Section 20 records a conditional tension with Theorem A. We do not identify prescribed-table membership with game perfection.

Earlier versions of Theorems A, B and C, of Theorems 3.1 and 8.1, of the trace-class correspondence, of the summability threshold, of the commuting row and of the I-only and infinite-entropy-only classifications appeared in our preprints [24, 22, 21], which this paper consolidates and supersedes. Apart from those preprints we know of no prior statement of

these results; Theorem D and the completion of the semifinite-but-not-spatial cells are new here. The Π^0_1-completeness of commuting perfection is known [53, 42]; what is new in Theorem B is that every fixed type-III subtype has exactly that language.

1.4 Organization

Section 2 fixes the models, languages and notation. Section 3 proves the Σ^0_2 upper bounds for spatial, finite-entropy, finite and semifinite perfection; Section 9.1 treats moment rungs and Section 16.4 reduces codes to games. Section 4 builds the finite compiler shared by all hardness proofs. Sections 5–7 prove the spatial hardness of Theorem A, and Section 8 the Bell variants. Sections 9–10 prove Theorem D, Sections 11–12 together with Section 13 prove Theorem B; Section 14 compares the type languages. Sections 15–19 prove Theorem C and the other communication rows. Section 20 discusses open problems.

2 Preliminaries

2.1 Games, strategies and models

A finite rational Boolean game G is a finite list of questions and answers, a rational distribution $\pi(x, y)$, and a predicate $V(a, b \mid x, y) \in \{0, 1\}$. Perfection means zero probability for every losing event of positive question weight. A spatial strategy consists of positive operator-valued measurements (POVMs) $E_{a|x}$ and $F_{b|y}$ and a normal state, represented by a positive trace-class density ρ of trace one, on $\mathcal{H}_A \otimes \mathcal{H}_B$. Its winning probability is

$$\sum_{xyab} \pi(x, y)V(a, b \mid x, y) \operatorname{Tr} \rho(E_{a|x} \otimes F_{b|y}).$$

We write C_q, C_{qs}, C_{qc} for finite-dimensional, normal spatial, and commuting correlations. For a strategy $\mathcal{S}$, write $\omega(G, \mathcal{S})$ for the displayed winning probability and $\operatorname{loss}_G(\mathcal{S}) = 1 - \omega(G, \mathcal{S})$. The value ω_q is the supremum over C_q. Finite Schmidt truncations show that C_q and C_{qs} have the same closure and the same supremum for every Bell functional.

The set PERF_q consists of games with a perfect finite-dimensional strategy. The set $\mathrm{PERF}_{\mathrm{qs}}$ consists of games with a perfect normal spatial strategy. Its stratum $\mathrm{PERF}_{\mathrm{qs}}(\kappa)$ requires a perfect pure strategy whose largest Schmidt probability is at least $1/\kappa$. The set $\mathrm{PERF}_{\mathrm{finS}}$ requires a perfect strategy with finite marginal entropy. Commuting perfection, $\mathrm{PERF}_{\mathrm{co}}$, uses two commuting families of POVMs on one Hilbert space and a state.

Hosts. For a class X of von Neumann algebras, $\mathrm{PERF}(X)$ denotes existence of a perfect strategy normal on an ambient von Neumann algebra of class X; Bob acts in its commutant. The ambient algebra need not be generated by the measurements. The state may be chosen along with the host. Finite-host perfection permits non-tracial normal states; this is consistent with $\omega_{\mathrm{tr}}(G_{\mathrm{FMS}}) < 1$ in [37, Theorem 3]. These are existence languages, not tests of a prescribed algebra, state, or full correlation table. Write $\mathrm{PERF}_{\mathrm{fin}}$ for finite-algebra hosting and $\mathrm{PERF}_{\mathrm{sf}}$ for semifinite hosting. Factor hosting gives the same existence languages as hosting in algebras purely of the indicated type.

We use separable hosts when decomposing. For commuting input this follows from the joint GNS representation of the finitely generated measurement C^*-algebra. For finite or semifinite input the finite-corner overlap construction of Section 3.5 gives a separable host in the same class: generate the two unital C^*-algebras by the finitely many effects and the finite band projection, and take the closure of their orbit of the trace vector. It is separable; Lemma 3.8 preserves a nonzero perfect component and proves its represented host finite or semifinite. One may then standardize Bob and decompose over the center. Every losing

probability is an integral of nonnegative probabilities, so the finitely many zero integrals vanish on a common conull set of factor fibres. A finite or semifinite fibre remains in that class; the amplifications in (10) give the required pure type-II factor. For type I use its direct-integral decomposition into spatial factor strategies; in particular the type-I language PERF(I) is the spatial language $\mathrm{PERF}_{\mathrm{qs}}$. The type-III constructions of Section 12 start with an unrestricted commuting factor fibre. Thus no group algebra constructed below is implicitly assumed to be a factor.

When several languages are compared at once we abbreviate

$$\mathbf{q} = \mathrm{PERF}_q, \quad \mathbf{E} = \mathrm{PERF}_{\mathrm{finS}}, \quad \mathbf{I} = \mathrm{PERF(I)},$$

$$\mathbf{J} = \mathrm{PERF}(\mathrm{II}_1), \quad \mathbf{U} = \mathrm{PERF}_{\mathrm{sf}}, \quad \mathbf{C} = \mathrm{PERF}_{\mathrm{co}}.$$

Here $\mathbf{q}$ requires attainment in finite dimension; it is not the value-one language. The inclusions are $\mathbf{q} \subseteq \mathbf{E} \subseteq \mathbf{I} \subseteq \mathbf{U} \subseteq \mathbf{C}$ and $\mathbf{q} \subseteq \mathbf{J} \subseteq \mathbf{U}$; every displayed inclusion is strict (Section 14). We keep PERF strictly for attainment. Nezhadi's 2025 thesis [44] uses "perfectly winnable" for the value-one Π^0_2 problem of [42], so its terminology has a different scope.

2.2 Arithmetical classes

The set Σ^0_2 consists of predicates $\exists j\, \forall t\, \mathcal{R}_{\mathrm{dec}}(e, j, t)$ with decidable $\mathcal{R}_{\mathrm{dec}}$. Completeness means that a problem is itself of this form and every set of this form reduces computably to it. The classes Σ^0_1 and Π^0_1 use a single existential or universal number quantifier, respectively, and Π^0_2 consists of complements of Σ^0_2 sets. We also use the class of differences

$$D_2(\Sigma^0_2) = \{A \setminus B : A, B \in \Sigma^0_2\}.$$

This is a class of differences, not the intersection of the two pointclasses Σ^0_2 and Π^0_2. The canonical complete language is $\{(e, f) : e \in \mathrm{FIN},\ f \notin \mathrm{FIN}\}$, where FIN is the Σ^0_2-complete set of indices of finite c.e. sets. Indeed reduce the two Σ^0_2 predicates separately to FIN and pair the resulting indices.

All encodings and reductions in this paper are effective finite encodings and computable many-one reductions. Completeness is for an unrestricted finite scenario, not one fixed question-and-answer size.

2.3 Conventions

Entropy uses logarithms to base two: $S(h) = -\operatorname{Tr}(h \log_2 h)$ and $H(p) = -\sum_i p_i \log_2 p_i$, with $0 \log_2 0 = 0$. A projection-valued measurement (PVM) is a POVM consisting of orthogonal projections. Write $|\Phi_d\rangle = d^{-1/2} \sum_{j=0}^{d-1} |j, j\rangle$ for a finite maximally entangled vector. Vectorization of an operator T in fixed orthonormal bases is

$$|T\rangle\!\rangle = \sum_{i,j} T_{ij} |i\rangle \otimes |j\rangle, \qquad \| \, |T\rangle\!\rangle \|^2 = \operatorname{Tr} T^* T,$$

when this sum is square summable. A purification is a unit vector with the given state as its partial trace over an added environment. Normal means represented by a positive trace-class density of trace one; on a spatial tensor product this includes pure vectors and countable mixtures. The centralizer of a density consists of the bounded operators commuting with it; for a faithful normal state φ on a von Neumann algebra M it is $M_\varphi = \{x \in M : \varphi(xy) = \varphi(yx)\ \forall y \in M\}$. For a game written in a code or compiler, YES and NO instances are those with and without the specified perfect strategy or code. Mat_r denotes the scalar r-by-r matrix algebra.

2.4 Supports and synchronization

The compiler of Section 4 uses the following elementary facts, in a spatial and in a standard-form version.

Pure components. If a mixed normal strategy is perfect, write $\rho = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$. Each losing event is positive. Its expectation vanishes in every component with $\lambda_j > 0$, so every such component is perfect with the same measurements. Trace-class marginals have separable support. Thus all arguments may be made on separable state supports.

Matching and support. For effects on opposite sides, perfect binary matching gives, on a purification Ω,

$$0 = \langle E + F - 2EF\rangle = \langle (E-F)^2 + (E-E^2) + (F-F^2)\rangle.$$

Each positive term annihilates Ω. If T is the coefficient operator of the pure vector on the chosen Schmidt cut, matching gives $ET = TF^{\mathsf{T}}$; dense coefficient range first makes the local state support reducing. On that support the effects are projections and commute with the marginal density. A positive operator B with $\mathrm{Tr}(hB) = 0$ for faithful h is zero, since $B^{1/2}h^{1/2} = 0$ and $h^{1/2}$ has dense range.

Sharp compressions and joint measurements. If PEP is a projection for an effect E and support projection P, positivity shows that E acts as identity on its range and zero on its kernel inside $P\mathcal{H}$. Hence E reduces the support. A POVM with sharp coarse marginals is their joint spectral measurement: each fine effect is supported in the corresponding marginal ranges, and summation gives the commuting products. These facts apply after compression before projectivity is used.

Synchronization and forbidden events. We call a projection synchronized when its binary question is perfectly matched to the corresponding opposite question. If synchronized projections satisfy $P_A^{\mathrm{sync}}\Omega = P_B^{\mathrm{sync}}\Omega$ and a forbidden event gives $Q_A P_B^{\mathrm{sync}}\Omega = 0$, dense Schmidt range gives the local identity $Q_A P_A^{\mathrm{sync}} = 0$. The arguments work on either party and on purifications of mixed states.

Standard form. A standard-form strategy for a faithful normal state φ on M uses its natural-cone vector ξ_φ in $L^2(M)$, Alice effects in M, and Bob effects in $M' = \mathcal{J}M\mathcal{J}$, where $\mathcal{J}$ is the modular conjugation and σ_t^φ the modular group. Nonfaithful states are interpreted on their support corners. A general normal-marginal commuting strategy has this form without changing probabilities: after joint GNS, compress Bob to $\overline{M\psi}$ using the isometry $x\xi_\varphi \mapsto x\psi$. The compression lies in the standard commutant and is still a POVM. Semifiniteness passes to corners. This is [20, Lemma 1.6]; the argument includes mixed states.

Perfectly matched binary effects are sharp centralizer projections with mirror $\mathcal{J}E\mathcal{J}$. Indeed different matched classes satisfy $E_c F_d \xi_\varphi = 0$. Summing gives $E_c\xi_\varphi = F_c\xi_\varphi$ and $E_cE_d\xi_\varphi = 0$ for $c \neq d$. Separation gives orthogonality, and completeness gives idempotence. The self-adjoint mirror gives $\varphi(E_c x) = \varphi(xE_c)$. A fine positive effect below sharp marginals is their joint projection. These facts replace any finite-dimensional rigidity assumption. On faithful support a synchronized projection is therefore a centralizer projection with its standard commutant mirror.

2.5 Notation

Table 2 collects the notation used across sections. Distinct fonts separate question names (sans serif), languages (bold), automaton symbols (typewriter) and operators. Integer indices such as $i, j, k, \ell, m, n, r, t$ are local to the statement or proof in which they appear, except that n counts bridges in the compiler and in Sections 16–19. Each root presentation (the Fock clock of Sections 5–7, the selector of Section 11, the shift separator of Section 14 and the Catalan presentation of Section 19) has its own labels, introduced where it is defined.

Table 2: Notation.

Symbol	Meaning
Games, models and languages	
$G;\ \pi, V$	finite rational Boolean game; question distribution and predicate
$\mathcal{S}, \omega(G,\mathcal{S}), \omega_q$	strategy, its winning probability, finite-dimensional value
$\mathrm{PERF}_q, \mathrm{PERF}_{\mathrm{qs}};$ $\mathrm{PERF}_{\mathrm{finS}}, \mathrm{PERF}_{\mathrm{co}}$	finite-dimensional, spatial; finite-entropy, commuting perfection
$\mathrm{PERF}(X), \mathrm{PERF}_{\mathrm{fin}}, \mathrm{PERF}_{\mathrm{sf}}$	hosting in class X, in a finite algebra, in a semifinite algebra
$\mathbf{q}, \mathbf{E}, \mathbf{I}, \mathbf{J}, \mathbf{U}, \mathbf{C}$	abbreviations for six existence languages
$\kappa;\ \mathrm{PERF}_{\mathrm{qs}}(\kappa), \mathrm{PERF}_{\mathrm{finS}}(m)$	Schmidt stratum integer; stratum and integer entropy-budget stratum
$F_{\mathrm{qs}}, F_{\mathrm{sf}}, F_{\mathrm{Bell}}$	decidable stage predicates of the upper bounds
$\mathcal{R}_{\mathrm{dec}}, \mathcal{A}_{\mathrm{dec}}$	decidable predicate and its Σ_2^0 set
$G_e^{\mathrm{clk}}, G_e^{(n)}, G_e^{\mathrm{sel}}$	Weyl-clock reduction (four bridges), its n-bridge version, selector reduction
$G_{\mathrm{sh}}, G_{\mathrm{lin}}, G_{\mathrm{Cat}}^{(2)}, G_{\mathrm{III}}$	fixed separating games of Section 14
States and hosts	
D_{Sch}, D	largest-amplitude-normalized and $\mathrm{Tr}\, D^2 = 1$ Schmidt operators
$M, \varphi, \sigma_t^{\varphi};\ \mathcal{J}, \xi_{\varphi}$	host, faithful state, modular group; standard conjugation and state vector
τ, h	trace and density
$R_{\mathrm{hyp}}, R_\infty, R_\lambda$	injective II_1, III_1, III_λ factors
The compiler	
$\mathsf{R}_i, \mathsf{C}_j, \mathsf{T}_x, \mathsf{B}_y;\ f, \perp$	flagged magic-square questions; flag and inactive answers
$c_x, r_y;\ P_{\mathrm{mag}}, m_{ab}$	selected column and row triples; magic support and its column-zero projections
$\mathfrak{x}, \mathfrak{z};\ \mathsf{U}$	Pauli axis reflections; calibrated flag-to-magic transport
P^{lab}	synchronized label projection in a refinement
$F_i, F_{ij}, G_{ij}, P_\triangle$	qutrit frame, pair support, graph and triangle projections
$P^{\mathrm{par}}, J_{ab}, s_{\mathrm{par}}$	bound parameter, its joint partition and reflection label
$S, L, V_j;\ k_{\mathrm{row}}$	rational row, denominator, signed reflections; scalar qubits
$\Pi_j^{\pm}, \Pi_t^{\mathrm{row}}, \Pi_d^{\mathrm{ref}}, \mathcal{I}_t$	scalar parity events, row partition, reference projection, column isometries
$d_{\max};\ \varphi_{\mathrm{old}}, h_{\mathrm{old}}$	row-ancilla dimension; state and marginal before the row ancilla
$A_\alpha, P_\circ;\ \nu$	creation contractions and root projection; modular ratio 2^{-n}
$e_0, e_g, m_{g,ab}, m_g$	source and selected corners, magic corners and supports of bridge g
E_g^{act}	activity $e_{g-1} + m_g$ of bridge g
$\mathsf{U}_g, v;\ \mathfrak{c}_n, \mu_n$	bridges and their chain; scaffold cover constant and weights
$\Lambda, u_{r,\ell}, V_{g,\ell}, L_r$	transported encodings, corner copies, block unitaries, decoded copies
$a_\alpha;\ L_{\mathrm{end}}, R_{\mathrm{end}}$	lifted centralizer contractions; endpoint coefficients

Table 2: Notation (continued).

Symbol	Meaning
The Fock clock	
$\mathcal{W}, A_w, \mathcal{K}, h_{F,\nu}, h_F, \rho_{\mathrm{spec}}$	four-letter words, word isometries, root multiplicity, tree densities, spectator
$U_x, U_y; P_x, P_y$	cyclic coordinate shifts; zero-column and zero-row projections
$\Gamma_{\mathrm{CA}}, P_c;\ \mathtt{s}, \mathtt{b}, \mathtt{f}, \mathtt{z}, \mathtt{H}$	automaton alphabet and its projections; start, blank, front, mark, halt
$J_{\mathrm{W}}, K_{\mathrm{W}}, Z_{\mathrm{clk}}, \zeta, d_{\mathrm{clk}}$	Weyl pair, central clock phase, its value and order
$W, C_{\mathrm{ctr}}, P_{\mathrm{ret}}, T_{\mathrm{inv}}$	coordinate clock, counter, return pulse and bounded reader inverse
$h^{[m]}, P_m, t_m$	truncated density, top-depth projection and its mass
The quarter-plane register and comb (Section 10)	
$S, T;\ P_x, P_t;\ C_x, C_t$	shifts, boundary projections and dyadic counters
$Z, J, K, W;\ E_{\mathrm{ret}}, R_{\mathrm{ret}}$	clock labels, return projection and bounded reader
$d_{\mathrm{clk}};\ \rho_{\mathrm{spec}}, \rho_{\mathrm{reg}}$	clock order; spectator and geometric register densities
$V_{\mathrm{sp}}, B_0, B_1;\ Q, P, E_{\mathrm{guard}}$	spine and branch maps; spine, root and guard projections
$C_{\mathrm{sp}}, L;\ E_{C_{\mathrm{sp}}}, E_L, F_{C_{\mathrm{sp}}}, F_L$	comb counters, equality projections and their readers
$k_m, \rho_{\mathrm{comb}}, Z_{\mathrm{comb}}$	bush depths, comb density and its normalizer
$\mathcal{P}_\beta, \alpha_c(h), c(G);\ d_{\mathrm{cut}}$	moment language, witness and game critical indices; cutoff dimension
The nested assembly (Appendix A)	
$X_i;\ a_i, a_i'$	root modular contractions; inner and transported lifts
$f, e, E_{\mathrm{in}}^{\mathrm{act}}, E;\ f', e'$	inner corners and activity, outer corner; transported corners
$u, \mathsf{U};\ \mathsf{U}_0, \mathsf{U}_1;\ t, \ell$	inner/outer transports, outer restrictions, equal-density map and quadratic coefficient
$U_k, V_k, V_k^{\mathrm{blk}}$	inner encodings, outer copies and whole-activity block unitaries
$R_{\mathrm{tar}};\ K_L, K_M, K_R$	target centralizer expression; arbitrary polynomial coefficients
$\epsilon_{xy}, \varsigma;\ J_{\mathrm{sgn}}, Z_{ij}^{\mathrm{gr}}$	FMS signs, scalar axes; sign grading and graph reflection
Communication	
$N, m;\ \mathrm{ZE}_t(N, m)$	finite rational channel and message count; one-shot feasibility
$\Gamma(G);\ \mathcal{Q}_A, \mathcal{Q}_B;\ k$	tagged confusability graph; question sets; number of tagged questions
$R_{i,x}, \rho, \rho_{\mathrm{code}};\ \mathcal{C}_c$	assemblage and code marginal; a clique
$\hat{\mu}_n;\ c_n^{\mathrm{amp}}$	normalized square roots of μ_n; amplitude constant
$h_{\mathrm{Cat}}, k_{\mathrm{Cat}}, C_{\mathrm{Cat}}$	critical Catalan density, its normalized square, Catalan height counter

3 Upper bounds: one existential quantifier

Each tracial language in this paper is a union over one integer of Π_1^0 sets. For spatial play the integer bounds the inverse of the largest Schmidt probability; for semifinite play it bounds the inverse overlap of the state with a finite tracial corner. We first treat the spatial case and prove the following theorem.

Theorem 3.1 (Schmidt strata)**.** *Uniformly for integers $\kappa \geq 1$, the existence of a perfect pure spatial strategy with largest Schmidt probability at least $1/\kappa$ is a Π_1^0 property. Their union is $\mathrm{PERF}_{\mathrm{qs}} \in \Sigma_2^0$. No total computable function $\kappa(G)$ bounds a sufficient stratum for every spatially perfect game. If $\omega_q(G) = 1$ and no perfect spatial strategy exists, the largest Schmidt probability tends to zero along every sequence of pure spatial strategies whose winning probabilities tend to one.*

3.1 Normalize one Schmidt coefficient

For a pure strategy write $\psi = \sum_i s_i |i,i\rangle$ with $s_1 \geq s_2 \geq \cdots$ and $\sum_i s_i^2 = 1$. Set $d_i = s_i/s_1$ and $D_{\mathrm{Sch}} = \mathrm{diag}(d_i)$. For some integer κ,

$$d_1 = 1, \qquad d_i \downarrow 0, \qquad \operatorname{Tr} D_{\mathrm{Sch}}^2 \leq \kappa. \tag{1}$$

For positive effects E, F, vectorization gives

$$p(E,F) = \frac{\|\sqrt{E}\, D_{\mathrm{Sch}} \sqrt{F^{\mathsf{T}}}\|_{\mathrm{HS}}^2}{\operatorname{Tr} D_{\mathrm{Sch}}^2}, \qquad p(E,F) = 0 \iff E D_{\mathrm{Sch}} F^{\mathsf{T}} = 0.$$

To prove the reverse implication, $E D_{\mathrm{Sch}} F^{\mathsf{T}} = 0$ sends $D_{\mathrm{Sch}} \operatorname{ran} F^{\mathsf{T}}$ into $\ker E$. The closures of the ranges of F^{T} and $\sqrt{F^{\mathsf{T}}}$ agree, and $\ker E = \ker \sqrt{E}$. Continuity gives the square-root sandwich identity. The forward implication follows by multiplying the zero square-root sandwich by the square roots.

Let $P_{d_{\mathrm{cut}}}$ be the first d_{cut} coordinate projection. Monotonicity in (1) gives

$$\|D_{\mathrm{Sch}} - P_{d_{\mathrm{cut}}} D_{\mathrm{Sch}} P_{d_{\mathrm{cut}}}\| = d_{d_{\mathrm{cut}}+1} \leq \sqrt{\kappa/(d_{\mathrm{cut}}+1)}. \tag{2}$$

Consequently the family in (1) is compact in operator norm: take coordinatewise limits and use this common tail estimate; Fatou preserves the squared-sum bound. Finite families of POVMs on ℓ^2 are compact in the weak operator topology. Positivity and the finite sum-to-identity relations survive these limits.

Lemma 3.2 (Compact sandwiches)**.** *If $D_{\mathrm{Sch},j} \to D_{\mathrm{Sch}}$ in norm with compact D_{Sch}, and uniformly bounded E_j, F_j converge weakly to E, F, then $E_j D_{\mathrm{Sch},j} F_j$ converges weakly to $E D_{\mathrm{Sch}} F$.*

Proof. For fixed v, compactness makes $D_{\mathrm{Sch}} F_j v \to D_{\mathrm{Sch}} F v$ in norm. Pair this with the weakly convergent vectors $E_j^* u$. The replacement of D_{Sch} by $D_{\mathrm{Sch},j}$ has norm error tending to zero. □

3.2 An effective finite predicate

Let $\mathcal{L}_G$ be the losing tuples with $\pi(x,y) > 0$. Define $F_{\mathrm{qs}}(G, \kappa, d_{\mathrm{cut}})$ by the existence of d_{cut}-dimensional POVMs and real numbers $1 = d_1 \geq \cdots \geq d_{d_{\mathrm{cut}}} \geq 0$, $\sum_i d_i^2 \leq \kappa$, such that

$$|(E_{a|x} D_{\mathrm{Sch},d_{\mathrm{cut}}} F_{b|y}^{\mathsf{T}})_{ij}|^2 \leq \kappa/(d_{\mathrm{cut}}+1) \quad ((x,y,a,b) \in \mathcal{L}_G;\ i,j \leq d_{\mathrm{cut}}). \tag{3}$$

This is semialgebraic over the rationals. Splitting complex entries into real and imaginary parts and using principal minors for positivity makes real quantifier elimination a decision procedure [3].

A perfect strategy with $s_1^2 \geq 1/\kappa$ supplies all these witnesses. Compress its POVMs. Since $P_{d_{\mathrm{cut}}}$ commutes with D_{Sch} and $E D_{\mathrm{Sch}} F^{\mathsf{T}} = 0$, the finite sandwich is $-P_{d_{\mathrm{cut}}} E (D_{\mathrm{Sch}} - D_{\mathrm{Sch},d_{\mathrm{cut}}}) F^{\mathsf{T}} P_{d_{\mathrm{cut}}}$, with norm bounded by (2).

Conversely choose a witness for every d_{cut}. Extend $D_{\mathrm{Sch},d_{\mathrm{cut}}}$ by zero and each POVM by a fixed deterministic POVM on the complement. Pass to a subsequence with $D_{\mathrm{Sch},d_{\mathrm{cut}}} \to D_{\mathrm{Sch}}$ in norm and all effects weakly convergent. For every fixed matrix entry, (3) and Lemma 3.2 give $E D_{\mathrm{Sch}} F^{\mathsf{T}} = 0$. We retain

$$1 \leq \operatorname{Tr} D_{\mathrm{Sch}}^2 \leq \kappa, \qquad d_1 = 1.$$

The normalized vectorization of D_{Sch} is a perfect normal strategy with largest Schmidt probability $1/\operatorname{Tr} D_{\mathrm{Sch}}^2 \geq 1/\kappa$. Therefore

$$G \in \mathrm{PERF}_{\mathrm{qs}}(\kappa) \iff \forall d_{\mathrm{cut}}\ F_{\mathrm{qs}}(G, \kappa, d_{\mathrm{cut}}), \qquad \mathrm{PERF}_{\mathrm{qs}} = \bigcup_{\kappa \geq 1} \mathrm{PERF}_{\mathrm{qs}}(\kappa). \tag{4}$$

This proves the upper membership in Theorems A and 3.1.

3.3 No effective choice of stratum; vanishing largest weight

If a total computable $\kappa(G)$ were sufficient for every YES game, (4) would put spatial perfection in Π^0_1. Perfect completeness of [32, Theorem 12.6] already makes it Σ^0_1-hard. NO instances of that map have spatial value at most $1/2$, because the finite-dimensional and spatial suprema agree. Thus the same map reduces the halting set to spatial perfection. The halting set cannot be Π^0_1, so such a function does not exist.

Now suppose pure spatial strategies approach perfect play while $p_{\max} \geq 1/\kappa$ on a subsequence. Normalize their largest Schmidt amplitudes. For a losing event,

$$\|ED_{\mathrm{Sch}}F^{\mathsf{T}}\|_{\mathrm{HS}} \leq \|\sqrt{E}D_{\mathrm{Sch}}\sqrt{F^{\mathsf{T}}}\|_{\mathrm{HS}} = \sqrt{\operatorname{Tr} D^2_{\mathrm{Sch}}\, p(E,F)}.$$

Positive question weights force every such probability to zero. The same compact limit gives a perfect spatial strategy. Contraposition proves the final assertion of Theorem 3.1, including sequences of finite-dimensional pure strategies. Equivalently their Schmidt min-entropy tends to infinity.

3.4 Entropy budgets

Fix a finite Bell scenario and let K_H be its correlations realized by normal spatial strategies with $S(\rho_A) \leq H$, for $0 \leq H < \infty$. Purify onto Bob, the unconstrained party: Alice's marginal and every correlation are preserved. Thus pure strategies suffice, with the same budget, for entire correlation tables as well as winning probabilities.

Proposition 3.3 (Entropy-budget compactness)**.** *The set K_H is compact. It is an effectively closed (Π^0_1) subset of $[0,1]^t$ uniformly in the scenario and H: for rational budgets this is an ordinary effective presentation; for real budgets it is uniform relative to a name of H.*

Proof. For sorted Schmidt probabilities p_i, $p_i \leq 1/i$, so

$$\delta_n = \sum_{i>n} p_i \leq H/\log_2(n+1). \tag{5}$$

This gives tightness in ℓ^1, and finite partial entropy sums give lower semicontinuity. Hence the sorted distributions of entropy at most H are compact in ℓ^1. Their Schmidt vectors are compact in ℓ^2, since $\|\sqrt{p} - \sqrt{q}\|_2^2 \leq \|p - q\|_1$. Compress the POVMs to the Schmidt supports and absorb the basis changes into them, obtaining common copies of ℓ^2 (if the Schmidt rank is finite, extend each POVM by a fixed deterministic answer on the complement in ℓ^2). Finite POVM tuples are weak-operator sequentially compact there. Tensor products of convergent effects converge weakly, first on simple tensors and then by boundedness on all vectors. The norm-convergent Schmidt vectors therefore give convergence of every correlation entry. This is compactness of *correlations*; the original states on fixed local spaces need not form a compact set. This argument is the bounded-entropy compactness of [23, Section 9.1, "Divergence at bounded entropy"].

Here are explicit outer approximations. Fix an integer $\bar{H} \geq 1$ and work with $0 \leq H \leq \bar{H}$. Choose positive rational $\eta_{\bar{H},n} > \sqrt{\bar{H}/\log_2(n+1)}$ tending to zero. Compute rational polynomials h_n with

$$\sup_{p \in \mathsf{S}_n} |h_n(p) - S(p)| \leq 1/n, \qquad \mathsf{S}_n = \{p_i \geq 0 : \textstyle\sum_i p_i = 1\}.$$

For example, Bernstein approximation to $-t\log_2 t$ on $[0,1]$, with rationally rounded coefficients and error at most n^{-2} per summand, gives these polynomials. Let $A^{\bar{H}}_{H,n}$ consist of tables $c \in [0,1]^t$ (t is the number of entries) for which there are sorted $p \in \mathsf{S}_n$, nonnegative s_i

with $s_i^2 = p_i$, and n-dimensional POVMs (extended by a fixed deterministic answer on the complement in ℓ^2) satisfying

$$h_n(p) \le H + 1/n, \qquad |c_{ab|xy} - \langle \psi_s, E_{a|x} \otimes F_{b|y} \psi_s \rangle| \le \eta_{\bar{H},n}, \quad \psi_s = \sum_i s_i |i,i\rangle. \tag{6}$$

Split complex entries into real and imaginary parts, impose positivity by principal minors, and impose the POVM sums exactly. These are compact semialgebraic outer sets, uniformly decidable by real quantifier elimination [3, Chapter 14] for rational input data. The joint sets in (H, c) are semialgebraic over the rationals as well.

The normalized first-n truncation majorizes p, so its entropy does not increase. Its pure-state trace distance from the original state is $\sqrt{\delta_n}$; compressing the POVMs thus changes each entry by at most this amount. Consequently $K_H \subseteq A^{\bar{H}}_{H,n}$ for every n. Conversely witnesses in (6) have entropy at most $H + 2/n$. Along any sequence $n \to \infty$, the preceding tail and POVM argument gives a subsequential limit with entropy at most H, and the entrywise slack vanishes. Hence

$$K_H = \bigcap_{n \ge 1} A^{\bar{H}}_{H,n}.$$

Compact semialgebraic sets with rational data are effectively closed; the displayed intersection proves the uniform Π^0_1 presentation. It also gives a joint effectively closed set on $[0, \bar{H}] \times [0,1]^t$, which explains the assertion for real H. □

Corollary 3.4 (Exact tables and the vanishing-loss entropy limit)**.** *Membership of a prescribed rational correlation table in the class realizable with some finite marginal entropy is in Σ^0_2, uniformly over finite scenarios. The same holds for real-algebraic tables. For a finite rational game G, set*

$$E_G(\varepsilon) = \inf\{S(\rho_A) : \omega(G, \mathcal{S}) \ge 1 - \varepsilon\}, \qquad H_{\mathrm{ex}}(G) = \inf\{S(\rho_A) : \omega(G, \mathcal{S}) = 1\},$$

where the infima range over normal spatial strategies and $\inf \varnothing = \infty$*. In particular $E_G(\varepsilon) = \infty$ when $\varepsilon < 1 - \omega_q(G)$. Then*

$$\lim_{\varepsilon \downarrow 0} E_G(\varepsilon) = H_{\mathrm{ex}}(G), \qquad G \in \mathrm{PERF}_{\mathrm{finS}} \iff \sup_{\varepsilon > 0} E_G(\varepsilon) < \infty.$$

If $H_{\mathrm{ex}}(G)$ is finite, a perfect strategy attains it.

Proof. Exact membership in K_m is uniformly Π^0_1 for integer $m \ge 0$; take the union over m. The perfect-value constraint is closed too, so the games $\mathrm{PERF}_{\mathrm{finS}}(m)$ with perfect play of entropy at most m form a uniform Π^0_1 stratum, and $\mathrm{PERF}_{\mathrm{finS}} \in \Sigma^0_2$. Purification preserves the supplied marginal entropy, and a pure state's two marginal entropies agree.

The increasing limit $L = \lim_{\varepsilon \downarrow 0} E_G(\varepsilon)$ is at most $H_{\mathrm{ex}}(G)$. If $L < \infty$, choose strategies of loss at most $1/j$ and entropy at most $L + 1/j$. Proposition 3.3, with its lower-semicontinuity argument, gives a perfect limit of entropy at most L. Thus $H_{\mathrm{ex}}(G) \le L$, and equality follows, also in the extended reals. A sequence of perfect strategies whose entropies decrease to a finite $H_{\mathrm{ex}}(G)$ gives attainment by the same argument. □

Lemma 3.5 (Entropy of clock witnesses)**.** *Suppose $e \in \mathcal{A}_{\mathrm{dec}}$, and let j_* be the least witness to $\forall t\, \mathcal{R}_{\mathrm{dec}}(e, j, t)$; if $e \notin \mathcal{A}_{\mathrm{dec}}$, Theorem 7.1 gives no perfect strategy. Every perfect normal spatial strategy for G^{clk}_e has full Alice-marginal entropy at least $\log_2 p_{j_*+1}$.*

Proof. This lemma and the next corollary use the hardness construction of Sections 4–7, which does not use them. Purify onto Bob as in (N1) of Section 4; this preserves Alice's full marginal h and all correlations. Work on its support. Lemma 4.5 extracts a faithful normalized source

density $h_0 = 121e_0he_0$. By (32), $h_0 = h_F \otimes \rho_{\mathrm{spec}}$. Every positive eigenspace of ρ_{spec} splits into Z_{clk} phase sectors. The soundness proof of Theorem 7.1 applies to each nonzero sector: its word descendants persist at every depth, so an invalid order or a false check would contradict $P_{\mathtt{H}} = 0$. Thus each phase order is p_{j+1} for a valid witness $j \geq j_*$. Equation (41) makes the dimension of that sector divisible by its order. Consequently every positive eigenvalue of ρ_{spec} has multiplicity at least p_{j_*+1}, and hence is at most $1/p_{j_*+1}$. Therefore $S(\rho_{\mathrm{spec}}) \geq \log_2 p_{j_*+1}$.

To pass to the full marginal, use the entire synchronized scaffold cover, rather than an entropy inequality for a compression. Set $v_0 = e_0$ and $v_g = \mathsf{U}_g \cdots \mathsf{U}_1$ for $1 \leq g \leq 4$. Let $t_{g,ab} = \mathfrak{x}_g^a \mathfrak{z}_g^b e_g$ be the synchronized magic matrix units, and put $T_{g,ab} = t_{g,ab} v_g$. These partial isometries all have initial projection e_0 and final projections $m_{g,ab}$, respectively. Together with e_0, their final projections are orthogonal and sum to the identity on the full support. Moreover $ht_{g,ab} = t_{g,ab}h$ and $hv_g = 2^g v_g h$, so

$$T_{g,ab}^* h T_{g,ab} = 2^g e_0 h e_0.$$

Write $T_0 = e_0$. Since each corner commutes with h and the corners are orthogonal, $T_k^* h T_l = 0$ for distinct members $k \neq l$ of the seventeen-element list $(T_0; T_{g,ab})$. Hence $U(\delta_k \otimes \xi) = T_k \xi$ is a unitary from $\mathbb{C}^{17} \otimes e_0 P_A \mathcal{H}_A$ onto $P_A \mathcal{H}_A$, and by (23)

$$h \simeq \mathrm{diag}(\mu_4) \otimes h_0, \qquad \mu_4 = 121^{-1}(1; 2^{\oplus 4}; 4^{\oplus 4}; 8^{\oplus 4}; 16^{\oplus 4}).$$

Here h_0 is the density of the extracted realization on the whole source corner $e_0 P_A \mathcal{H}_A$; by (31) any further tensor factor present there lies in $\mathcal{K}$ and is counted in ρ_{spec}. Nonnegative spectral sums now give, also when an entropy is infinite,

$$S(h) = S(\mathrm{diag}(\mu_4)) + S(h_F) + S(\rho_{\mathrm{spec}}) \geq \log_2 p_{j_*+1}.$$

No monotonicity under compression is used. □

Corollary 3.6 (The entropy busy beaver)**.** *Fix a binary encoding of finite rational games, with decidable validity, and write $|G|$ for its bit length. Put*

$$m_*(G) = \min\{m \in \mathbb{N} : G \in \mathrm{PERF}_{\mathrm{finS}}(m)\},$$
$$B(n) = \max(\{m_*(G) : |G| \leq n,\ m_*(G) < \infty\} \cup \{0\}),$$

where $\mathbb{N}$ includes zero and an empty minimum is infinity. For finite $m_*(G)$, attainment gives $m_*(G) = \lceil H_{\mathrm{ex}}(G) \rceil$. There is no total $0'$-computable upper bound on B, and

$$B \equiv_T 0'', \qquad B \oplus 0' \equiv_T 0''.$$

Moreover B eventually strictly dominates every total $0'$-computable function, and hence exceeds each such function infinitely often. This is the profile of Radó's busy beaver function [50] relativized to $0'$, the "super Busy Beaver" BB_1 of [1, Section 3], with Beeping Busy Beavers giving an earlier natural example of this growth [1, Section 5.10]; the content of the corollary is that a resource of perfect play realizes it.

Proof. The fixed-budget test is uniformly Π_1^0, hence decidable in $0'$; at budget zero it reduces to the decidable product-state test. If a total $0'$-computable $b(n)$ bounded $B(n)$, testing the single budget $b(|G|)$ would decide $\mathrm{PERF}_{\mathrm{finS}}$ in $0'$, contradicting its Σ_2^0-completeness in Theorem A. Using $0''$, decide which of the finitely many games of size at most n lie in $\mathrm{PERF}_{\mathrm{finS}}$, then search their budgets using $0'$ to compute $B(n)$. Conversely $B \oplus 0'$ decides $\mathrm{PERF}_{\mathrm{finS}}$ by testing budget $B(|G|)$, so computes $0''$. Finally, an eventual upper bound could be patched at finitely many inputs to a total $0'$-computable upper bound, which has just been excluded.

For the reverse reduction $0' \leq_T B$, use the clock games for the decidable predicate "the eth machine on input e halts within j steps." If it halts at time T_e, the valid witnesses are exactly $j \geq T_e$, and Lemma 3.5 gives

$$T_e < p_{T_e+1} \leq 2^{m_*(G_e^{\mathrm{clk}})} \leq 2^{B(|G_e^{\mathrm{clk}}|)}.$$

Thus simulating $2^{B(|G_e^{\mathrm{clk}}|)}$ steps decides the halting set from B: a computation that has not halted by then never halts. Combining this with $B \oplus 0' \equiv_T 0''$ proves $B \equiv_T 0''$.

For eventual domination, fix a computable array $g(a, n, s)$ obtained by finite-stage simulations of oracle machines with the standard finite approximations to $0'$. Every total $0'$-computable function occurs as $n \mapsto \lim_s g(a, n, s)$ for some a. Compile the decidable predicates

$$\mathcal{R}_{\mathrm{dec}}(a, n, \langle s_0, v\rangle, t) \iff \forall s \in [s_0, t]\ g(a, n, s) = v$$

into clock games $G_{a,n}$, with hardwired input $e = \langle a, n\rangle$, using a computable pairing with $\langle s_0, v\rangle \geq v$. Pad by harmless always-accepted questions so that $|G_{a,n}| \geq n + 1$. A perfect strategy for a padded game restricts to one for the unpadded game with the same state, and conversely extends by arbitrary answers; padding therefore changes neither h nor the bound of Lemma 3.5. Define the computable, nondecreasing, unbounded size envelope

$$q(n) = \max_{a,k \leq n} |G_{a,k}|.$$

For a given total $0'$-computable $F : \mathbb{N} \to \mathbb{N}$, the limit lemma [51] supplies an index a_0 with

$$\lim_s g(a_0, n, s) = f(n), \qquad f(n) = 2^{1+\max_{m \leq q(n+1)} F(m)}.$$

The envelope was defined using all indices before selecting a_0, so there is no circular size definition. Each $G_{a_0,n}$ is a YES instance, and every witness index is at least $f(n)$. For $n \geq a_0$ and $q(n) \leq m < q(n + 1)$, Lemma 3.5 therefore gives

$$B(m) \geq m_*(G_{a_0,n}) \geq \log_2 p_{f(n)+1} > \log_2 f(n) > F(m).$$

The intervals cover every sufficiently large integer since q is nondecreasing and unbounded. This proves eventual strict domination. □

Remark (Finite strata and jump bounds). Let $n \geq 2$ and suppose a Σ^0_n-complete attainment language has $e \in L \iff \exists m\, R(e, m)$ with R uniformly Π^0_{n-1}. No total $0^{(n-2)}$-computable function bounds a sufficient stratum index on all YES inputs; in fact no total $0^{(n-1)}$-computable function does. Indeed the latter oracle decides each stratum test, so a bounded finite search would put L in Δ^0_n, contradicting completeness. The entropy busy beaver is the case $n = 2$. The Schmidt resource has the full profile as well.

Corollary 3.7 (The Schmidt busy beaver)**.** *For the same encoding, put*

$$\kappa_{\min}(G) = \min\{\kappa \geq 1 : G \in \mathrm{PERF}_{\mathrm{qs}}(\kappa)\},$$
$$K(n) = \max(\{\kappa_{\min}(G) : |G| \leq n,\ G \in \mathrm{PERF}_{\mathrm{qs}}\} \cup \{0\}).$$

An empty minimum is infinity. For each YES clock game,

$$\kappa_{\min}(G_e^{\mathrm{clk}}) \geq \frac{121}{12} p_{j_*+1}.$$

Moreover $K \equiv_T 0''$, and K eventually strictly dominates every total $0'$-computable function.

Proof. For every perfect pure strategy, the full-marginal equivalence in Lemma 3.5 and its spectral-multiplicity argument give

$$\|h\| \le \|\operatorname{diag}(\mu_4)\|\,\|h_F\|\,\|\rho_{\mathrm{spec}}\| \le \frac{16}{121}\frac{3}{4}\frac{1}{p_{j_*+1}} = \frac{12}{121 p_{j_*+1}}.$$

The largest Schmidt probability is $\|h\|$. A strategy in stratum κ therefore requires $\kappa \ge (121/12)p_{j_*+1}$, proving the bound. Every spatially perfect game lies in some finite stratum by Theorem 3.1. Using $0''$, decide which of the finitely many valid games of length at most n are spatially perfect, then use $0'$ to test their uniform Π^0_1 strata and search for each least one. This computes K. Conversely $K \oplus 0'$ decides spatial perfection by testing the single stratum $K(|G|)$ (or stratum one if that value is zero), and hence computes $0''$ by Theorem A.

To remove the extra $0'$ oracle, use the halting clock games from the proof of Corollary 3.6. If machine e halts at time T_e, its least valid witness is T_e, so

$$T_e < p_{T_e+1} \le \frac{12}{121}\kappa_{\min}(G_e^{\mathrm{clk}}) \le \frac{12}{121}K(|G_e^{\mathrm{clk}}|) \le K(|G_e^{\mathrm{clk}}|).$$

Simulating $K(|G_e^{\mathrm{clk}}|)$ steps therefore decides whether it halts: if it has not halted by then, it never halts. Thus $0' \le_T K$, and the preceding reductions give $K \equiv_T 0''$.

Finally use the same computable array, padded games and nondecreasing unbounded envelope q as in Corollary 3.6; padding preserves the state and hence the Schmidt probabilities. Given total $0'$-computable F, the limit lemma supplies a_0 for $f(n) = 2^{1+\max_{r\le q(n+1)} F(r)}$. Every witness for $G_{a_0,n}$ is at least $f(n)$. For $n \ge a_0$ and $q(n) \le m < q(n+1)$,

$$K(m) \ge \kappa_{\min}(G_{a_0,n}) \ge \frac{121}{12}p_{f(n)+1} > f(n) > F(m).$$

The intervals cover all sufficiently large integers, proving eventual strict domination. □

Remark (Least entropy and the least witness). For a YES clock game, let $d_{\max}(e)$ be the computable maximum row ancilla dimension in its compiler. Then

$$H_{\mathrm{ex}}(G_e^{\mathrm{clk}}) = \log_2 p_{j_*+1} + H(\mu_4) + S(h_F) + \delta_e, \qquad 0 \le \delta_e \le \log_2(3d_{\max}(e)).$$

Indeed the full-marginal identity and multiplicity bound in Lemma 3.5 give the lower bound $S(h) \ge H(\mu_4)+S(h_F)+\log_2 p_{j_*+1}$ for every perfect normal strategy, including mixed strategies by purification onto Bob. For the upper bound choose the least witness in the explicit pure strategy of Theorem 7.1: its marginal is the tensor product of $\operatorname{diag}(\mu_4)$, h_F, $I_{p_{j_*+1}}/p_{j_*+1}$, $I_3/3$ and $I_{d_{\max}(e)}/d_{\max}(e)$. Entropy additivity gives exactly the same lower-bound expression plus $\log_2(3d_{\max}(e))$. Taking the infimum proves the display. In particular the least entropy determines $\log_2 p_{j_*+1}$ up to a computable additive error.

3.5 A finite tracial corner

The same quantifier pattern governs every semifinite host. Here the existential integer bounds the overlap of the state with a finite tracial corner.

Lemma 3.8 (Finite-corner overlap)**.** *Let $\mathcal{A}, \mathcal{B}$ be commuting unital C^*-algebras containing a game's effects. Suppose unit vectors ψ, η and $e = e^2 = e^* \in \mathcal{A}$ satisfy: ψ is perfect, $e\eta = \eta$, $\langle \eta, \psi\rangle \ne 0$, and the vector state of η is tracial on $e\mathcal{A}e$. Then the game has a perfect normal semifinite strategy. If $e = 1$, it has a finite-algebra strategy.*

Proof. Let $\mathcal{H}_\eta = \overline{\operatorname{span} \mathcal{AB}\eta}$. It reduces both algebras because they are self-adjoint and commute. Put $M = (\mathcal{A}|_{\mathcal{H}_\eta})''$ and $p = e|_{\mathcal{H}_\eta}$. The vector functional of η is normal on M and tracial on pMp: Kaplansky density approximates each corner element by a bounded net from $p\mathcal{A}p$, and separate ultraweak continuity extends $\tau_\eta(xy) = \tau_\eta(yx)$ one variable at a time. Moreover

$$p\mathcal{H}_\eta = \overline{\operatorname{span} \mathcal{B}(pMp)\eta}. \tag{7}$$

Indeed $pab\eta = bpap\eta$, and bounded strong approximation supplies the reverse closure. If $x \geq 0$ in pMp has $\tau_\eta(x) = 0$, then, for $a \in pMp$,

$$\|x^{1/2}a\eta\|^2 = \tau_\eta(a^*xa) = \tau_\eta(x^{1/2}aa^*x^{1/2}) \leq \|a\|^2\tau_\eta(x) = 0.$$

Since x commutes with $\mathcal{B}$, (7) gives $x = 0$. Thus pMp has a faithful normal finite trace. If a central projection z of M annihilates p, it annihilates η, and hence $\mathcal{AB}\eta$ and $\mathcal{H}_\eta$; so $z = 0$. Thus p has full central support. A type-III summand has no nonzero finite projection, so its intersection with p is zero and fullness excludes that summand. Consequently M is semifinite.

The orthogonal projection onto $\mathcal{H}_\eta$ commutes with both parties. $\zeta = P_{\mathcal{H}_\eta}\psi$ is nonzero by its overlap with η. Each losing positive product annihilates ψ and hence ζ. Normalize ζ. Its state is normal on M and Bob's compressed effects commute with M. If $e = 1$, the faithful finite trace is on M. □

Conversely standardize a normal semifinite strategy and write $\psi = h^{1/2} \in L^2(M, \tau)$, with $\tau(h) = 1$. Some band $e = 1_{[a,b]}(h)$, $0 < a < b < \infty$, is nonzero. It has $0 < \tau(e) \leq 1/a$. For $\eta = e/\sqrt{\tau(e)}$,

$$\langle \eta, \psi \rangle = \frac{\tau(eh^{1/2}e)}{\sqrt{\tau(e)}} \geq \sqrt{a}\sqrt{\tau(e)} > 0. \tag{8}$$

Its corner state is tracial. In a finite algebra take a faithful normalized trace and $e = 1$; then $\tau(h^{1/2}) > 0$. Adjoin e to Alice's algebra if necessary.

Theorem 3.9. $\mathrm{PERF}_{\mathrm{fin}}$ *and* $\mathrm{PERF}_{\mathrm{sf}}$ *belong to* Σ_2^0.

Proof. For $k \geq 1$ seek moments $m_{ij}(w) = \langle \xi_i, w\xi_j \rangle$, where $\xi_0 = \psi, \xi_1 = \eta$. Letters are the self-adjoint effects and Alice's extra projection e. Impose positivity for all polynomial vector lists, localizing positivity for each effect and its complement, POVM normalization, Alice–Bob commutation *including* e, $e^2 = e$, $e\eta = \eta$, unit norms, zero losing expectations on ψ, and

$$\mathrm{Re}\langle \eta, \psi \rangle \geq 1/k, \qquad \langle \eta, eue\, eve\, \eta \rangle = \langle \eta, eve\, eue\, \eta \rangle$$

for all Alice words u, v, including e. A vector equality means vanishing squared norm. These form a recursive list of closed rational polynomial constraints. Impose $|m_{ij}(w)| \leq 1$ directly. All genuine contraction moments obey this bound, and the ambient product of closed disks is compact. Localizing positivity gives bounded GNS effects $0 \leq E \leq 1$. One can also use the certificate $E - E^2 = (I - E)E(I - E) + E(I - E)E$.

At stage m take the first m constraints and their coordinates. Feasibility $F_{\mathrm{sf}}(G, k, m)$ is a non-strict rational semialgebraic question, decidable by real quantifier elimination [3]. Its solution set is a closed cylinder in the compact product of all moment disks. The cylinders are nested because stages retain previous constraints. If every stage is feasible, their intersection is nonempty: otherwise their open complements have a finite subcover, already making one stage empty. A point of the intersection gives a positive matrix-valued functional. Quotient polynomial pairs by its null space and complete. Left multiplication yields the bounded GNS effects and the two vectors. Relations hold on the polynomial dense space and extend by

boundedness. Corner traciality holds on the linear span of eue and then in norm on $e\mathcal{A}e$. Apply Lemma 3.8. Conversely (8) supplies some k for every semifinite strategy. Thus

$$G \in \mathrm{PERF}_{\mathrm{sf}} \iff \exists k\, \forall m\, F_{\mathrm{sf}}(G, k, m). \tag{9}$$

Set $e = 1$ for the finite-algebra version. □

Tensoring a finite factor host with R_{hyp} gives a II_1 host. Tensoring a semifinite factor host with $R_{\mathrm{hyp}}\bar{\otimes}B(\ell^2)$ gives a II_∞ host. Product-state expectations, or standard-form tensoring, preserve the probabilities. Factor decomposition as in Section 2 therefore gives

$$\mathrm{PERF}_{\mathrm{fin}} = \mathrm{PERF}(\mathrm{II}_1), \qquad \mathrm{PERF}_{\mathrm{sf}} = \mathrm{PERF}(\mathrm{II}) = \mathrm{PERF}(\mathrm{II}_\infty). \tag{10}$$

These are equalities of existence languages.

Remark (One parameter for both corners). In a type I factor $B(\mathcal{H})$ with its trace, the state vector is $\psi = h^{1/2}$. If e is a rank-one subprojection of the spectral projection of h for its largest eigenvalue, then $\langle e, \psi\rangle = \mathrm{Tr}(eh^{1/2}) = s_1$ is the largest Schmidt amplitude. This rank-one choice recovers the largest Schmidt amplitude from the corner overlap. General finite-corner overlaps need not equal it, and the parameters κ and k need not agree. Both upper bounds put one existential quantifier on that amount and one universal quantifier on finite stages. Without a trace there is no such corner. Commuting perfection is in Π_1^0 by the convergent hierarchy of Navascués, Pironio and Acín [43]: a NO instance eventually has an infeasible perfect-value relaxation, and feasibility at every order gives a commuting strategy by compactness. Section 12 shows that every type-III subtype has exactly this commuting language.

4 A finite compiler for normal strategies

The spatial, semifinite and communication hardness proofs compile operator relations into one finite game. This section builds that compiler once. It works for normal spatial strategies and for normal commuting strategies; the finite predicates are the same in both models, and the proofs say where separation in standard form replaces Schmidt-range cancellation.

The modular operators below are not synchronized. A synchronized operator commutes with the marginal density, so it cannot itself change a density eigenvalue. We retain conditional flagged magic-square questions to supply that change. Everything submitted to the graph compiler lies in the density centralizer.

We first restate the bridge, label and rational-row tools from [23, Lemmas 4.1 and 4.2, Theorem B, Eq. (32)], with their proofs, and extend them from spatial strategies to normal commuting strategies. These arguments concern arbitrary normal supports and finite word lists. We then give the n-bridge assembly. No product of two modular transports is submitted as a centralizer word.

From a presentation to a game

This overview says what the compiler does. The proofs follow in Sections 4.1–4.7.

Input. An admissible presentation (Section 4.4) is a finite rational list: four contraction symbols A_α, a root projection $P_\circ$, finitely many typed centralizer labels and finitely many rational $*$-polynomial equations. Its intended solutions are normal realizations: bounded operators on a Hilbert space $\mathcal{H}_0$ with a faithful trace-class density h_0, such that the labels commute with h_0, the equations hold, and $h_0 A_\alpha = A_\alpha h_0/16$.

Output. A finite game with a rational question distribution and a 0/1 predicate. Its perfect normal spatial strategies and the normal realizations of the presentation correspond in both directions (Lemma 4.5).

The pipeline.

1. *Labels become unitaries.* A projection P is encoded by $2P - I$, a positive contraction T_+ by a unitary u_+ with $T_+ = (2I + u_+ + u_+^*)/4$, and a general contraction by an average of two unitaries.
2. *A scaffold carries the factor* 16. Seventeen synchronized corners and four conditional flagged magic-square bridges certify partial isometries U_g with $h\mathsf{U}_g = 2\mathsf{U}_g h$ (Lemma 4.1). Their product v maps the source corner e_0 onto e_4 with $hv = 16vh$, and the cover fixes the source mass $1/121$.
3. *Labels travel through the bridges.* Corner copies, block unitaries and shared-question refinements (Lemma 4.2) force $L_4 v = vL_0$ for every transported label L.
4. *The modular operators disappear.* Each creation symbol is written $A_\alpha = v^* a_\alpha$ with a centralizer contraction a_α from e_0 to e_4. Every application equation becomes an equation among centralizer labels in which v does not occur.
5. *Words become projections.* A qutrit frame, graph projections and triangle projections certify actual products of unitary labels. A rational equation among words becomes a rational linear equation among synchronized projections (Section 4.2).
6. *Linear equations become Boolean tests.* Each such equation is enforced by finitely many orthogonality tests against a certified scalar matrix algebra (Lemma 4.3).
7. *Assembly.* All questions are kept, the question sets are padded, and question pairs are sampled uniformly; unspecified pairs are accepted.

Soundness. In a perfect normal strategy every losing event of positive weight has probability zero. This makes each synchronized event a projection on the state support that commutes with the marginal density, and it turns each submitted equation into an operator identity on that support. The bridges then give the realization on the source corner, with density $h_0 = 121 e_0 h e_0$. No step inverts a density.

Completeness. A realization is copied into the scaffold density (27). Contractions are written as averages of unitaries inside the finite-dimensional positive eigenspaces of that density. A qutrit frame and a finite maximally entangled row ancilla are appended, and Bob uses transposes in an eigenbasis of the density. Every test then holds exactly. Only finite-dimensional factors are added, so finite entropy is preserved.

What was compiled. Section 4.7 bounds the loss when the application equations hold only approximately, and Section 7 bounds the size of the reduction game. The one compiled instance, described in Appendix C, is minimal: four bridges, one modular isometry and no transported labels. A full reduction instance has been dry-counted, but no full Boolean game was materialized; Section 7 states the two counts and their encoding conventions.

Normal states, supports and exact cancellation

The proofs in this section use only the following facts. They apply to arbitrary, possibly nonseparable, Hilbert spaces. None of them uses a dimension count, a determinant or a bounded inverse of a density.

(N1) Normal states and purifications. A normal state on $\mathcal{H}_A \otimes \mathcal{H}_B$ is a positive trace-class operator ρ of trace one. By the spectral theorem for compact operators, $\rho = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$ with countably many $\lambda_j > 0$. Let J be the countable index set, put $\mathcal{H}_{\mathrm{env}} = \ell^2(J)$, and put $\Omega = \sum_j \lambda_j^{1/2} \psi_j \otimes |j\rangle$. Then

$$\operatorname{Tr} \rho(E \otimes F) = \langle \Omega, (E \otimes F \otimes I_{\mathrm{env}})\Omega \rangle$$

for all bounded E, F, and no measurement acts on $\mathcal{H}_{\mathrm{env}}$. The marginals $h_A = \mathrm{Tr}_B\,\rho$ and $h_B = \mathrm{Tr}_A\,\rho$ are positive trace-class operators of trace one. Write P_A, P_B for their support projections, the projections onto the closures of their ranges; these closures are separable. The restriction of h_A to $P_A\mathcal{H}_A$ is injective with dense range. Its positive eigenspaces are finite-dimensional and span a dense subspace of $P_A\mathcal{H}_A$. The same holds for h_B. If P_A has infinite rank, the inverse of h_A on its range is unbounded; it is never used.

(N2) Coefficient operators. Put $\mathcal{H}_{B,\mathrm{env}} = \mathcal{H}_B \otimes \mathcal{H}_{\mathrm{env}}$. Fix orthonormal bases (b_j) of $\mathcal{H}_B$ and (e_ℓ) of $\mathcal{H}_{\mathrm{env}}$, and use their tensor-product basis $(f_{j\ell}) = (b_j \otimes e_\ell)$ of $\mathcal{H}_{B,\mathrm{env}}$, and let $Y^{\mathsf{T}} = CY^*C$ be the transpose of $Y \in B(\mathcal{H}_{B,\mathrm{env}})$, where C is the complex conjugation fixing every $f_{j\ell}$. Bob's transpose F^{T} is defined in the basis (b_j). Transposition preserves norms, adjoints, positivity and projections, reverses products, and satisfies $(F \otimes I_{\mathrm{env}})^{\mathsf{T}} = F^{\mathsf{T}} \otimes I_{\mathrm{env}}$. The map $T \mapsto \sum_{j,\ell} T f_{j\ell} \otimes f_{j\ell}$ is a unitary from the Hilbert–Schmidt operators $\mathcal{H}_{B,\mathrm{env}} \to \mathcal{H}_A$ onto $\mathcal{H}_A \otimes \mathcal{H}_{B,\mathrm{env}}$. If it sends T_Ω to Ω, it sends $XT_\Omega Y^{\mathsf{T}}$ to $(X \otimes Y)\Omega$, and $T_\Omega T_\Omega^* = h_A$. We call T_Ω the coefficient operator. Identities such as $ET = TF^{\mathsf{T}}$ below are coefficient forms of vector identities, and we state both forms where it helps. For a mixed state this is the cut $A \mid B\,\mathrm{env}$, and every Bob operator enters as $F \otimes I_{\mathrm{env}}$.

(N3) Exact cancellation. For $X \in B(\mathcal{H}_A)$,

$$\|(X \otimes I)\Omega\|^2 = \mathrm{Tr}(h_A X^* X) = \|X h_A^{1/2}\|_{\mathrm{HS}}^2.$$

Hence $(X \otimes I)\Omega = 0$, equivalently $XT_\Omega = 0$, holds exactly when $XP_A = 0$. Indeed the range of $h_A^{1/2}$ is dense in $P_A\mathcal{H}_A$, and a bounded operator that vanishes on a dense subspace vanishes on its closure. Likewise $(I \otimes F \otimes I_{\mathrm{env}})\Omega = 0$, equivalently $T_\Omega(F \otimes I_{\mathrm{env}})^{\mathsf{T}} = 0$, holds exactly when $FP_B = 0$, because its squared norm is $\mathrm{Tr}(h_B F^* F)$. These are the dense-range cancellations used throughout. The same statements hold for any vector in place of Ω, with the supports of its own marginals.

(N4) Mirrors. Call $X \in B(\mathcal{H}_A)$ and $Y \in B(\mathcal{H}_B)$ *mirrors* if $(X \otimes I)\Omega = (I \otimes Y \otimes I_{\mathrm{env}})\Omega$, that is, $XT_\Omega = T_\Omega(Y \otimes I_{\mathrm{env}})^{\mathsf{T}}$.

(a) If X_k and Y_k are mirrors for $k = 1, \ldots, n$, then $X_1 \cdots X_n$ and $Y_n \cdots Y_1$ are mirrors. Operators on different tensor factors commute, so $(X_1X_2 \otimes I)\Omega = (X_1 \otimes Y_2 \otimes I)\Omega = (I \otimes Y_2Y_1 \otimes I)\Omega$; induct. Linear combinations of mirror pairs are mirror pairs.

(b) If X and X' are both mirrors of Y, then $(X - X')P_A = 0$ by (N3). If Y and Y' are both mirrors of X, then $(Y - Y')P_B = 0$.

(c) Suppose X, Y are mirrors and X^*, Y^* are mirrors. This holds for self-adjoint mirrors, and for unitary mirrors because applying $X^* \otimes Y^*$ to their mirror identity gives $(I \otimes Y^* \otimes I)\Omega = (X^* \otimes I)\Omega$. Then $Xh_A = h_AX$ and $Yh_B = h_BY$. For $Z \in B(\mathcal{H}_A)$,

$$\mathrm{Tr}(h_A ZX) = \langle (I \otimes Y^* \otimes I)\Omega, (Z \otimes I)\Omega \rangle = \langle (X^* \otimes I)\Omega, (Z \otimes I)\Omega \rangle = \mathrm{Tr}(h_A XZ).$$

Cyclicity of the trace, valid because h_A is trace class and X, Z are bounded, gives $\mathrm{Tr}((Xh_A - h_AX)Z) = 0$. Taking Z of rank one shows that the trace-class operator $Xh_A - h_AX$ is zero. Bob's statement is symmetric.

(d) An operator commuting with h_A commutes with P_A and with every spectral projection of h_A. In particular it maps each finite-dimensional positive eigenspace of h_A into itself.

(N5) Perfect matching. Let E be an Alice effect and F a Bob effect whose two disagreement events $E \otimes (I - F)$ and $(I - E) \otimes F$ have probability zero. By the matching identity, the sum

of these two probabilities is the expectation of

$$(E \otimes I - I \otimes F)^2 + (E - E^2) \otimes I + I \otimes (F - F^2),$$

a sum of three positive operators. Each expectation vanishes, so

$$(E \otimes I)\Omega = (I \otimes F \otimes I)\Omega, \qquad ((E - E^2)^{1/2} \otimes I)\Omega = 0, \qquad (I \otimes (F - F^2)^{1/2} \otimes I)\Omega = 0.$$

Thus E and F are mirrors. Next, E reduces P_A. Since $((I - P_A) \otimes I)\Omega = 0$,

$$((I - P_A)E \otimes I)\Omega = (I \otimes F \otimes I)((I - P_A) \otimes I)\Omega = 0,$$

so $(I - P_A)EP_A = 0$ by (N3), and self-adjointness gives $EP_A = P_AEP_A = P_AE$. Then (N3) gives $(E - E^2)P_A = 0$, so EP_A is a projection, and E commutes with h_A by (N4c). The same holds for F on Bob's side. We call EP_A a *synchronized projection* and mean this support-restricted operator in all local identities. Reduction of the support is proved before projectivity is used.

(N6) Compressions and joint measurements.

(a) If $0 \le q \le Z$ with Z a projection, then $q = ZqZ$. Indeed $0 \le (I - Z)q(I - Z) \le (I - Z)Z(I - Z) = 0$, so $q^{1/2}(I - Z) = 0$.

(b) *Projective compression.* If P is a projection, E an effect and PEP a projection, then E reduces $P\mathcal{H}$. For $u \in \mathrm{ran}(PEP)$, $\langle u, (I - E)u\rangle = 0$ and positivity give $Eu = u$. For $u \in P\mathcal{H}$ with $PEPu = 0$, $\langle u, Eu\rangle = 0$ gives $Eu = 0$. Thus E maps $P\mathcal{H}$ into itself, and self-adjointness gives reduction.

(c) *Only compressions matter.* Since $(P_A \otimes I)\Omega = \Omega = (I \otimes P_B \otimes I)\Omega$, every probability equals $\langle \Omega, (P_AEP_A \otimes P_BFP_B \otimes I)\Omega\rangle$. Identities may therefore be proved first for compressed effects. Changing effects outside the supports, while keeping them POVMs, changes no probability.

(d) *Sharp marginals.* Let (E_c) be a finite POVM indexed by tuples $c = (c_1, \dots, c_m)$ whose compressions to a reducing subspace $P\mathcal{H}$ have projections as coarse marginals $Q^k_s = \sum_{c:\, c_k = s} E_c$. By (a), each compressed E_c satisfies $E_c = Q^k_{c_k} E_c Q^k_{c_k}$, hence $E_c Q^k_s = \delta_{s,c_k} E_c$. Summing, $Q^1_{s_1} Q^2_{s_2} = \sum_{c_1 = s_1,\, c_2 = s_2} E_c$, which is self-adjoint, so the marginals commute. Iterating, $Q^1_{s_1} \cdots Q^m_{s_m}$ is the compressed effect of the outcome $(s_1, \dots, s_m)$, or zero if there is no such outcome. Each compressed fine effect is thus a product of commuting projections, hence a projection, and (b) makes the original fine effect reduce $P\mathcal{H}$.

(N7) Forbidden events. Let Q be an Alice effect and R a Bob effect whose compression $R_B = P_BRP_B$ is a synchronized projection with Alice mirror R_A. If the event $Q \otimes R$ has probability zero, then $QR_AP_A = 0$. If Q also reduces P_A, then $R_AQP_A = 0$ as well. Indeed $0 = \langle \Omega, (Q \otimes R_B \otimes I)\Omega\rangle = \|(Q^{1/2} \otimes R_B \otimes I)\Omega\|^2$, and $(Q^{1/2} \otimes R_B \otimes I)\Omega = (Q^{1/2}R_A \otimes I)\Omega$, so (N3) gives $Q^{1/2}R_AP_A = 0$. The second claim is the adjoint of the first, because P_A commutes with Q and R_A. No commutation with a density is used.

(N8) Canonical purifications for completeness. Let h be a faithful positive trace-class operator of trace one on $\mathcal{H}$. By (N1), $\mathcal{H}$ is separable and has an orthonormal basis (φ_i) with $h\varphi_i = \lambda_i\varphi_i$ and $\lambda_i > 0$. Put $\Omega_h = \sum_i \lambda_i^{1/2} \varphi_i \otimes \varphi_i$, and let X^{T} denote the transpose in (φ_i). Both marginals of Ω_h equal h. If X commutes with h, it commutes with $h^{1/2}$, and the rule of (N2) gives

$$(I \otimes X^{\mathsf{T}})\Omega_h = (X \otimes I)\Omega_h,$$

since both vectors correspond to $h^{1/2}X = Xh^{1/2}$. The map $X \mapsto X^{\mathsf{T}}$ is linear, reverses products, and preserves adjoints, positivity and projections; and X^{T} commutes with $h^{\mathsf{T}} = h$.

Hence a centralizer projection X for Alice and X^{T} for Bob are perfectly matched: their disagreement probability is $\langle \Omega_h, ((X(I-X)+(I-X)X)\otimes I)\Omega_h\rangle = 0$. For centralizer effects Q of Alice and R^{T} of Bob, the joint probability is $\mathrm{Tr}(hQR)$, so an event with $QR = 0$ has probability zero.

4.1 The normal flagged magic-square bridge

The magic-square relations originate with Mermin and Peres [40, 49]. We use Mančinska's flagged magic-square (FMS) geometry [37]. We spell out the predicate and prove its normal extension because the published finite-dimensional statement does not suffice here. All FMS question and answer indices start at zero; frame indices below start at one.

4.1.1 Questions, calibration and exact bindings

Alice has $\mathsf{R}_0, \mathsf{R}_1, \mathsf{R}_2, \mathsf{T}_0, \mathsf{T}_1$ and Bob has $\mathsf{C}_0, \mathsf{C}_1, \mathsf{C}_2, \mathsf{B}_0, \mathsf{B}_1$. A row question has a flag answer f or a triple of product $+1$. A column question has f or a triple of product $+1, +1, -1$ according to its column. The magic square is

$$\begin{pmatrix} XI & IX & XX \\ IZ & ZI & ZZ \\ XZ & ZX & YY \end{pmatrix}. \tag{11}$$

Here X, Y, Z denote the Pauli matrices, I the 2×2 identity, and adjacent letters their tensor product. The displayed matrices describe a canonical realization; the predicate only checks the corresponding signs. Select column-zero outcomes $c_0 = (+++)$, $c_1 = (+--)$ and row-zero outcomes $r_0 = (+++)$, $r_1 = (+--)$. Question T_x has answers $+, -$ and the three column-zero triples other than c_x; B_y analogously retains all row-zero triples other than r_y.

Table 3 and Figure 1 summarize the rules. On $\mathsf{R}_i, \mathsf{C}_j$, reject flag/nonflag mismatch and otherwise check the intersecting cell; two flags are accepted. On $\mathsf{T}_x, \mathsf{C}_0$, a sign is allowed only against f or c_x; a retained triple must equal the opposite triple. On $\mathsf{R}_0, \mathsf{B}_y$ use the analogous rule with r_y. On $\mathsf{T}_x, \mathsf{B}_y$, reject two signs $a, b \in \{+1, -1\}$ unless $ab = (-1)^{xy}$; all pairs with a nonsign answer are accepted. Other question pairs are accepted. For a conditional instance add an inactive answer $\perp$, accept two inactive answers, and reject exactly one inactive answer. Bind each noninactive event to its synchronized active projection and all row/column flags to the synchronized flag.

Let P_{mag} be the nonflag magic support. Its four column-zero projections $m_{00}, m_{01}, m_{10}, m_{11}$ correspond respectively to $(+++), (+--), (-+-), (--+)$. On Alice, m_{00} is read from the retained $(+++)$ answer of T_1 and the other three from T_0. On Bob all four come from C_0. Each of these chosen occurrences is matched to its opposite synchronized binary question. The positive nonflag ZI event is read from Alice's R_1 component 1 and Bob's C_1 component 1. The positive IX event comes from Alice's R_0 component 1 and Bob's C_1 component 0. Bind both positive nonflag events to synchronized binary questions, denoted $E_{ZI,+}$ and $E_{IX,+}$. Put

$$\mathfrak{x} = I - 2P_{\mathrm{mag}} + 2E_{ZI,+}, \quad \mathfrak{z} = I - 2P_{\mathrm{mag}} + 2E_{IX,+}, \quad t_{ab} = \mathfrak{x}^a \mathfrak{z}^b m_{00}.$$

These are reflections extended by identity outside P_{mag}, and $t_{ab}^* t_{ab} = m_{00}$, $t_{ab} t_{ab}^* = m_{ab}$.

4.1.2 Normal bridge theorem

Lemma 4.1. *A perfect normal conditional FMS strategy has on its active supports the decomposition*

$$\mathcal{H}_{\mathrm{active}} = \mathcal{H}_f \oplus (\mathbb{C}^4 \otimes \mathcal{H}_{\mathrm{mag}}), \qquad h = h_f \oplus (I_4 \otimes h_{\mathrm{mag}}).$$

Questions	Answers	Tested rule
$\mathsf{R}_i, \mathsf{C}_j$	flag or parity triple	flags match; nonflags agree at (i, j)
$\mathsf{T}_x, \mathsf{C}_0$	sign or retained triple	signs pair only with f, c_x; retained triples match
$\mathsf{R}_0, \mathsf{B}_y$	triple or sign	symmetric calibration at r_y
$\mathsf{T}_x, \mathsf{B}_y$	signs or triples	two signs satisfy $ab = (-1)^{xy}$
Conditional copies	also inactive $\perp$	activity matches; flags and events bind to named projections

Table 3: Flagged magic-square questions, answers and rules. Unspecified pairs are accepted.

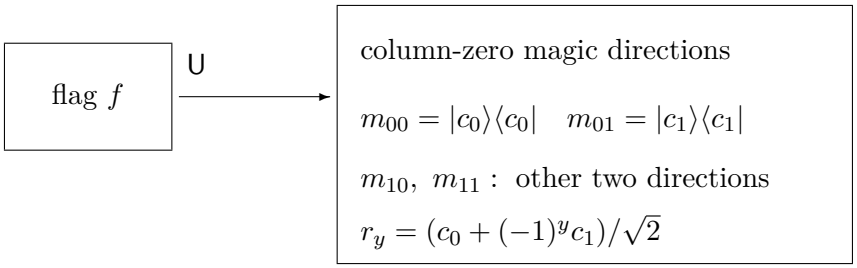


Figure 1: Selected column and row directions. The flag transports to c_0; m_{ab} denote projections.

The calibrated transports $\mathsf{U}_A, \mathsf{U}_B$ *from the flag to the selected* c_0 *corner satisfy* $\mathsf{U}^*\mathsf{U} = f$, $\mathsf{U}\mathsf{U}^* = m_{00}$ *and* $h\mathsf{U} = 2\mathsf{U}h$. *On a purification their matched components satisfy*

$$\Omega_{c_0 c_0} = \sqrt{2}(\mathsf{U}_A \otimes \mathsf{U}_B \otimes I_{\mathrm{env}})\Omega_{ff}. \tag{12}$$

The flag answer and its projection are both denoted f. *The other three magic directions obey the transported versions of this equation.*

The proof first extracts the magic matrix algebra from the synchronized cells. Calibration then identifies the two binary sign supports. The four sign equations force an off-diagonal unitary between the flag and a magic corner, and commutation with the calibrated marginal gives the density ratio two.

Proof. First apply Section 2.4 to the activity, flags and checked cells. They reduce the original state supports and are sharp there. On the compressed nonflag space each compressed fine effect lies below its assigned sharp cell marginals. Thus it is their joint spectral projection. The projective-compression fact then makes the original fine effects reduce the support. Alice's row relations and Bob's column relations transfer through the dense coefficient range, in reversed product order; within a context the products commute.

Fix a purification Ω as in (N1). The predicate makes the following pairs of events perfectly matched in the sense of (N5): an inactive answer on one side and the inactive answer on the other; each noninactive event and the opposite synchronized activity projection; each row or column flag and the opposite synchronized flag; and, for every cell (i, j) and sign s, Alice's event "R_i is not a flag and its entry j equals s" and Bob's event "C_j is not a flag and its entry i equals s". The last pair is matched because the rule on $\mathsf{R}_i, \mathsf{C}_j$ rejects a flag against a nonflag and checks the intersecting cell otherwise. By (N5), each coarse effect in this list reduces its party's support, its compression is a projection commuting with that party's marginal density, and matched compressions are mirrors. Let a_A and f_A be Alice's activity and flag projections, put $N_A = a_A - f_A$, and let Q^s_{ij} be the compression of the cell event above, so that $Q^+_{ij} + Q^-_{ij} = N_A$. The cell observables $\mathsf{M}^A_{ij} = Q^+_{ij} - Q^-_{ij}$ are self-adjoint unitaries on $N_A P_A \mathcal{H}_A$. Each is a mirror of Bob's operator M^B_{ij}, built in the same way from C_j.

For a nonflag answer $t = (t_0, t_1, t_2)$ of R_i, the compressed effect lies below $Q_{ij}^{t_j}$ for each j, since the other nonflag effects of R_i with entry j equal to t_j are positive. By (N6d) it equals $Q_{i0}^{t_0} Q_{i1}^{t_1} Q_{i2}^{t_2}$, the three cell observables of row i commute, and the original fine effect reduces P_A. Summing over the triples of product $+1$ gives $\mathsf{M}_{i0}^A \mathsf{M}_{i1}^A \mathsf{M}_{i2}^A = N_A$ on $P_A \mathcal{H}_A$. The same argument for Bob's columns gives commuting $\mathsf{M}_{0j}^B, \mathsf{M}_{1j}^B, \mathsf{M}_{2j}^B$ with product $\epsilon_j N_B$, where $\epsilon_0 = \epsilon_1 = 1$ and $\epsilon_2 = -1$. By (N4a), $\mathsf{M}_{0j}^A \mathsf{M}_{1j}^A \mathsf{M}_{2j}^A$ is a mirror of $\mathsf{M}_{2j}^B \mathsf{M}_{1j}^B \mathsf{M}_{0j}^B = \epsilon_j N_B$, which is a mirror of $\epsilon_j N_A$. Then (N4b) gives $\mathsf{M}_{0j}^A \mathsf{M}_{1j}^A \mathsf{M}_{2j}^A = \epsilon_j N_A$ on $P_A \mathcal{H}_A$. In the same way $\mathsf{M}_{ij}^A \mathsf{M}_{i'j}^A$ and $\mathsf{M}_{i'j}^A \mathsf{M}_{ij}^A$ are mirrors of the equal operators $\mathsf{M}_{i'j}^B \mathsf{M}_{ij}^B$ and $\mathsf{M}_{ij}^B \mathsf{M}_{i'j}^B$, so they agree on $P_A \mathcal{H}_A$. The nine cell observables therefore satisfy every row and column relation of (11) as operator identities on $N_A P_A \mathcal{H}_A$. The symmetric argument gives the same for Bob on $N_B P_B \mathcal{H}_B$.

For completeness, write $a = \mathsf{M}_{00}, b = \mathsf{M}_{01}, c = \mathsf{M}_{10}, d = \mathsf{M}_{11}$ for the four top-left cell reflections of the square, where M_{ij} denotes the checked cell observable. Context relations give

$$[a, b] = [c, d] = [a, c] = [b, d] = 0, \qquad (ac)(bd) = -(ab)(cd).$$

Cancel the outside a, d to obtain $cb = -bc$. Commutation of ac and bd then gives $da = -ad$. The pairs (a, d) and (b, c) are anticommuting Pauli pairs commuting across pairs. Each pair exchanges its two eigenspaces unitarily. Hence they generate $\mathrm{Mat}_4 \otimes I_{\mathcal{H}_{\mathrm{mag}}}$. Marginal commutation gives the stated density form. The synchronized magic units identify its coefficient block as $I_4 \otimes T_{\mathrm{mag}}$; write the flag coefficient block as T_f. The coefficient maps have dense range on the corresponding local marginal supports.

Put $\mathcal{N}_A = N_A P_A \mathcal{H}_A$, and let $\mathcal{N}_\pm$ be the ± 1 eigenspaces of a, so that $\mathcal{N}_A = \mathcal{N}_+ \oplus \mathcal{N}_-$. Since $da = -ad$, d maps $\mathcal{N}_\pm$ into $\mathcal{N}_\mp$, and $d^2 = I$ makes $d|_{\mathcal{N}_+}$ unitary onto $\mathcal{N}_-$. This identifies $\mathcal{N}_A$ with $\mathbb{C}^2 \otimes \mathcal{N}_+$, with $a = Z \otimes I$ and $d = X \otimes I$. The operators b, c commute with a and d, hence with the algebra $\mathrm{Mat}_2 \otimes I$ they generate, so $b = I \otimes b'$ and $c = I \otimes c'$ for anticommuting self-adjoint unitaries b', c'. Repeating the argument for c', b' gives $\mathcal{N}_+ = \mathbb{C}^2 \otimes \mathcal{H}_{\mathrm{mag}}$. No dimension count is used. The space $\mathcal{H}_{\mathrm{mag}}$ may be infinite-dimensional; it is separable because it lies in $P_A \mathcal{H}_A$.

For later use we record the identification through the units t_{ab}. The projection m_{00} is the joint $(+1, +1)$ eigenprojection of the column-zero cells $a = \mathsf{M}_{00}$ and $c = \mathsf{M}_{10}$, and m_{ab} is their joint $((-1)^a, (-1)^b)$ eigenprojection; this is the stated correspondence with $(++), (+-), (-+), (--)$. On $\mathcal{N}_A$ the reflection $\mathfrak{x}$ equals $d = \mathsf{M}_{11}$, which anticommutes with a and commutes with c, and $\mathfrak{z}$ equals $b = \mathsf{M}_{01}$, which commutes with a and anticommutes with c. Hence $t_{ab} = \mathfrak{x}^a \mathfrak{z}^b m_{00}$ maps $m_{00} \mathcal{N}_A$ isometrically onto $m_{ab} \mathcal{N}_A$, and

$$U_A : \mathbb{C}^4 \otimes \mathcal{H}_{\mathrm{mag}}^A \to \mathcal{N}_A, \qquad U_A(u_{ab} \otimes \xi) = t_{ab} \xi, \qquad \mathcal{H}_{\mathrm{mag}}^A = m_{00} \mathcal{N}_A,$$

is unitary. With $u_{ab} = |a\rangle \otimes |b\rangle$, these coordinates give $m_{ab} = |u_{ab}\rangle\langle u_{ab}| \otimes I$, $a = Z \otimes I$, $c = I \otimes Z$, $d = X \otimes I$ and $b = I \otimes X$. The operators $\mathfrak{x}, \mathfrak{z}, m_{00}$ are built from synchronized projections, so they commute with h_A by (N5), and so does each t_{ab}. Therefore $U_A^* h_A U_A = I_4 \otimes h_{\mathrm{mag}}^A$, where h_{mag}^A is the restriction of h_A to $\mathcal{H}_{\mathrm{mag}}^A$. Together with $[f_A, h_A] = 0$ this is the decomposition in the lemma, with $\mathcal{H}_f = f_A P_A \mathcal{H}_A$. Both blocks are faithful, because h_A is faithful on $P_A \mathcal{H}_A$ and they are its restrictions to reducing subspaces. The same construction on Bob's side gives U_B and h_{mag}^B.

Now the coefficient blocks. Flags and activity are matched, so $(f_A \otimes I)\Omega = (I \otimes f_B \otimes I)\Omega$ and $(N_A \otimes I)\Omega = (I \otimes N_B \otimes I)\Omega$. Put $\Omega_{ff} = (f_A \otimes f_B \otimes I)\Omega$ and $\Omega_{NN} = (N_A \otimes N_B \otimes I)\Omega$. The mixed components vanish; for example $(f_A \otimes N_B \otimes I)\Omega = (f_A N_A \otimes I)\Omega = 0$. Each of $\mathfrak{x}, \mathfrak{z}, m_{00}$ is a linear combination of matched events, hence a mirror of its Bob counterpart by (N4a); for $\mathfrak{x}$ the events are the activity, the flag and the positive ZI event, which the rule on $\mathsf{R}_1, \mathsf{C}_1$ matches. The reflections are self-adjoint, so (N4a) makes t_{ab}^A a mirror of $(t_{ab}^B)^*$, and the matrix unit $t_{ab}^A (t_{a'b'}^A)^*$ a mirror of $t_{a'b'}^B (t_{ab}^B)^*$. In the coordinates $U_A \otimes U_B$ write $\Omega_{NN} = \sum_{k,l} u_k \otimes u_l \otimes \omega_{kl}$,

where k, l range over the four labels ab and $\omega_{kl} \in \mathcal{H}^A_{\rm mag} \otimes \mathcal{H}^B_{\rm mag} \otimes \mathcal{H}_{\rm env}$. The mirror identity for the units $|u_k\rangle\langle u_l|$ and $|u_l\rangle\langle u_k|$ reads

$$\sum_{l'} u_k \otimes u_{l'} \otimes \omega_{ll'} = \sum_{k'} u_{k'} \otimes u_l \otimes \omega_{k'k},$$

so $\omega_{ll'} = 0$ for $l' \neq l$ and $\omega_{ll} = \omega_{kk}$. Hence

$$\Omega_{NN} = \sum_k u_k \otimes u_k \otimes \omega_{\rm mag}$$

for a single vector $\omega_{\rm mag}$, whose two marginals are $h^A_{\rm mag}$ and $h^B_{\rm mag}$. In coefficient form, the coefficient operator of Ω_{NN} is $I_4 \otimes T_{\rm mag}$, where $T_{\rm mag}$ is that of $\omega_{\rm mag}$, and T_f is that of Ω_{ff}. Here $T_{\rm mag} : \mathcal{H}^B_{\rm mag} \otimes \mathcal{H}_{\rm env} \to \mathcal{H}^A_{\rm mag}$ and $T_f : \mathcal{H}^B_f \otimes \mathcal{H}_{\rm env} \to \mathcal{H}^A_f$ are the coefficient maps in the fixed product coordinates. They satisfy $T_{\rm mag}T^*_{\rm mag} = h^A_{\rm mag}$ and $T_f T^*_f = h^A_f$, and have dense range. All cancellations below use (N3) on the corresponding original local support.

For a column-zero triple c, let Π_c be the joint spectral projection of its three cell reflections, and Π_r the joint projection of Bob's row-zero cells. Calibration makes Alice's sign support $S^A_x = f + \Pi_{c_x}$ and Bob's $S^B_y = f + \Pi_{r_y}$. The selected directions have overlap $\langle c_x, r_y\rangle = \epsilon_{xy}/\sqrt{2}$, where $\epsilon_{xy} = (-1)^{xy}$. Their compressed coefficient operator is

$$W_{xy} = \mathrm{diag}(T_f, \epsilon_{xy} T_{\rm mag}/\sqrt{2}).$$

For compressed sign effects E_a, F_b, a forbidden pair gives $E_a^{1/2} W_{xy} (F_b^T)^{1/2} = 0$, and therefore $E_a W_{xy} F_b^T = 0$. Summing the two complete binary effects gives

$$E_a W_{xy} = W_{xy} F^T_{\epsilon_{xy} a} = E_a W_{xy} F^T_{\epsilon_{xy} a}.$$

The vector calculation below, followed by (N3) on each original local sign support, gives $F_b^2 = F_b$ and $E_a^2 = E_a$ there. Their zero extensions give projective compressions; thus the original sign effects reduce the state supports. Their differences $\mathcal{X}_x, \mathcal{Y}_y$ obey

$$\mathcal{X}_x W_{xy} = \epsilon_{xy} W_{xy} \mathcal{Y}_y^T. \tag{13}$$

The rule on $\mathsf{T}_x, \mathsf{C}_0$ matches Alice's coarse sign event of T_x with Bob's event $\{f, c_x\}$ of C_0, and each retained triple of T_x with the same triple of C_0. Bob's C_0 projection for a triple is the joint projection of his column-zero cells, and by (N4a) its mirror is Alice's joint projection Π_c of her cells $\mathsf{M}^A_{00}, \mathsf{M}^A_{10}, \mathsf{M}^A_{20}$. Thus (N4b) gives $S^A_x = f_A + \Pi_{c_x}$ on $P_A\mathcal{H}_A$, and the retained effects of T_x are the other column-zero projections. Symmetrically, the rule on $\mathsf{R}_0, \mathsf{B}_y$ gives $S^B_y = f_B + \Pi_{r_y}$, where Π_{r_y} is Bob's joint projection of his row-zero cells. In the coordinates above, $c_0 = u_{00}$ and $c_1 = u_{01}$, while $r_y = (u_{00} + (-1)^y u_{01})/\sqrt{2}$, because row zero consists of $a = Z \otimes I$, $b = I \otimes X$ and ab. Hence $\langle c_x, r_y\rangle = \epsilon_{xy}/\sqrt{2}$, and

$$\Omega_{xy} := (S^A_x \otimes S^B_y \otimes I)\Omega = \Omega_{ff} + \frac{\epsilon_{xy}}{\sqrt{2}}\, c_x \otimes r_y \otimes \omega_{\rm mag}.$$

The maps $V^A_x(\xi \oplus \eta) = \xi + c_x \otimes \eta$ and $V^B_y(\xi' \oplus \eta') = \xi' + r_y \otimes \eta'$ identify $\mathcal{H}_f \oplus \mathcal{H}^A_{\rm mag}$ and $\mathcal{H}^B_f \oplus \mathcal{H}^B_{\rm mag}$ with the two sign supports. Under them Ω_{xy} becomes $\Omega_{ff} + \epsilon_{xy}\omega_{\rm mag}/\sqrt{2}$, with the two terms in the flag-flag and the magic-magic blocks; its coefficient operator is W_{xy}. Its Alice marginal is $h_f \oplus h^A_{\rm mag}/2$ and its Bob marginal is $h^B_f \oplus h^B_{\rm mag}/2$; both are faithful.

Let $E'_a = P_A E_a P_A$ and $F'_b = P_B F_b P_B$ be the compressed sign effects of T_x and B_y. Then $0 \le E'_a \le S^A_x$ and $0 \le F'_b \le S^B_y$, so $E'_a = E'_a S^A_x$ and $F'_b = F'_b S^B_y$ by (N6a). If $ab \neq \epsilon_{xy}$, the pair is rejected, so

$$0 = \langle \Omega, (E'_a \otimes F'_b \otimes I)\Omega\rangle = \|(E'^{1/2}_a \otimes F'^{1/2}_b \otimes I)\Omega\|^2,$$

whence $(E'_a \otimes F'_b \otimes I)\Omega = 0$. These are the vector forms of $E_a^{1/2} W_{xy} (F_b^T)^{1/2} = 0$ and $E_a W_{xy} F_b^T = 0$. Writing $\epsilon = \epsilon_{xy}$ and summing the complete binary effects,

$$(E'_a \otimes I)\Omega_{xy} = (E'_a \otimes S_y^B \otimes I)\Omega = (E'_a \otimes F'_{\epsilon a} \otimes I)\Omega = (S_x^A \otimes F'_{\epsilon a} \otimes I)\Omega = (I \otimes F'_{\epsilon a} \otimes I)\Omega_{xy}.$$

Next, $(E'_a E'_{-a} \otimes I)\Omega_{xy} = (E'_a \otimes F'_{-\epsilon a} \otimes I)\Omega_{xy} = (E'_a \otimes F'_{-\epsilon a} \otimes I)\Omega = 0$. Since $E'_{-a} = S_x^A - E'_a$, this says $((E'_a - E'^2_a) \otimes I)\Omega_{xy} = 0$. The Alice marginal of Ω_{xy} is faithful on $S_x^A P_A \mathcal{H}_A$, so (N3), applied to Ω_{xy}, gives $E'^2_a = E'_a$. This is where dense range enters. Bob's case is symmetric. By (N6b) the original sign effects reduce the supports. The differences $\mathcal{X}_x = E'_+ - E'_-$ and $\mathcal{Y}_y = F'_+ - F'_-$ are self-adjoint unitaries on $S_x^A P_A \mathcal{H}_A$ and $S_y^B P_B \mathcal{H}_B$, and

$$(\mathcal{X}_x \otimes I)\Omega_{xy} = \epsilon_{xy}(I \otimes \mathcal{Y}_y \otimes I)\Omega_{xy}$$

is the vector form of (13).

Use the matrix units $t_{ab} t^*_{cd}$ to identify each selected magic direction with the same multiplicity space, and use $\sqrt{2}\Pi_{c_x}\Pi_{r_y}$ with its sign ϵ_{xy} for the selected row directions. Set $T_{\mathrm{cal}} = \mathrm{diag}(T_f, T_{\mathrm{mag}}/\sqrt{2})$ and $J_{\mathrm{sgn}} = \mathrm{diag}(I, -I)$. For $(x,y) = (0,0), (0,1), (1,0)$, Eq. (13) reads respectively $\mathcal{X}_0 T_{\mathrm{cal}} = T_{\mathrm{cal}} \mathcal{Y}_0^T$, $\mathcal{X}_0 T_{\mathrm{cal}} = T_{\mathrm{cal}} \mathcal{Y}_1^T$, and $\mathcal{X}_1 T_{\mathrm{cal}} = T_{\mathrm{cal}} \mathcal{Y}_0^T$. Writing these three equations in vector form and applying (N3) on Alice's and Bob's respective local sign supports gives $\mathcal{X}_0 = \mathcal{X}_1 = \mathcal{X}$ and $\mathcal{Y}_0 = \mathcal{Y}_1 = \mathcal{Y}$. For $(1,1)$ it reads $\mathcal{X} J_{\mathrm{sgn}} T_{\mathrm{cal}} = -J_{\mathrm{sgn}} T_{\mathrm{cal}} \mathcal{Y}^T = -J_{\mathrm{sgn}} \mathcal{X} T_{\mathrm{cal}}$, so dense range gives $\mathcal{X} J_{\mathrm{sgn}} = -J_{\mathrm{sgn}} \mathcal{X}$. In blocks $\mathcal{X} = (\mathcal{X}_{ij})$, this says $\mathcal{X}_{11} = \mathcal{X}_{22} = 0$. Since $\mathcal{X} = \mathcal{X}^*$ and $\mathcal{X}^2 = I$, its remaining block $\beta = \mathcal{X}_{21}$ is unitary. Thus

$$\mathcal{X} = \begin{pmatrix} 0 & \beta^* \\ \beta & 0 \end{pmatrix}, \qquad \beta : \mathcal{H}_f \longrightarrow \mathcal{H}_{\mathrm{mag}} \text{ unitary.}$$

Self-adjointness and $\mathcal{X} T_{\mathrm{cal}} = T_{\mathrm{cal}} \mathcal{Y}^T$ imply $[\mathcal{X}, T_{\mathrm{cal}} T^*_{\mathrm{cal}}] = 0$. Here $h_f = T_f T_f^*$ and $h_{\mathrm{mag}} = T_{\mathrm{mag}} T^*_{\mathrm{mag}}$ are Alice's density blocks. The off-diagonal block gives $h_{\mathrm{mag}} \beta = 2\beta h_f$. The corresponding corner transport has the required ratio.

Put $\widehat{\mathcal{X}}_x = V_x^{A*} \mathcal{X}_x V_x^A$ and $\widehat{\mathcal{Y}}_y = V_y^{B*} \mathcal{Y}_y V_y^B$, and let $\Omega_{\mathrm{cal}} = \Omega_{ff} + \omega_{\mathrm{mag}}/\sqrt{2}$, whose coefficient operator is T_{cal}. For $(x,y) \neq (1,1)$ the vector Ω_{xy} becomes Ω_{cal}, and for $(1,1)$ it becomes $(J_{\mathrm{sgn}} \otimes I)\Omega_{\mathrm{cal}} = (I \otimes J_{\mathrm{sgn}} \otimes I)\Omega_{\mathrm{cal}}$. The cases $(0,0)$ and $(1,0)$ give $((\widehat{\mathcal{X}}_0 - \widehat{\mathcal{X}}_1) \otimes I)\Omega_{\mathrm{cal}} = 0$, and the cases $(0,0)$ and $(0,1)$ give $(I \otimes (\widehat{\mathcal{Y}}_0 - \widehat{\mathcal{Y}}_1) \otimes I)\Omega_{\mathrm{cal}} = 0$. Both marginals of Ω_{cal}, namely $h_f \oplus h^A_{\mathrm{mag}}/2$ and $h_f^B \oplus h^B_{\mathrm{mag}}/2$, are faithful, so (N3) gives $\widehat{\mathcal{X}}_0 = \widehat{\mathcal{X}}_1 = \mathcal{X}$ and $\widehat{\mathcal{Y}}_0 = \widehat{\mathcal{Y}}_1 = \mathcal{Y}$. These are applications of (N3) to the two original local marginals. The case $(1,1)$ then gives

$$(\mathcal{X} J_{\mathrm{sgn}} \otimes I)\Omega_{\mathrm{cal}} = -(I \otimes \mathcal{Y} \otimes I)(J_{\mathrm{sgn}} \otimes I)\Omega_{\mathrm{cal}} = -(J_{\mathrm{sgn}} \mathcal{X} \otimes I)\Omega_{\mathrm{cal}},$$

so $\mathcal{X} J_{\mathrm{sgn}} + J_{\mathrm{sgn}} \mathcal{X} = 0$ by (N3). A self-adjoint unitary anticommuting with $J_{\mathrm{sgn}} = \mathrm{diag}(I, -I)$ has zero diagonal blocks, and $\mathcal{X}^2 = I$ gives its lower block β the identities $\beta^* \beta = I_{\mathcal{H}_f}$ and $\beta \beta^* = I_{\mathcal{H}_{\mathrm{mag}}}$. The operators $\mathcal{X}, \mathcal{Y}$ are self-adjoint mirrors for Ω_{cal}, so (N4c) gives $[\mathcal{X}, h_f \oplus h^A_{\mathrm{mag}}/2] = 0$. In coefficient form this is the displayed commutator: from $\mathcal{X} T_{\mathrm{cal}} = T_{\mathrm{cal}} \mathcal{Y}^T$, the operator $\mathcal{X} T_{\mathrm{cal}} T^*_{\mathrm{cal}} = T_{\mathrm{cal}} \mathcal{Y}^T T^*_{\mathrm{cal}}$ is self-adjoint. Its off-diagonal block is $\beta h_f = \frac{1}{2} h^A_{\mathrm{mag}} \beta$. On Bob's side the same argument gives a unitary β_B with $h^B_{\mathrm{mag}} \beta_B = 2\beta_B h^B_f$.

The actual Alice extraction is $\mathsf{U}_A = \Pi_{c_0} \mathcal{X}_0 f$. Bob's r_0 projection is the retained $(+++)$ answer of B_1, equivalently the sign support of B_0 minus its flag. His extraction to the same c_0 direction is

$$\mathsf{U}_B = \sqrt{2}\Pi_{c_0} \Pi_{r_0} \mathcal{Y}_0 f.$$

The coefficient $\sqrt{2}$ occurs in extraction, not in the predicate. To see the phase in (12) directly, put $\Omega' = (S_0^A \otimes S_0^B)\Omega$. The zero sign events give $\mathcal{X}_0 \Omega' = \mathcal{Y}_0 \Omega'$. Project onto $c_0 \otimes f$. The left side is $(\mathsf{U}_A \otimes I)\Omega_{ff}$; the right side is

$$2^{-1/2}(I \otimes f \mathcal{Y}_0 |r_0\rangle\langle c_0|)\Omega_{c_0 c_0}.$$

Apply Bob's flag-to-r_0 block and then $|c_0\rangle\langle r_0| = \sqrt{2}\Pi_{c_0}\Pi_{r_0}$. This gives (12).

In the coordinates V_0^A, U_A is the map $\xi \mapsto c_0 \otimes \beta\xi$ from $\mathcal{H}_f$ into $m_{00}P_A\mathcal{H}_A$. Thus $\mathsf{U}_A^*\mathsf{U}_A = f_A$ and $\mathsf{U}_A\mathsf{U}_A^* = m_{00}$ on the support, and $h_A\mathsf{U}_A = 2\mathsf{U}_A h_A$, because h_A acts as h_f on the flag and as h^A_{mag} on $c_0 \otimes \mathcal{H}^A_{\mathrm{mag}}$. On Bob's side $\mathcal{Y}_0 f_B$ is $\xi' \mapsto r_0 \otimes \beta_B\xi'$, and $\sqrt{2}\Pi_{c_0}\Pi_{r_0}$ restricts to the isometry $r_0 \otimes \eta \mapsto c_0 \otimes \eta$ from $\Pi_{r_0}P_B\mathcal{H}_B$ onto $m_{00}P_B\mathcal{H}_B$, since $\langle c_0, r_0\rangle = 2^{-1/2}$. So $\mathsf{U}_B = c_0 \otimes \beta_B$ has the same three properties. The components in $\mathcal{H}^A_{\mathrm{mag}} \otimes \mathcal{H}^B_f \otimes \mathcal{H}_{\mathrm{env}}$ of the two sides of $(\mathcal{X} \otimes I)\Omega_{\mathrm{cal}} = (I \otimes \mathcal{Y} \otimes I)\Omega_{\mathrm{cal}}$ give $(\beta \otimes I)\Omega_{ff} = (I \otimes \beta_B^* \otimes I)\omega_{\mathrm{mag}}/\sqrt{2}$, that is, $\omega_{\mathrm{mag}} = \sqrt{2}(\beta \otimes \beta_B \otimes I)\Omega_{ff}$. Since $\Omega_{c_0c_0} = c_0 \otimes c_0 \otimes \omega_{\mathrm{mag}}$, this is (12). The formula for Ω_{NN} also gives $\Omega_{m_{ab}m_{ab}} = (t^A_{ab} \otimes t^B_{ab} \otimes I)\Omega_{c_0c_0}$, and the components for two different magic directions vanish. These are the transported versions for the other three directions.

For a mixed state, first apply the pure-vector calculation on the cut $A \mid B\,\mathrm{env}$. The synchronized opposite effects reduce the Schmidt support, and the sign effects do so after their compressed sharpness is proved. Thus every compressed product used above is the restriction of an original Bob-only product tensored with I_{env}. The resulting Alice identities hold on her original marginal support. Repeat on $B \mid A\,\mathrm{env}$ for Bob's identities. The component calculation then gives Eq. (12) on the original purification, using $\mathsf{U}_A \otimes \mathsf{U}_B \otimes I_{\mathrm{env}}$. Tracing out the environment preserves the displayed component identity.

The calculation above already used the purification Ω on the cut $A \mid B\,\mathrm{env}$, with every Bob operator of the form $F \otimes I_{\mathrm{env}}$; the cut $B \mid A\,\mathrm{env}$ is the Bob form of (N3). The supports P_A, P_B are those of $\mathrm{Tr}_B\,\rho$ and $\mathrm{Tr}_A\,\rho$, so every identity holds on the original marginal supports. The partial trace over $\mathcal{H}_{\mathrm{env}}$ of the rank-one operators of the two sides of (12) gives $(m_{00} \otimes m_{00})\rho(m_{00} \otimes m_{00}) = 2(\mathsf{U}_A \otimes \mathsf{U}_B)(f \otimes f)\rho(f \otimes f)(\mathsf{U}_A \otimes \mathsf{U}_B)^*$. □

4.1.3 The commuting form

On a normal commuting host the same predicate calibrates a modular eigenoperator. We recall the arbitrary-normal bridge argument of [20, Lemma 2.2]. Matching makes the cells sharp centralizer effects. Write the upper-left four cell reflections as a, b, c, d. The row/column relations give $[a, b] = [a, c] = [b, d] = [c, d] = 0$, $(ac)(bd) = -(ab)(cd)$, and $[ac, bd] = 0$. Canceling the outside a, d gives $cb = -bc$; substitution into the last commutator gives $ad = -da$. The two commuting Pauli pairs therefore generate centralizer Mat_4. In $\mathbb{C}^2 \otimes \mathbb{C}^2$ the calibrated unit vectors are $\chi_x = |+\rangle \otimes |x\rangle$ and $\rho_y^{\mathrm{dir}} = |+\rangle \otimes |(-1)^y\rangle$, where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|(-1)^y\rangle = (|0\rangle + (-1)^y|1\rangle)/\sqrt{2}$. Choose a minimal projection q of the magic matrix algebra Mat_4 and centralizer partial isometries u_x^A, u_y^B with initial projection q and ranges the projections onto $\chi_x, \rho_y^{\mathrm{dir}}$ (tensored with the multiplicity identity). Choose their phases so that their overlap is $(u_x^A)^* u_y^B = (-1)^{xy} q/\sqrt{2}$. If Σ_x^A, Σ_y^B are the original sign differences, their pull-backs on the standard corner $(f+q)\mathcal{J}(f+q)\mathcal{J}L^2(M)$ are

$$\mathcal{X}_x = (f + (u_x^A)^*)\Sigma_x^A(f + u_x^A), \qquad \mathcal{Y}_y = \mathcal{J}(f + (u_y^B)^*)\mathcal{J}\Sigma_y^B\mathcal{J}(f + u_y^B)\mathcal{J}.$$

They are commuting self-adjoint contractions. Their joint-support vector is

$$\theta_{xy} = (f + (-1)^{xy} q/\sqrt{2})\xi_\varphi.$$

Forbidden sign events give $\mathcal{X}_x\theta_{xy} = (-1)^{xy}\mathcal{Y}_y\theta_{xy}$ and $\mathcal{X}_x\mathcal{Y}_y\theta_{xy} = (-1)^{xy}\theta_{xy}$. The multiplier $f \pm q/\sqrt{2}$ is bounded invertible in the corner centralizer, so θ_{xy} separates both corner algebras. Commuting the opposite contraction through the first identity gives $\mathcal{X}_x^2\theta_{xy} = \theta_{xy}$ and $\mathcal{Y}_y^2\theta_{xy} = \theta_{xy}$; separation makes them reflections. Put $c_{\mathrm{cal}} = f + q/\sqrt{2}$. The three positive cells give $\mathcal{X}c_{\mathrm{cal}}\xi_\varphi = \mathcal{Y}c_{\mathrm{cal}}\xi_\varphi$ and identify $\mathcal{X}_0 = \mathcal{X}_1 = \mathcal{X}$ and $\mathcal{Y}_0 = \mathcal{Y}_1 = \mathcal{Y}$ by separation on $c_{\mathrm{cal}}\xi_\varphi$. The fourth vector is $J_{\mathrm{sgn}}c_{\mathrm{cal}}\xi_\varphi$, $J_{\mathrm{sgn}} = f - q$. Its equation is $\mathcal{X}J_{\mathrm{sgn}}c_{\mathrm{cal}}\xi_\varphi = -\mathcal{Y}J_{\mathrm{sgn}}c_{\mathrm{cal}}\xi_\varphi = -J_{\mathrm{sgn}}\mathcal{Y}c_{\mathrm{cal}}\xi_\varphi = -J_{\mathrm{sgn}}\mathcal{X}c_{\mathrm{cal}}\xi_\varphi$, since opposite algebras commute. Separation gives $\mathcal{X}J_{\mathrm{sgn}} =$

$-J_{\mathrm{sgn}}\mathcal{X}$. Thus $\mathcal{X} = \mathsf{U} + \mathsf{U}^*$, $\mathsf{U} = q\mathcal{X}f$, $\mathsf{U}^*\mathsf{U} = f$, $\mathsf{U}\mathsf{U}^* = q$. The mirror equation means that $\mathcal{X}$ centralizes $\omega(\cdot) = \varphi(d_{\mathrm{cal}}\,\cdot)$, $d_{\mathrm{cal}} = f + q/2$. The bounded centralizer perturbation formula of Pedersen–Takesaki [48, Lemma 4.5] $\sigma_t^{\omega}(x) = d_{\mathrm{cal}}^{it}\sigma_t^{\varphi}(x)d_{\mathrm{cal}}^{-it}$ gives on the q–f block

$$\sigma_t^{\varphi}(\mathsf{U}) = 2^{it}\mathsf{U}, \qquad \varphi(q) = 2\varphi(f). \tag{14}$$

The last equality is KMS. Multiplication by the magic matrix units gives all four transports. If one sector vanished, the off-diagonal reflection would kill the other; hence on a nonzero active corner both sectors are nonzero. No finite multiplicity is used.

4.1.4 A shared question fixes transported labels

Lemma 4.2 (Label refinement)**.** *Retain the original FMS questions. For a synchronized projection P^{lab}, add Alice's $\mathsf{T}_0[P^{\mathrm{lab}}]$ with answers (a, z). Ignore z against the unchanged $\mathsf{B}_0, \mathsf{B}_1, \mathsf{C}_0$ and use the original predicates; match z to Bob's binary P^{lab}. Add the symmetric $\mathsf{B}_0[P^{\mathrm{lab}}]$ against unchanged $\mathsf{T}_0, \mathsf{T}_1, \mathsf{R}_0$. Also retain activity matching. In every perfect normal strategy the coarse signs equal the original signs and commute with P^{lab} on their state supports. Conversely commuting labels give these refinements. Distinct labels require no mutual commutation.*

Proof. Let $q_a = \sum_z q_{a,z}$ be the compressed coarse effects of $\mathsf{T}_0[P^{\mathrm{lab}}]$. Against C_0 the original rule is applied to a, so the matching argument in the proof of Lemma 4.1 gives, by (N5) and (N4b), $q_t = \Pi_t$ for each retained triple t and $q_+ + q_- = R_A = f_A + \Pi_{c_0}$. Against $\mathsf{B}_0, \mathsf{B}_1$ the forbidden sign pairs are the original ones. The calculation in that proof which gave $(E'_a \otimes I)\Omega_{0y} = (I \otimes F'_a \otimes I)\Omega_{0y}$ and $E'^2_a = E'_a$ used only these pairs and the projection R_A, so it applies verbatim to $q_\pm$. With $\mathcal{X}' = q_+ - q_-$ it gives

$$(\mathcal{X}' \otimes I)\Omega_{00} = (I \otimes \mathcal{Y}_0 \otimes I)\Omega_{00} = (\mathcal{X}_0 \otimes I)\Omega_{00},$$

the vector form of $\mathcal{X}'W_{00} = W_{00}\mathcal{Y}_0^T$. The Alice marginal of Ω_{00} is faithful on $R_AP_A\mathcal{H}_A$, so (N3) gives $\mathcal{X}' = \mathcal{X}_0$ and $q_\pm = (R_A \pm \mathcal{X}_0)/2$.

The z-marginal $\sum_a Q_{a,1}$ is perfectly matched with Bob's binary question for P^{lab}. By (N5) its compression is a projection and a mirror of Bob's synchronized projection, whose Alice mirror is P_A^{lab}; (N4b) gives equality. For every a, positivity gives $0 \le q_{a,1} \le P_A^{\mathrm{lab}}$ and $0 \le q_{a,0} \le P_A - P_A^{\mathrm{lab}}$. By (N6a), $q_{a,1} = P_A^{\mathrm{lab}}q_{a,1}P_A^{\mathrm{lab}}$ and $q_{a,0} = (P_A - P_A^{\mathrm{lab}})q_{a,0}(P_A - P_A^{\mathrm{lab}})$, so $P_A^{\mathrm{lab}}q_a = q_{a,1} = q_aP_A^{\mathrm{lab}}$. All of these are identities on $P_A\mathcal{H}_A$. For the converse, if the original effects E_a commute with P^{lab}, then E_aP^{lab} and $E_a(I - P^{\mathrm{lab}})$ are positive. Summing over z gives E_a, and summing the $z = 1$ effects over a gives P^{lab}. Every probability involving only a is an original probability, and the z-test is the original perfect matching of P^{lab}. By (N6c) one may first replace P^{lab} by its support-restricted projection. No two different labels are measured jointly, so no commutation between them enters either direction. □

If a label is block diagonal with flag block P_f^{lab} and identical magic blocks P_m^{lab}, the commutator says $\beta P_f^{\mathrm{lab}} = P_m^{\mathrm{lab}}\beta$. In particular, for a graph projection of a block unitary it certifies transport of that same unitary by the calibrated U. It does not supply a fresh bridge with an independent gauge. Indeed, with respect to $R_A = f \oplus \Pi_{c_0}$ the commutator $[\mathcal{X}_0, P^{\mathrm{lab}}]$ has the (c_0, f) block $\beta P_f^{\mathrm{lab}} - P_m^{\mathrm{lab}}\beta$.

Proof in the commuting model. Calibration fixes the retained effects and the sign support. Pull the new coarse sign difference $\mathcal{X}'$ to the same corner. Against the unchanged opposite sign it satisfies $\mathcal{X}'\theta_{0y} = \mathcal{Y}_y\theta_{0y} = \mathcal{X}_0\theta_{0y}$. The separating vector of Section 4.1.3 gives $\mathcal{X}' = \mathcal{X}_0$. Thus its coarse effects E_a are the old projections. For the fine effects $Q_{a,z}$, label matching gives $\sum_a Q_{a,1} = P^{\mathrm{lab}}$ and $\sum_a Q_{a,0} = I - P^{\mathrm{lab}}$ on faithful support. Positivity gives $0 \le Q_{a,1} \le$

E_a, P^{lab} and $0 \le Q_{a,0} \le E_a, I - P^{\mathrm{lab}}$. Hence $E_a P^{\mathrm{lab}} = Q_{a,1} = P^{\mathrm{lab}} E_a$, using completeness of each marginal. This proves commutation, and the symmetric argument handles Bob. Conversely use the joint products $E_a P^{\mathrm{lab}}, E_a(I - P^{\mathrm{lab}})$. Each label has its own refined question with the same unchanged opposite signs, so there is no cross-label commutation constraint. □

4.2 Rational projection equations that certify products

All projections in this section are synchronized. We give the finite rules used in the compiler. The projection-partition viewpoint is standard [25]; the formulas here provide the needed actual multiplication identities. See also the graph-projection encoding in [8, Section 4].

4.2.1 Graphs and triangles

Let F_i, F_j be orthogonal, $S = F_i + F_j$, and introduce projections $G, H, G^\perp, H^\perp$ with

$$G + G^\perp = S, \quad H + H^\perp = S, \quad 5H + S = 3F_i + 4G. \tag{15}$$

On S, put $Z_{ij}^{\mathrm{gr}} = F_i - F_j$, $\Xi_{ij} = 2G - S$. These are reflections, and $2H - S = (3Z_{ij}^{\mathrm{gr}} + 4\Xi_{ij})/5$. Squaring forces $Z_{ij}^{\mathrm{gr}}\Xi_{ij} + \Xi_{ij}Z_{ij}^{\mathrm{gr}} = 0$. Thus

$$\Xi_{ij} = \begin{pmatrix} 0 & u_{ij} \\ u_{ij}^* & 0 \end{pmatrix}, \qquad u_{ij}^* u_{ij} = F_j, \quad u_{ij} u_{ij}^* = F_i.$$

The graph blocks are certified, rather than assumed. Conversely any such unitary corner map gives the four projections in (15).

For a partition $F_1 + F_2 + F_3 = I$ impose the graph tests on all three pairs and add a projection $P_\triangle$ with

$$3P_\triangle = 2(G_{12} + G_{23} + G_{13}) - I. \tag{16}$$

Its block matrix has diagonal $I/3$ and off-diagonal $u_{ij}/3$. The $(1,3)$ block of $P_\triangle^2 = P_\triangle$ gives $2u_{13} + u_{12}u_{23} = 3u_{13}$, hence $u_{13} = u_{12}u_{23}$. Conversely this product identity makes the entire matrix a projection.

Start with a reference triangle for identity. Its scalar matrix algebra is called the *frame*. It gives scalar matrix units and $\mathcal{H} = \mathbb{C}^3 \otimes \mathcal{H}_{\mathrm{base}}$. For every unitary label Θ, use graphs in all three positions. Triangles $(\Theta, 1, \Theta)$ and $(1, \Theta, \Theta)$ identify the placements. A triangle $(\Theta, \Theta_2, \Theta_3)$ then certifies $\Theta_3 = \Theta\Theta_2$; a triangle with product identity certifies inverse labels. Build each word from its proper prefix and final token. Finally a real rational equation $\sum_w q_w \Theta_w = 0$ is precisely

$$2\sum_w q_w G_{12}(\Theta_w) - \Big(\sum_w q_w\Big)(F_1 + F_2) = 0. \tag{17}$$

Its two off-diagonal blocks are the equation and its adjoint. All products therefore refer to actual common operators. Synchronization makes the extracted unitaries commute with the base marginal density.

4.2.2 Binding prescribed parameters

A prescribed synchronized projection P^{par} is linked to the frame by projections $P^{\mathrm{par}}F_i, NF_i$ with

$$P^{\mathrm{par}}F_i + NF_i = F_i, \quad \sum_i P^{\mathrm{par}}F_i = P^{\mathrm{par}}, \quad \sum_i NF_i = I - P^{\mathrm{par}}.$$

These symbols are names for projections before the equations identify their products; the names NF_i denote complements of $P^{\mathrm{par}}F_i$ inside F_i. For every pair ij, introduce $J_{11}, J_{10}, J_{01}, J_{00}$ and impose

$$J_{11} + J_{10} = P^{\mathrm{par}}F_i + P^{\mathrm{par}}F_j, \qquad J_{00} + J_{01} = NF_i + NF_j,$$
$$J_{11} + J_{01} = G_{ij}(1), \qquad G_{ij}(s_{\mathrm{par}}) = J_{11} + J_{00}.$$

Projection sums give the indicated joint partitions. The symbol s_{par} is a unitary parameter label. These equations force P^{par} to commute with the frame and identify the extracted s_{par} with $2e - I$, where $P^{\mathrm{par}} = I_3 \otimes e$. In particular these parameters are the specified FMS events, not independent replacements. The four-bridge assembly uses twenty-five such parameters.

These rules are a deterministic finite compiler for the finite word list. Cancellation of adjacent inverse tokens is performed before taking proper prefixes. All coefficients stay rational. Conversely a centralizer unitary assignment with the prescribed parameter projections gives every graph, complement, H, triangle and joint projection simultaneously by these formulas.

Scope in infinite dimensions. All projections in this section are support-restricted synchronized projections on Alice's support $P_A\mathcal{H}_A$, as in (N5), and every displayed equation holds there as an operator identity by Lemma 4.3. The derivations are block-matrix calculations for orthogonal decompositions of $P_A\mathcal{H}_A$. Projections whose sum is a projection are mutually orthogonal, and a self-adjoint unitary anticommuting with $\mathrm{diag}(I, -I)$ has the off-diagonal form used for Ξ_{ij}. No trace, rank or dimension is used. The frame identification is the unitary

$$\mathbb{C}^3 \otimes \mathcal{H}_{\mathrm{base}} \to P_A\mathcal{H}_A, \qquad e_i \otimes \xi \mapsto u_{i1}\xi, \qquad \mathcal{H}_{\mathrm{base}} = F_1 P_A \mathcal{H}_A,$$

where $u_{11} = F_1$, $u_{21} = u_{12}^*$, $u_{31} = u_{13}^*$, and $u_{ij} = F_i(2G_{ij}(1) - F_i - F_j)F_j$. The frame projections and the maps u_{ij} commute with h_A by (N5), so $h_A = I_3 \otimes h_{\mathrm{base}}$ in these coordinates. An extracted label Θ is defined by $F_1(2G_{12}(\Theta) - F_1 - F_2)F_2 = e_{12} \otimes \Theta$, and it commutes with h_{base}.

The binding equations are solved in the same way. Write $\pi_i^{\mathrm{par}} = P^{\mathrm{par}}F_i$ and $\eta_i^{\mathrm{par}} = NF_i$. These are projections below F_i with $\pi_i^{\mathrm{par}} + \eta_i^{\mathrm{par}} = F_i$, and $\sum_i \pi_i^{\mathrm{par}} = P^{\mathrm{par}}$, so $P^{\mathrm{par}}F_i = \pi_i^{\mathrm{par}} = F_i P^{\mathrm{par}}$ and P^{par} commutes with every F_i. For a pair ij the four projections J_{ab} sum to $S = F_i + F_j$ and are therefore mutually orthogonal. Both $\pi^{\mathrm{par}} = \pi_i^{\mathrm{par}} + \pi_j^{\mathrm{par}}$ and $G_{ij}(1)$ are sums of them, so they commute and $\pi^{\mathrm{par}}G_{ij}(1) = J_{11}$. In frame coordinates write $\pi_i^{\mathrm{par}} = e_{ii} \otimes p_i$. Commutation of π^{par} with $\Xi_{ij}(1) = (e_{ij} + e_{ji}) \otimes I$ gives $p_i = p_j$; over all pairs, $P^{\mathrm{par}} = I_3 \otimes e$. Finally $J_{00} = (S - \pi^{\mathrm{par}})(S - G_{ij}(1))$, so

$$G_{ij}(s_{\mathrm{par}}) = \pi^{\mathrm{par}}G_{ij}(1) + (S - \pi^{\mathrm{par}})(S - G_{ij}(1)) = \tfrac{1}{2}(S + (e_{ij} + e_{ji}) \otimes (2e - I)),$$

which identifies s_{par} with $2e - I$.

4.3 Rational rows as Boolean winning conditions

We implement each rational row by a subspace orthogonality test. This is the general rational-row theorem of [23, Theorem B], restated here to specify the extension and its quantitative loss.

Lemma 4.3 (Boolean rational rows)**.** *Let a finite strategy specification name finitely many synchronized binary projections and finitely many rational linear identities among them. There is a finite extension by Boolean tests with the following properties.*

Soundness: *in every perfect normal spatial strategy for the extended tests, the identities hold as operator identities for the named support-restricted projections on each local state support.*

Completeness: *any normal spatial strategy with these synchronized projections, satisfying the identities on its supports, extends to satisfy all the added tests perfectly after adjoining a finite maximally entangled state. The old measurements act trivially on the new factor. The rest of the old strategy, including nonsynchronous questions, is retained; if its old tests were perfect, the whole extension is perfect.*

Remark. The same proof, paragraph "The commuting form of rational rows", gives the commuting form used in Section 4.6.

4.3.1 A certified scalar matrix algebra

Choose $d_{\max} = 2^{2r}$ large enough for every row construction below. Use r synchronized magic squares, binding each context's cell marginals to synchronized binary cells. The Pauli-pair calculation in Lemma 4.1 gives two anticommuting pairs per square. Four-answer joint questions for all pairs of axes from distinct squares force cross-square commutation. Hence their algebra is $\mathrm{Mat}_{d_{\max}} \otimes I$ on the support. Synchronization makes the marginal $I_{d_{\max}}/d_{\max} \otimes h$.

Add joint questions between every named old projection and every scalar axis. Their sharp marginals force the old projections into the commutant: $E_i = I_{d_{\max}} \otimes e_i$. This step concerns only synchronized projections, never an FMS sign. Add global Z and global X measurements with all bit outcomes, binding each bit marginal to its axis. They are the actual joint spectral measurements.

The Pauli-pair calculation is the one beginning "For completeness, write $a = \mathsf{M}_{00}$" in the proof of Lemma 4.1, without flags. A four-answer joint question for axes ς, ς' of distinct squares has its two binary marginals bound to synchronized cells, so the axes commute on $P_A\mathcal{H}_A$ by (N6d). Iterating the eigenspace argument of that proof over the $2r$ mutually commuting anticommuting pairs identifies $P_A\mathcal{H}_A$ with $\mathbb{C}^{d_{\max}} \otimes \mathcal{H}'$, the axes generating $\mathrm{Mat}_{d_{\max}} \otimes I$; the space $\mathcal{H}'$ may be infinite-dimensional. The axes commute with h_A by (N5), so h_A lies in $(\mathrm{Mat}_{d_{\max}} \otimes I)' = I \otimes B(\mathcal{H}')$, which is the stated form. A joint question between a named projection E_i and an axis makes them commute by (N6d). The axes generate $\mathrm{Mat}_{d_{\max}} \otimes I$ as an algebra, so E_i lies in its commutant, and $E_i = I_{d_{\max}} \otimes e_i$ for a projection e_i on $\mathcal{H}'$. The global Z and X measurements have sharp bit marginals, so by (N6d) they are the joint spectral measures of the commuting Z axes and of the commuting X axes.

For a row dimension $d = 2^{k_{\mathrm{row}}} \le d_{\max}$ use the first k_{row} qubits; the rest are spectators. For each nonzero $j \in \mathbb{F}_2^{k_{\mathrm{row}}}$, choose a basis $\ell_1, \dots, \ell_{k_{\mathrm{row}}-1}$ of $j^\perp$. Add a joint context for $X^j, Z^{\ell_1}, \dots, Z^{\ell_{k_{\mathrm{row}}-1}}$, binding its marginals to the corresponding global parity events. These commute because $\ell \cdot j = 0$. The zero-Z-syndrome outcomes with X^j sign $\pm$ are

$$\Pi_j^{\pm} = \tfrac{1}{2}(|0\rangle \pm |j\rangle)(\langle 0| \pm \langle j|) \otimes I_{\mathrm{spectator}}.$$

Write $|+_d\rangle = d^{-1/2} \sum_{j=0}^{d-1} |j\rangle$. The reference $\Pi_d^{\mathrm{ref}} = |+_d\rangle\langle +_d| \otimes I_{\mathrm{spectator}}$ is the event that the first k_{row} global X bits are positive. It includes all spectator outcomes.

4.3.2 Signed reflections and column graphs

For one row $S = \alpha_0 I + \sum_i \alpha_i e_i$, let L clear all coefficient denominators and set

$$s_i = 2e_i - I, \quad c_i = L\alpha_i, \quad c_0 = 2L\alpha_0 + \sum_i L\alpha_i.$$

Then $2LS = c_0 I + \sum_i c_i s_i$. Expand this into signed reflections with integer repetitions, add a canceling $I, -I$ pair with $V_0 = I$, and pad by further such pairs to a power of four d. The list has even length because its coefficient sum is even. Thus

$$\sum_{j=0}^{d-1} V_j = 2LS, \qquad V_0 = I, \qquad V_j \in \{\pm I,\ \pm(2e_i - I)\}. \tag{18}$$

For each $j > 0$ use a six-answer joint question of e_i and the three-way scalar measurement $\Pi_j^+, \Pi_j^-, I - \Pi_j^+ - \Pi_j^-$. For $V_j = \pm I$, bind its binary bit to a synchronized deterministic anchor: its 1-projection is I and its 0-projection is zero. Matching both marginals identifies all joint effects. Appropriate coarse events are the two-coordinate graph projections

$$G_j^{\pm} = \frac{1}{2}\begin{pmatrix} I & \pm V_j \\ \pm V_j & I \end{pmatrix}$$

on coordinates $0, j$, zero elsewhere. A negative reflection sign exchanges the two coarse events.

Add a synchronized d-answer question with projections Π_t^{row}, indexed by $t \in \mathbb{F}_2^{k_{\mathrm{row}}}$. Put $\chi_t(j) = (-1)^{t \cdot j}$. Forbid Π_t^{row} together with G_j^- when $\chi_t(j) = 1$, or G_j^+ when $\chi_t(j) = -1$. The forbidden-event support rule gives actual orthogonality. A vector in $\operatorname{ran} \Pi_t^{\mathrm{row}}$ therefore has coordinates $v_j = \chi_t(j) V_j v_0$.

Define

$$\mathcal{I}_t \xi = d^{-1/2} \big(\chi_t(j) V_j \xi\big)_{j=0}^{d-1}, \qquad \mathcal{P}_t = \mathcal{I}_t \mathcal{I}_t^*.$$

Unitarity and character orthogonality give $\mathcal{I}_t^* \mathcal{I}_u = \delta_{tu} I$ and $\sum_t \mathcal{P}_t = I$. Neither equation requires different V_j to commute. The kernel conditions give $\Pi_t^{\mathrm{row}} \le \mathcal{P}_t$. Since both families sum to identity, $\Pi_t^{\mathrm{row}} = \mathcal{P}_t$ for every t.

Finally forbid Π_0^{row} jointly with the opposite Π_d^{ref}. Then $\Pi_0^{\mathrm{row}} \Pi_d^{\mathrm{ref}} = 0$, equivalently $\Pi_d^{\mathrm{ref}} \mathcal{I}_0 = 0$. The column overlap is

$$\mathcal{I}_0^* \Pi_d^{\mathrm{ref}} \mathcal{I}_0 = d^{-2} \Big(\sum_j V_j\Big)^* \Big(\sum_j V_j\Big).$$

It vanishes exactly when $S = 0$. This proves soundness on arbitrary normal supports.

The forbidden pairs are handled by (N7). Take Q to be the coarse event $G_j^{\mp}$ of the party that asks it and R the opposite synchronized Π_t^{row}. Both reduce the support, so $G_j^{\mp} \Pi_t^{\mathrm{row}} = \Pi_t^{\mathrm{row}} G_j^{\mp} = 0$ on that party's support. The events $G_j^{\pm}$ and Π_t^{row} are products of mirrored projections, so (N4a) and (N3) transfer this identity to the other party. In particular it holds on $P_A \mathcal{H}_A$. Write a vector of $P_A \mathcal{H}_A = \mathbb{C}^d \otimes \mathbb{C}^{d_{\max}/d} \otimes \mathcal{H}'$ as $(v_j)_{j \in \mathbb{F}_2^{k_{\mathrm{row}}}}$, and regard each V_j as a support-restricted reflection, a self-adjoint unitary on $\mathbb{C}^{d_{\max}/d} \otimes \mathcal{H}'$. If $\chi_t(j) = 1$, the equation $G_j^- v = 0$ says $v_j = V_j v_0$. If $\chi_t(j) = -1$, the equation $G_j^+ v = 0$ says $v_j = -V_j v_0$. Thus $\operatorname{ran} \Pi_t^{\mathrm{row}} \subseteq \operatorname{ran} \mathcal{I}_t$, that is, $\Pi_t^{\mathrm{row}} \le \mathcal{P}_t$. The two families are complete, so $\sum_t (\mathcal{P}_t - \Pi_t^{\mathrm{row}}) = 0$ with positive terms, and $\Pi_t^{\mathrm{row}} = \mathcal{P}_t$. The forbidden pair $(\Pi_0^{\mathrm{row}}, \Pi_d^{\mathrm{ref}})$ gives $\Pi_d^{\mathrm{ref}} \mathcal{P}_0 = 0$ by (N7), and $\Pi_d^{\mathrm{ref}} \mathcal{I}_0 = \Pi_d^{\mathrm{ref}} \mathcal{P}_0 \mathcal{I}_0 = 0$. Finally $(\langle +_d | \otimes I) \mathcal{I}_0 = d^{-1} \sum_j V_j = (2L/d) S$, so

$$\|\Pi_d^{\mathrm{ref}} \mathcal{I}_0 \xi\|^2 = \frac{4L^2}{d^2} \|S\xi\|^2$$

for every ξ in the support. Hence $\Pi_d^{\mathrm{ref}} \mathcal{I}_0 = 0$ holds exactly when S vanishes on the support. No inverse of S or of a density is used.

For completeness append $\Phi_{d_{\max}}$, use the scalar Paulis and the actual joint products, and define the column partition by the $\mathcal{I}_t$. Synchronized original projections commute with the marginal and transfer across a purification. Products transfer in reversed order. Define Bob's column effects by those transferred products, equivalently by transpose in canonical Schmidt coordinates. This order matters for noncommuting e_i. All column and synchrony tests hold. The row-zero test holds precisely because $S = 0$. These constructions act trivially on every old measurement's new factor and prove completeness, including for mixed original states.

Let the old strategy have purification Ω and supports P_A, P_B. By (N6c) we may replace every named synchronized effect by its support-restricted projection, completed by a fixed outcome off the support; no probability changes. Adjoin $\Phi_{d_{\max}}$, so that the new purification is $\Omega' = \Omega \otimes \Phi_{d_{\max}}$ and the new state is again normal. Alice's new effects are the displayed

operators: scalar Paulis, joint products $E_i\varsigma$ and $E_i\Pi_j^{\pm}$ of commuting projections, and the column projections $\mathcal{P}_t = \mathcal{I}_t\mathcal{I}_t^*$, all defined on the support and completed by a fixed outcome off it. For an Alice operator $\sum_k z_k \otimes p_k(e)$, with scalar factors z_k and products $p_k(e)$ of named projections, Bob uses $\sum_k z_k^{\mathsf{T}} \otimes p_k^{\mathrm{rev}}(e^B)$, where $p_k^{\mathrm{rev}}(e^B)$ is the reversed product of his matched projections. By (N4a) and $(z \otimes I)\Phi_{d_{\max}} = (I \otimes z^{\mathsf{T}})\Phi_{d_{\max}}$, it is a mirror of the Alice operator. This assignment commutes with adjoints, so the counterpart Y of an Alice projection P is self-adjoint. Moreover $(I \otimes (Y^2 - Y) \otimes I)\Omega' = ((P^2 - P) \otimes I)\Omega' = 0$, so Y is a projection on Bob's support by (N3), and the counterparts of the column projections sum to the identity. Every matching test therefore has disagreement probability zero. Every forbidden pair has probability zero because the corresponding Alice product vanishes, and the row-zero pair does so because $S = 0$ on the support. The old measurements act as $E \otimes I_{d_{\max}}$, so all old tests, including nonsynchronous ones, keep their probabilities.

Write h_{old} for the trace-one marginal before the row ancilla is appended (including the qutrit frame when present). For the honest column construction, without assuming $S = 0$, the sole row-zero event has probability

$$p(\Pi_0^{\mathrm{row}}, \Pi_d^{\mathrm{ref}}) = \frac{4L^2}{d^3} \operatorname{Tr}(h_{\mathrm{old}} S^2). \tag{19}$$

Indeed the scalar density is I_d/d, the column isometry is normalized by $d^{-1/2}$, and the reference vector by $d^{-1/2}$. The overlap computed above supplies $(2LS)^*(2LS)$. Spectator factors have normalized trace one. All column-graph and matching tests remain exact even when the row is nonzero. This formula is [23, Eq. (32)] in the general row setting.

Bob's reference effect is the transpose of Π_d^{ref} on the scalar factor and is a mirror of Π_d^{ref}. The probability is therefore $\|(\mathcal{P}_0 \otimes (\Pi_d^{\mathrm{ref}})^{\mathsf{T}})\Omega'\|^2 = \operatorname{Tr}(h'\Pi_d^{\mathrm{ref}}\mathcal{P}_0\Pi_d^{\mathrm{ref}})$, where $h' = (I_d/d) \otimes (d/d_{\max}) I_{\mathrm{spectator}} \otimes h_{\mathrm{old}}$. Since $(\langle +_d| \otimes I)\mathcal{I}_0 = (2L/d)S$, we have $\Pi_d^{\mathrm{ref}}\mathcal{P}_0\Pi_d^{\mathrm{ref}} = |+_d\rangle\langle +_d| \otimes I_{\mathrm{spectator}} \otimes (4L^2/d^2)S^2$. The trace against h' is (19): the factor $1/d$ is $\langle +_d|I_d/d|+_d\rangle$, and the spectator trace is one. The calculation uses only that the V_j are self-adjoint unitaries, so it holds for every row, zero or not.

The commuting form of rational rows. On a normal commuting host the same finite extension is used. The Pauli calculation of Section 4.1.3 identifies the tensor matrix algebra, and old projections are $I \otimes e_i$ with centralizer scalar matrix units. A forbidden pair with mirror $\mathcal{J}G\mathcal{J}$ gives $\Pi_t^{\mathrm{row}} G\xi_\varphi = 0$, since $\mathcal{J}G\mathcal{J}\xi_\varphi = G\xi_\varphi$; separation gives $\Pi_t^{\mathrm{row}} G = 0$. The remaining soundness argument is unchanged. Conversely amplify by $\mathrm{Mat}_{d_{\max}}$ with normalized trace. Use the actual scalar Paulis, joint products and projections $\mathcal{P}_t$. All are centralizer projections. Bob uses $\mathcal{J}_{\mathrm{new}} X \mathcal{J}_{\mathrm{new}}$ for each projection X. For a centralizer word xy, $\mathcal{J}(xy)\mathcal{J}\xi_\varphi = (xy)^*\xi_\varphi$; adjunction supplies the reversed order. Thus Bob's products are the actual transferred products, even when the old projections do not commute. Every structural forbidden event is zero, and the final event is zero precisely for a zero row. Old measurements act trivially on the new matrix factor, preserving all old probabilities. Finite matrix amplification preserves all indicated factor types. This proves Lemma 4.3.

4.4 Admissible presentations

Definition 4.4 (Admissible presentation; normal realization)**.** Fix an integer $n \geq 3$ and put $\nu = 2^{-n}$. The four-isometry normal model requires $4\nu < 1$; the generic chained-bridge statement of Lemma 4.6 separately permits $n \geq 1$. An admissible presentation is a finite rational list with four contraction symbols A_α ($\alpha \in \{0,1\}^2$), a centralizer projection label $P_\circ$ for the root, and finitely many centralizer labels, each typed as a unitary, projection, positive contraction ($0 \leq T \leq I$), or general contraction ($\|T\| \leq 1$). Its application equations (the

identities enforced by the final row tests) are rational $*$-polynomials in the centralizer labels, the creation and range equations

$$A_\alpha^* A_\beta = \delta_{\alpha\beta} I, \qquad \sum_\alpha A_\alpha A_\alpha^* = I - P_\circ,$$

and endpoint equations $L_{\mathrm{end}} A_\alpha = A_\beta R_{\mathrm{end}}$ or $L_{\mathrm{end}} A_\alpha = 0$. Finite sums $\sum_\iota L_\iota A_{\alpha_\iota} L'_\iota = 0$ are allowed. Here $L_{\mathrm{end}}, R_{\mathrm{end}}, L_\iota, L'_\iota$ are centralizer polynomial expressions. Apart from the displayed creation and range equations, no monomial contains more than one creation symbol. Every label used on a left endpoint is included in the transported unitary list Λ defined in the proof of Lemma 4.5, and the root projection is transported even when it occurs in no endpoint equation.

A normal realization consists of a nonzero Hilbert space $\mathcal{H}_0$, a faithful positive trace-class operator h_0 with $\operatorname{Tr} h_0 = 1$, and bounded operators of the declared types on $\mathcal{H}_0$. Every centralizer label, including $P_\circ$, commutes with h_0. All application equations hold as operator identities on $\mathcal{H}_0$, and

$$h_0 A_\alpha = \nu A_\alpha h_0. \tag{20}$$

Thus all identities are imposed on the state support, where the density is faithful. No positive lower spectral bound is required for a positive contraction. Finite entropy of a realization means $S(h_0) < \infty$. The ratio is imposed by the assembly, rather than by a real coefficient in the Boolean game. The spatial hardness proof of Sections 5–8 uses $n = 4$, so $\nu = 1/16$; Section 16 uses $n = 5$.

4.5 The n-bridge compiler

Each FMS bridge forces a density ratio of two between consecutive selected corners (Figure 2). A chain of n bridges therefore gives the ratio 2^n from source to far corner, and transporting a source-to-far creation operator back to the source gives $\nu = 2^{-n}$. Set

$$\mathfrak{c}_n = 1 + 4\sum_{g=1}^{n} 2^g = 2^{n+3} - 7, \qquad \mu_n = \mathfrak{c}_n^{-1}(1; 2^{\oplus 4}; 4^{\oplus 4}; \ldots; (2^n)^{\oplus 4}). \tag{21}$$

The notation $t^{\oplus 4}$ denotes four entries equal to t.

Lemma 4.5 (n-bridge compiler)**.** *Every admissible presentation effectively yields a finite rational Boolean game $G^{(n)}$. Every perfect normal spatial strategy yields a normal realization in the sense of Definition 4.4, on the nonzero source corner with faithful trace-one density $h_0 = \mathfrak{c}_n e_0 h e_0$, where h is the marginal density. Every such normal realization extends to a perfect normal spatial strategy after adjoining the scaffold and finite ancillary factors (scaffold, qutrit and row factors). Finite marginal entropy is preserved by the completeness construction.*

The assembly parameters are

quantity	value
orthogonal scaffold corners	$4n + 1$
frame-bound parameters	$6n + 1$
cover constant	$\mathfrak{c}_n = 2^{n+3} - 7$
chain covariance	$hv = 2^n vh$
normalized source density	$\mathfrak{c}_n e_0 h e_0$
far-corner weight relative to source	2^n

For $n = 4$ there are seventeen corners and twenty-five parameters, $\mathfrak{c}_4 = 121$, $hv = 16vh$ and $h_0 = 121 e_0 h e_0$.

Proof. We give the assembly and both directions.

Corners and parameters. Use $4n+1$ orthogonal synchronized corners

$$e_0; \quad m_{g,00}, m_{g,01}, m_{g,10}, m_{g,11} \quad (1 \le g \le n).$$

Put $e_g = m_{g,00}$ and $m_g = \sum_{ab} m_{g,ab}$, and impose the cover $e_0 + \sum_g m_g = I$. Conditional bridge g has flag e_{g-1}, magic support m_g and activity $E_g^{\mathrm{act}} = e_{g-1} + m_g$. Use all event bindings described above in both orientations, including the flag/four-magic/inactive partition. Bind to the qutrit frame the $4n+1$ corners and the $2n$ positive ZI, IX events. These are the $6n+1$ frame-bound parameters. Introduce $2n$ axis labels $\mathfrak{x}_g, \mathfrak{z}_g$ with the explicit equations

$$\mathfrak{x}_g = I - 2m_g + 2E_{g,ZI,+}, \qquad \mathfrak{z}_g = I - 2m_g + 2E_{g,IX,+},$$

where $E_{g,ZI,+}, E_{g,IX,+}$ are the positive nonflag events of bridge g, bound to synchronized binary questions as above. The calibrated partial isometries satisfy $\mathsf{U}_g^*\mathsf{U}_g = e_{g-1}$, $\mathsf{U}_g\mathsf{U}_g^* = e_g$ and $h\mathsf{U}_g = 2\mathsf{U}_g h$. The other three magic corners are identified by synchronized Pauli matrix units and have the same density block as e_g. Thus

$$v = \mathsf{U}_n \cdots \mathsf{U}_1, \quad v^*v = e_0, \quad vv^* = e_n, \quad hv = 2^n vh. \tag{22}$$

Trace cyclicity gives four magic masses twice the preceding flag mass in each bridge. The cover fixes

$$\mathrm{Tr}(he_0) = 1/\mathfrak{c}_n; \qquad 121 = \mathfrak{c}_4 = 1 + 4(2 + 4 + 8 + 16). \tag{23}$$

In particular the source corner is nonzero. Let a perfect normal spatial strategy be given, with purification and supports as in (N1). By (N5) every corner, frame and graph question yields a synchronized projection on $P_A\mathcal{H}_A$ commuting with h_A, and by Section 4.2 and Lemma 4.3 every submitted equation, including the cover, holds on $P_A\mathcal{H}_A$. Projections with sum I are mutually orthogonal, so the $4n+1$ corners partition $P_A\mathcal{H}_A$. Bridge g is a perfect conditional FMS strategy with activity $e_{g-1} + m_g$, flag e_{g-1} and magic support m_g, whose magic directions are the bound corners $m_{g,ab}$. Lemma 4.1 gives partial isometries U_g with $\mathsf{U}_g^*\mathsf{U}_g = e_{g-1}$, $\mathsf{U}_g\mathsf{U}_g^* = e_g$ and $h_A\mathsf{U}_g = 2\mathsf{U}_g h_A$ on the support. Hence $v^*v = e_0$ and $vv^* = e_n$, and moving h_A through the n factors gives $h_A v = 2^n v h_A$. For the masses, h_A is trace class and U_g is bounded, so cyclicity gives

$$\mathrm{Tr}(h_A e_g) = \mathrm{Tr}(h_A \mathsf{U}_g \mathsf{U}_g^*) = \mathrm{Tr}(\mathsf{U}_g^* h_A \mathsf{U}_g) = 2\,\mathrm{Tr}(\mathsf{U}_g^* \mathsf{U}_g h_A) = 2\,\mathrm{Tr}(h_A e_{g-1}).$$

Each t_{ab} of bridge g commutes with h_A, so $\mathrm{Tr}(h_A m_{g,ab}) = \mathrm{Tr}(h_A e_g)$. Summing over the cover, with $\mathrm{Tr}\, h_A = 1$, gives (23).

$$e_0 \xrightarrow{\mathsf{U}_1} \boxed{m_1 \supset e_1} \xrightarrow{\mathsf{U}_2} \boxed{m_2 \supset e_2} \xrightarrow{\mathsf{U}_3} \boxed{m_3 \supset e_3} \xrightarrow{\mathsf{U}_4} \boxed{m_4 \supset e_4}$$

corner	e_0	e_1	e_2	e_3	e_4
$121\,\mathrm{Tr}(he_g)$	1	2	4	8	16
$121\,\mathrm{Tr}(hm_g)$	–	8	16	32	64

Figure 2: Four calibrated bridges ($n = 4$). Each m_g has four equal-mass magic corners; its selected e_g is the next flag. For n bridges the selected masses are $2^g/\mathfrak{c}_n$, and returning along the chain gives the creation ratio $\nu = 2^{-n}$.

Every transported label. Encode a unitary directly, a projection P by $2P - I$, a positive contraction T_+ by a unitary u_+ with $T_+ = (2I + u_+ + u_+^*)/4$, and a general contraction by an average of two unitaries. Let Λ be the resulting finite unitary label list that must be

transported; it contains the encoding of $P_\circ$ and all unitary encodings required at left endpoints. For every $\ell \in \Lambda$, $r = 0, \dots, n$, introduce a unitary $u_{r,\ell}$ and impose

$$(u_{r,\ell} - I)(I - e_r) = 0. \tag{24}$$

The unitary reduces e_r and is identity on its complement. For a typed label L, its corner copy L_r is explicitly decoded as

type of L	L_r
unitary	$e_r u_{r,L}$
projection	$e_r(I + u_{r,L})/2, \quad u_{r,L}^2 = I$
positive contraction	$e_r(2I + u_{r,L} + u_{r,L}^*)/4$
general contraction	$e_r(u_{r,L,+} + u_{r,L,-})/2$

Thus $(P_\circ)_n$ denotes the decoded root projection at e_n. The two encodings of a general contraction are transported separately when that label occurs at a left endpoint.

For each $g = 1, \dots, n$ and every $\ell \in \Lambda$ introduce a unitary $V_{g,\ell}$ with

$$V_{g,\ell} = I - E_g^{\mathrm{act}} + e_{g-1} u_{g-1,\ell} + \sum_{a,b} \mathfrak{x}_g^a \mathfrak{z}_g^b e_g u_{g,\ell} \mathfrak{z}_g^{-b} \mathfrak{x}_g^{-a}. \tag{25}$$

For this bridge, apply Lemma 4.2 to the six projections

$$F_1, F_2, F_3, G_{12}(1), G_{13}(1), G_{23}(1)$$

and the $|\Lambda|$ graph projections $G_{12}(V_{g,\ell})$. Thus there are $6 + |\Lambda|$ refinements of each retained sign question per bridge, on each party. Each label has its own duplicated question; the unchanged opposite questions are always retained. There are $(n+1)|\Lambda|$ support equations and $n|\Lambda|$ block equations.

The six frame commutators make $\mathsf{U}_g = I_3 \otimes \widetilde{\mathsf{U}}_g$. The commutator of the original sign with $G_{12}(V_{g,\ell})$, in its (c_0, f) block, is precisely

$$\mathsf{U}_g u_{g-1,\ell} = u_{g,\ell} \mathsf{U}_g. \tag{26}$$

For clarity the sign has off-diagonal block U_g between flag and selected magic direction, while the block unitary has the respective diagonal blocks $u_{g-1,\ell}, u_{g,\ell}$. Their commutator equates these two products. Consequently $L_n v = v L_0$ for every decoded transported label, and for every expression built from them. Here L_r means its decoded copy at e_r, whose corner identity is e_r. The same actual U_g works for every label. Distinct transported labels need not commute with each other.

Equation (24) says $u_{r,\ell}(I - e_r) = I - e_r$. Multiplying by $u_{r,\ell}^*$ gives $u_{r,\ell}^*(I - e_r) = I - e_r$, and the adjoint of this is $(I - e_r)u_{r,\ell} = I - e_r$. Hence $u_{r,\ell}$ commutes with e_r and is the identity on $(I - e_r)P_A\mathcal{H}_A$. On the active space of bridge g, the (a, b) term of (25) is $t_{ab}(e_g u_{g,\ell} e_g)t_{ab}^*$, because $\mathfrak{x}_g, \mathfrak{z}_g$ are reflections and $\mathfrak{x}_g^a \mathfrak{z}_g^b e_g = t_{ab}$. So $V_{g,\ell}$ is block diagonal for $e_{g-1} \oplus \bigoplus_{ab} m_{g,ab}$, with the block $u_{g-1,\ell}$ on the flag and transported copies of $u_{g,\ell}$ on the magic corners; in particular it reduces e_{g-1} and e_g. The six frame refinements make the sign $\mathcal{X}_0$ of bridge g commute with F_1, F_2, F_3 and $G_{ij}(1)$. These generate $\mathrm{Mat}_3 \otimes I$ in the frame coordinates of Section 4.2, whose commutant is $I_3 \otimes B(\mathcal{H}_{\mathrm{base}})$, and the corners are frame-bound; hence $\mathsf{U}_g = I_3 \otimes \widetilde{\mathsf{U}}_g$. The refinement by $G_{12}(V_{g,\ell}) = \frac{1}{2}(F_1 + F_2 + e_{12} \otimes V_{g,\ell} + e_{21} \otimes V_{g,\ell}^*)$ makes $\mathcal{X}_0$ commute with it, and the $(1, 2)$ frame block gives $[\widetilde{\mathcal{X}}_0, V_{g,\ell}] = 0$ on the base. With respect to $e_{g-1} \oplus e_g$,

$$\widetilde{\mathcal{X}}_0 = \begin{pmatrix} 0 & \beta^* \\ \beta & 0 \end{pmatrix}, \qquad V_{g,\ell} = \begin{pmatrix} u_{g-1,\ell} & 0 \\ 0 & u_{g,\ell} \end{pmatrix},$$

and the lower-left block of the commutator is $\beta u_{g-1,\ell} - u_{g,\ell}\beta$. This is (26). Induction over g gives $u_{n,\ell}v = vu_{0,\ell}$. Multiplying on the left by $u^*_{n,\ell}$ and on the right by $u^*_{0,\ell}$ gives $u^*_{n,\ell}v = vu^*_{0,\ell}$. The decoded copies in the table are polynomials in $e_r, u_{r,L}, u^*_{r,L}$ with the same coefficients for every r, and $e_n v = v = ve_0$. Hence $L_n v = vL_0$ and $L^*_n v = vL^*_0$, and the same holds for sums and products of decoded labels.

Extraction and exact translation. Introduce four centralizer contractions $a_\alpha = (a_{\alpha,+} + a_{\alpha,-})/2$ and impose $a_\alpha = e_n a_\alpha e_0$. On the root corner extract $A_\alpha = v^* a_\alpha$. The adjoint of (22) gives $hv^* = 2^{-n}v^*h$, so $h_0 A_\alpha = \nu A_\alpha h_0$, where $h_0 = \mathfrak{c}_n e_0 h e_0$. Supply the following polynomials to the graph compiler:

Root identity	Centralizer identity
$A^*_\alpha A_\beta = \delta_{\alpha\beta} I$	$a^*_\alpha a_\beta = \delta_{\alpha\beta} e_0$
$\sum_\alpha A_\alpha A^*_\alpha = I - P_\circ$	$\sum_\alpha a_\alpha a^*_\alpha = e_n - (P_\circ)_n$
$L_{\mathrm{end}} A_\alpha = A_\beta R_{\mathrm{end}}$	$(L_{\mathrm{end}})_n a_\alpha = a_\beta (R_{\mathrm{end}})_0$
$L_{\mathrm{end}} A_\alpha = 0$	$(L_{\mathrm{end}})_n a_\alpha = 0$
$\sum_\iota L_\iota A_{\alpha_\iota} L'_\iota = 0$	$\sum_\iota (L_\iota)_n a_{\alpha_\iota} (L'_\iota)_0 = 0$
Root-only polynomial	Same polynomial in source copies

Expand the finite words, bind their parameters, apply graphs and triangles, and then Lemma 4.3. Retain all FMS and refinement questions. Pad the two question sets if necessary and sample uniformly from all pairs, accepting every unspecified pair. The result has rational question distribution and a Boolean predicate. The support arguments prove soundness on a nonzero corner, using Alice alone; mirror equations are unnecessary.

Put $\mathcal{H}_0 = e_0 P_A \mathcal{H}_A$ and let h_0 be the restriction of $\mathfrak{c}_n e_0 h_A e_0$ to $\mathcal{H}_0$. It is positive and trace class, it is faithful because h_A is faithful on $P_A\mathcal{H}_A$ and commutes with e_0, and it has trace one by (23); in particular $\mathcal{H}_0 \neq 0$. Every source copy L_0 maps $\mathcal{H}_0$ into itself and commutes with h_0, because $u_{0,L}$ commutes with e_0 and with h_A. The declared types hold: $e_0 u_{0,L}$ is unitary on $\mathcal{H}_0$; $e_0(I + u_{0,L})/2$ is a projection when $u^2_{0,L} = I$; $e_0(2I + u + u^*)/4$ is a positive contraction; and an average of two unitaries is a contraction. The contraction $a_\alpha = e_n a_\alpha e_0$ commutes with h_A, so $A_\alpha = v^* a_\alpha$ maps $\mathcal{H}_0$ into itself, and

$$h_0 A_\alpha = \mathfrak{c}_n\, e_0 h_A v^* a_\alpha = 2^{-n}\mathfrak{c}_n\, v^* h_A a_\alpha = 2^{-n}\mathfrak{c}_n\, v^* a_\alpha e_0 h_A e_0 = 2^{-n} A_\alpha h_0,$$

using $e_0 v^* = v^*$, $h_A v^* = 2^{-n} v^* h_A$, $a_\alpha = a_\alpha e_0$ and $[e_0, h_A] = 0$. The translations are exact. For the creation equations, $A^*_\alpha A_\beta = a^*_\alpha v v^* a_\beta = a^*_\alpha e_n a_\beta = a^*_\alpha a_\beta$. For the range sum, $\sum_\alpha A_\alpha A^*_\alpha = v^*(\sum_\alpha a_\alpha a^*_\alpha)v$. If $\sum_\alpha a_\alpha a^*_\alpha = e_n - (P_\circ)_n$, this equals $e_0 - v^*(P_\circ)_n v = e_0 - (P_\circ)_0$; conversely, conjugation by v recovers the centralizer form, because $vv^* = e_n$ and $a_\alpha = e_n a_\alpha$. For an endpoint equation, $v^* L_n = L_0 v^*$ gives $L_0 A_\alpha - A_\beta R_0 = v^*(L_n a_\alpha - a_\beta R_0)$. The bracket has range in $e_n P_A \mathcal{H}_A$, where v^* is isometric, so either side vanishes exactly when the other does. Sums $\sum_\iota L_\iota A_{\alpha_\iota} L'_\iota$ translate in the same way, with $(L_\iota)_n$ on the left. Root-only polynomials involve only source copies. Every application equation therefore holds as an operator identity on $\mathcal{H}_0$.

One simultaneous realization. Given a root realization with density h_0, take

$$h_{\mathrm{sc}} = \mathrm{diag}(\mu_n) \otimes h_0 = \frac{1}{\mathfrak{c}_n} \mathrm{diag}(h_0;\, 2h_0^{\oplus 4};\, 4h_0^{\oplus 4};\, \ldots;\, 2^n h_0^{\oplus 4}). \tag{27}$$

Use identity corner identifications in the bridges and $a_\alpha = |e_n\rangle\langle e_0| \otimes A_\alpha$. Its commutation with h_{sc} is exactly the modular relation. Every positive eigenspace of a faithful trace-class

density is finite-dimensional. In each such block the polar partial isometry of a contraction T extends to a unitary V_{pol}, and

$$T = (U_+ + U_-)/2, \qquad U_\pm = V_{\mathrm{pol}}(|T| \pm i\sqrt{I - |T|^2}).$$

Direct sums give centralizer encodings for all four a_α at once. The root isometries A_α are never themselves written as averages of centralizer unitaries. The contraction completion takes place on the scaffold. Encode each source contraction inside the source density blocks, choose $2P - I$ for projections, and use

$$u_+ = 2T_+ - I + 2i\sqrt{T_+(I - T_+)}$$

for a positive contraction T_+. Copy these same choices through the corner identifications and extend by identity outside the corners. The block equations now hold simultaneously. Append the scalar qutrit frame, use the graph formulas, and take actual joint products for all refinements. Their block unitaries commute with the coarse FMS effects. Separate labels need not commute. Finally append the finite maximally entangled row factor of dimension $d_{\max}$. All tests hold with one state and one assignment. The scaffold and ancillary dimensions are finite, so they add only finite entropy. Let $(\mathcal{H}_0, h_0)$ be a normal realization, and put $\mathcal{H}_{\mathrm{sc}} = \mathbb{C}^{4n+1} \otimes \mathcal{H}_0$ with the density (27). It is faithful and trace class, of trace one by the definition of $\mathfrak{c}_n$. Its positive eigenspaces are spanned by corner vectors tensored with positive eigenvectors of h_0, so they are finite-dimensional, and a bounded operator commuting with h_{sc} maps each of them into itself by (N4d). On such a block $\mathcal{E}$ the contraction $T = a_\alpha|_{\mathcal{E}}$ has a polar decomposition $T = V|T|$. Because $\dim\ker T = \dim\ker T^*$ in finite dimension, V extends to a unitary V_{pol} with $T = V_{\mathrm{pol}}|T|$, and $|T| \pm i(I - |T|^2)^{1/2}$ are commuting unitaries. The direct sums of the resulting unitaries over all blocks are unitaries commuting with h_{sc}. Finite-dimensional blocks are needed here: a nonunitary isometry, such as a root isometry A_α, is an extreme point of the unit ball and is therefore not an average of two unitaries. That is why A_α is never encoded directly. The other encodings are functions of operators commuting with h_0, so they commute with h_0. The copies $u_{r,\ell} = |e_r\rangle\langle e_r| \otimes U_\ell + (I - |e_r\rangle\langle e_r|) \otimes I$ satisfy (24); the block unitaries (25) act as U_ℓ on every corner of bridge g and as I elsewhere; and $\mathsf{U}_g = |e_g\rangle\langle e_{g-1}| \otimes I$ gives (26). The translated equations hold because, for example, $L_n a_\alpha - a_\beta R_0 = |e_n\rangle\langle e_0| \otimes (LA_\alpha - A_\beta R)$.

The strategy is explicit. Take $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^3 \otimes \mathcal{H}_{\mathrm{sc}} \otimes \mathbb{C}^{d_{\max}}$, the density $h = (I_3/3) \otimes h_{\mathrm{sc}} \otimes (I_{d_{\max}}/d_{\max})$ and the pure state Ω_h of (N8), formed in a product eigenbasis. Every synchronized question of Alice measures a projection commuting with h: corner, frame, graph, triangle and binding projections built from the centralizer unitaries by the formulas of Section 4.2; the FMS cell, activity and corner projections; and the row projections of Lemma 4.3. Bob measures their transposes. By (N8) every matching test passes. Every forbidden pair between centralizer events has probability $\mathrm{Tr}(hQR) = 0$, because the defining identities give $QR = 0$. The FMS sign questions are not in the centralizer. For bridge g, Alice's sign observable for T_x is the reflection $|f\rangle\langle c_x| + |c_x\rangle\langle f|$ on the span of the flag corner e_{g-1} and the corner c_x, tensored with the identity of $\mathcal{H}_0$. Bob's sign observable for B_y is $|f\rangle\langle r_y| + |r_y\rangle\langle f|$, with $r_y = (c_0 + (-1)^y c_1)/\sqrt{2}$. Their retained answers are the remaining column-zero and row-zero projections. If μ is the mass of e_{g-1}, each magic corner has mass 2μ, and $\langle c_x, r_y\rangle = \epsilon_{xy}/\sqrt{2}$. So, up to the frame and row factors, the compressed state is $\Omega_{xy} = \mu^{1/2}(|f, f\rangle + \epsilon_{xy}|c_x, r_y\rangle) \otimes \Omega_{h_0}$. Substitution gives $(\mathcal{X}_x \otimes I)\Omega_{xy} = \epsilon_{xy}(I \otimes \mathcal{Y}_y)\Omega_{xy}$, so every forbidden sign pair has probability zero; the calibrated rules against C_0 and R_0 hold by construction. The FMS effects are corner operators tensored with the identity of $\mathcal{H}_0$. They commute with every block unitary $V_{g,\ell}$ and with the frame, on Alice's side and, after transposition in the product eigenbasis, on Bob's. The refinements $E_a P^{\mathrm{lab}}$ of Lemma 4.2 are therefore POVMs. The marginal entropy is $\log_2 3 + S(h_{\mathrm{sc}}) + \log_2 d_{\max}$, where $S(h_{\mathrm{sc}}) = H(\mu_n) + S(h_0)$. □

4.6 Chained bridges on normal commuting hosts

The same assembly works on arbitrary normal commuting supports. Here the edges may be any finite family and the range equation may carry a transported defect.

Lemma 4.6 (Chained bridges)**.** *For any fixed $n \geq 1$, the conditional FMS predicates, individual label refinements and rational rows above compile the following finite root specifications on arbitrary normal commuting supports. The centralizer labels have prescribed types (projection, positive contraction, contraction or unitary). Each edge A_α has modular ratio 2^{-n}, with isometry and range equations and endpoint relations $L_{\mathrm{end}}A_\alpha = A_\beta R_{\mathrm{end}}$ or $L_{\mathrm{end}}A_\alpha = 0$; $L_{\mathrm{end}}, R_{\mathrm{end}}$ are real rational polynomials in the typed centralizer labels (for example $R_{\mathrm{end}} = C^2/2$). Other relations are root centralizer polynomials with real rational coefficients. Every perfect strategy extracts that root specification on a nonzero corner. Every faithful normal root realization extends simultaneously to a perfect strategy after finite amplification, preserving its factor type and, for a spatial finite-entropy model, finiteness of entropy. Only labels occurring at a left endpoint (including a range-equation label) need transport; transporting further labels is permitted and changes neither direction.*

Proof. Use the synchronized partition

$$e_0 + \sum_{g=1}^{n} m_g = I, \qquad m_g = \sum_{\epsilon,\delta\in\{0,1\}} m_{g,\epsilon\delta}, \qquad e_g = m_{g,00}.$$

For bridge g use a conditional FMS copy with flag e_{g-1}, magic support m_g, and activity $E_g^{\mathrm{act}} = e_{g-1} + m_g$. The inactive effect is $I - E_g^{\mathrm{act}}$. Match activity on every conditional question and bind the flag and four chosen magic events to their named binary projections, in both party orientations. The flag/four-magic/inactive column partition then has precisely these six marginals. Different copies have disjoint question names; no agreement between their nonsynchronous signs is imposed.

The commuting bridge of Section 4.1.3 applies in each $E_g^{\mathrm{act}} M E_g^{\mathrm{act}}$. Indeed E_g^{act} is centralizer, so its normalized restriction has the restricted modular group. It gives $\mathsf{U}_g^*\mathsf{U}_g = e_{g-1}$, $\mathsf{U}_g\mathsf{U}_g^* = e_g$ and $\sigma_t^{\varphi}(\mathsf{U}_g) = 2^{it}\mathsf{U}_g$. These equations extend by zero off E_g^{act} and imply

$$v = \mathsf{U}_n \cdots \mathsf{U}_1, \quad v^*v = e_0, \quad vv^* = e_n, \quad \sigma_t^{\varphi}(v) = 2^{nit}v.$$

This is an extracted operator, never a centralizer word submitted to the graph compiler. KMS and the magic matrix units give

$$\varphi(m_{g,\epsilon\delta}) = 2^g \varphi(e_0), \qquad \varphi(e_0) = \mathfrak{c}_n^{-1}.$$

If an active corner vanishes its flag and all four magic corners vanish together; otherwise the local bridge makes all five nonzero. The recurrence therefore holds also before nonvanishing is known, and the cover proves that $e_0 \neq 0$.

Bind every corner and every positive Pauli parameter to the same qutrit frame. Let Λ be the unitary encodings of the labels to be transported, including every label used on a left endpoint: use a unitary directly, a reflection for a projection, the displayed functional calculus for a positive contraction, and two unitaries for a general contraction. Introduce the axis labels $\mathfrak{x}_g, \mathfrak{z}_g$ of each bridge as in Lemma 4.5. For each $\ell \in \Lambda$ and $r = 0, \ldots, n$, impose $(u_{r,\ell} - I)(I - e_r) = 0$, and introduce the unitary label $V_{g,\ell}$ of (25). The orthogonal block supports make this a unitary. At bridge g apply Lemma 4.2 separately to the six frame projections and to each $G_{12}(V_{g,\ell})$. There is one refinement of each retained sign question per label per bridge, on each party, always against the unchanged opposite questions. There is no

joint refinement of distinct labels. The frame refinements give $\mathsf{U}_g = I_3 \otimes \widetilde{\mathsf{U}}_g$. In the selected magic–flag block the graph commutator is exactly

$$\mathsf{U}_g u_{g-1,\ell} = u_{g,\ell} \mathsf{U}_g.$$

Induction gives $(L_{\mathrm{end}})_n v = v (L_{\mathrm{end}})_0$ for every decoded transported label and its adjoints and words. Labels used only at the source need one source encoding and no refinement.

For each edge introduce a centralizer contraction $a_\alpha = e_n a_\alpha e_0$ and extract $A_\alpha = v^* a_\alpha$. Compile the centralizer equations

$$a_\alpha^* a_\beta = \delta_{\alpha\beta} e_0, \qquad \sum_\alpha a_\alpha a_\alpha^* = e_n - (P_\circ)_n, \qquad (L_{\mathrm{end}})_n a_\alpha = a_\beta (R_{\mathrm{end}})_0$$

for the corresponding root equations; the zero endpoint uses zero on the right. The equations with a specified initial defect use that defect in the first right side. More generally a range equation $\sum_\alpha A_\alpha A_\alpha^* = R_{\mathrm{ran}}$, with R_{ran} a transported centralizer polynomial, is compiled as $\sum_\alpha a_\alpha a_\alpha^* = (R_{\mathrm{ran}})_n$. For the Catalan input of [23, Section 6] this is $R_{\mathrm{ran}} = I - BB^* - P$; its centralizer contraction B and projection P are already in the transported list. Multiplying by v or its adjoint proves both directions, since its initial and final projections are e_0, e_n. Every root-only polynomial stays at e_0. In particular the edge has modular ratio 2^{-n} and all labels refer to the same extracted root operators. Graphs, triangles and Lemma 4.3 enforce these finitely many centralizer polynomials without changing the conditional tests. This proves commuting soundness.

For completeness take a faithful normal root (M_0, φ_0) and on $\mathrm{Mat}_{1+4n} \bar{\otimes} M_0$ put

$$\varphi_{\mathrm{sc}} = \mathrm{diag}(\mu_n) \otimes \varphi_0 = \mathfrak{c}_n^{-1} \,\mathrm{diag}(1;\, 2^{\oplus 4};\, 4^{\oplus 4};\, \ldots;\, (2^n)^{\oplus 4}) \otimes \varphi_0.$$

Choose the actual matrix corner maps for U_g, the ideal real FMS effects on E_g^{act}, and $I - E_g^{\mathrm{act}}$ for every inactive answer. Take $a_\alpha = |e_n\rangle\langle e_0| \otimes A_\alpha$. Its modular ratio is $2^n 2^{-n} = 1$, so it lies in the finite centralizer. Its polar partial isometry therefore has a unitary completion, giving its two-unitary encoding. Choose the root centralizer encodings once and copy them identically to the required corners. All $V_{g,\ell}$ then commute with the FMS effects at their own bridge, even though adjacent activities overlap at e_g. Use their actual joint products for the separate refinements. Append the common scalar frame and row factors and use standard commutant mirrors as in Lemma 4.3. All old conditional probabilities and every added row are simultaneously correct. Only finite matrix factors have been appended. Their state entropy is finite; in the spatial case it adds to the root entropy. This proves completeness and the stated preservation properties. □

4.7 A dimension-independent completeness estimate

The exact construction gives a useful quantitative statement for the four-bridge compiler ($n = 4$). Keep the label types, centralizer conditions, contractivity and modular covariance exact, but allow residuals r_j in the application equations of Definition 4.4. Write $\|T\|_{2,h_0}^2 = \mathrm{Tr}(h_0 T^* T)$. The constructed strategy satisfies

$$1 - \omega(G, \mathcal{S}) \le C_{\mathrm{comp}} \sum_j \|r_j\|_{2,h_0}^2, \tag{28}$$

where C_{comp} depends only on the finite presentation and the compiler.

Here is the proof, including the density factors. The same scaffold, exact unitary encodings and actual products make all structural tests perfect. Only the final application row-zero tests

may lose. Let p_j be the lifted polynomial residual. For a source or root-polynomial row it is the source copy of r_j; for an endpoint row it is vr_j; for a range row it is vr_jv^*. Consequently

$$\|p_j\|^2_{2,h_{\mathrm{sc}}} = \frac{\beta_j}{121}\|r_j\|^2_{2,h_0}, \qquad \beta_j = \begin{cases} 16, & \text{range row at } e_4, \\ 1, & \text{otherwise.} \end{cases}$$

For the endpoint case use $v^*v = e_0$. For the range case use $v^*h_{\mathrm{sc}}v = 16h_0/121$. These are equalities with no density-gap estimate.

The qutrit graph row S_j has off-diagonal blocks p_j, p_j^*. Since p_j commutes with h_{sc},

$$\mathrm{Tr}(h_{\mathrm{old}}S_j^2) = \frac{1}{3}\,\mathrm{Tr}[h_{\mathrm{sc}}(p_j^*p_j + p_jp_j^*)] = \frac{2\beta_j}{363}\|r_j\|^2_{2,h_0}.$$

Apply (19), with L_j and d_j the denominator and slot dimension of row j. For uniform Q-by-Q sampling one may take

$$C_{\mathrm{comp}} = \max_j \frac{8L_j^2\beta_j}{363d_j^3Q^2}. \tag{29}$$

Summing event probabilities gives an upper bound even if rejecting rules overlap. The dimension, clock order, density spectrum and spectator dimensions do not enter this constant. No continuity of polar completions or approximate dense-range cancellation is used: the structural realization is built exactly. This is a completeness estimate for constructed strategies under the stated exact structural hypotheses. It is not a robust extraction claim for arbitrary nearly perfect strategies.

5 The rooted tree and computation squares

Sections 5–7 prove the hardness half of Theorem A. The game is the four-bridge compilation of one admissible presentation, the Fock clock. Its root carries a four-ary tree, each depth of the tree is a finite square, and a deterministic computation runs on the squares. A Weyl clock supplies the existential witness in Section 6.

5.1 Normality removes every rootless sector

Require four isometries and a root projection with

$$A_{ab}^*A_{cd} = \delta_{ac}\delta_{bd}I, \qquad \sum_{ab} A_{ab}A_{ab}^* = I - P_\circ, \qquad h_0A_{ab} = A_{ab}h_0/16. \tag{30}$$

Here h_0 is a faithful density on the extracted root-corner Hilbert space. For $w = w_1\cdots w_m$ set $A_w = A_{w_1}\cdots A_{w_m}$ (the empty product is I) and put $Q_m = \sum_{|w|=m} A_wA_w^*$. Orthogonality and trace cyclicity give $\mathrm{Tr}(h_0Q_m) = 4^{-m}$. The projections Q_m decrease. Their strong limit has zero h_0-weight, hence is zero by faithfulness. Also $Q_m - Q_{m+1} = \sum_{|w|=m} A_wP_\circ A_w^*$. Therefore

$$\mathcal{H} = \bigoplus_w A_w\mathcal{K} \simeq \ell^2(\mathcal{W}) \otimes \mathcal{K}, \qquad \mathcal{K} = P_\circ\mathcal{H} \neq 0. \tag{31}$$

Here $\mathcal{W}$ denotes finite words in the four-letter alphabet $\{0,1\}^2$. The modular equation makes h_0 commute with every $A_wP_\circ A_w^*$. Since $\mathrm{Tr}(h_0P_\circ) = 3/4$, transporting its root block gives

$$h_0 = h_F \otimes \rho_{\mathrm{spec}}, \qquad h_F = \frac{3}{4}\sum_w 16^{-|w|}|w\rangle\langle w|, \qquad \mathrm{Tr}\,\rho_{\mathrm{spec}} = 1. \tag{32}$$

In particular

$$S(h_F) = \log_2(4/3) + 4/3. \tag{33}$$

Indeed the depth-m mass is $(3/4)4^{-m}$ and its mean depth is $1/3$; the log probability at each word is $\log_2(4/3) + 4m$. This proves the formula and excludes a rootless normal summand. Section 16.2 repeats this calculation at a general ratio; there $h_F = h_{F,1/16}$.

5.2 Each depth is a finite square

For $w = (a_1, b_1) \cdots (a_m, b_m)$ define

$$x(w) = \sum_{j=1}^{m} a_j 2^{j-1}, \qquad y(w) = \sum_{j=1}^{m} b_j 2^{j-1}.$$

The first letter is the least significant bit. Prefixing A_{ab} maps (x, y) to $(2x + a, 2y + b)$. Depth m is a $2^m \times 2^m$ square. Require centralizer unitaries U_x, U_y and projections P_x, P_y satisfying, for $a, b \in \{0, 1\}$,

$$U_x A_{0b} = A_{1b}, \qquad U_x A_{1b} = A_{0b} U_x, \qquad U_x P_\circ = P_\circ, \tag{34}$$
$$U_y A_{a0} = A_{a1}, \qquad U_y A_{a1} = A_{a0} U_y, \qquad U_y P_\circ = P_\circ, \tag{35}$$
$$P_x A_{0b} = A_{0b} P_x, \qquad P_x A_{1b} = 0, \qquad P_x P_\circ = P_\circ, \tag{36}$$
$$P_y A_{a0} = A_{a0} P_y, \qquad P_y A_{a1} = 0, \qquad P_y P_\circ = P_\circ. \tag{37}$$

Induction on word length identifies U_x and U_y as cyclic increments of x and y respectively, acting identically on $\mathcal{K}$. The same induction makes P_x, P_y the indicators of $x = 0, y = 0$. In particular U_x, U_y commute, and

$$P_{x,\text{last}} = U_x^* P_x U_x, \qquad P_{y,\text{last}} = U_y^* P_y U_y$$

mark the final column and row. These are conclusions on the full root multiplicity, not just selected vectors.

5.3 A finite deterministic tableau

Choose a deterministic radius-one cellular automaton on the half-line. Its finite alphabet Γ_{CA} includes start `s`, blank `b` and halt `H`, and its update is $\Phi(\ell, c, r, \varepsilon)$ with left boundary symbol # and external pulse $\varepsilon \in \{0, 1\}$. The initial row is $\mathtt{s}, \mathtt{b}, \mathtt{b}, \ldots$. We use these conventions:

- $\Phi(\mathtt{b}, \mathtt{b}, \mathtt{b}, \varepsilon) = \mathtt{b}$ for both pulse values.
- The initial start state ignores the pulse at time zero. Later waiting-front states respond to positive-time pulses.
- Cells passed by the waiting front keep their unary marks. Frozen cells and all states of the later input-checking computation ignore subsequent pulses.
- A transition using the pulse at time y produces row $y + 1$. Thus the pulse at time d_{clk} records the front at coordinate d_{clk} in that next row.

Here is an explicit front layer, with alphabet $\{\mathtt{s}, \mathtt{b}, \mathtt{f}, \mathtt{z}, \mathtt{H}\}$; `f` is the front state. The right neighbour is unused in these six rules:

condition	next centre
$c = \mathtt{s}$	`b`
$c = \mathtt{f}$	`z` if $\varepsilon = 1$, `b` if $\varepsilon = 0$
$c = \mathtt{b},\ \ell = \mathtt{s}$	`f`
$c = \mathtt{b},\ \ell = \mathtt{f}$	`f` if $\varepsilon = 0$, `b` if $\varepsilon = 1$
$c = \mathtt{z}$	`z`
$c = \mathtt{H}$	`H`

All remaining blank centres stay blank. At time $y \geq 1$, until a return, the unique front is at $x = y$. The first positive pulse at d_{clk} writes `z` at $(d_{\text{clk}}, d_{\text{clk}} + 1)$ and suppresses the front's successor. The frozen mark's distance from the left boundary is the unary input d_{clk}; blank cells between the boundary and the mark represent its unary positions. Pulse-dependence is

confined to neighbourhoods containing the front. In particular a blank right of `f` may read the pulse.

Add a persistent unary bit $u \in \{0,1\}$, initially zero, with $u' = 1$ when the old front centre is `s` or `f`, and $u' = u$ otherwise. Thus passed cells retain their unary marks on a separate track, even though their front-layer component is blank. Take the product alphabet $\{\mathtt{s},\mathtt{b},\mathtt{f},\mathtt{z},\mathtt{H}\} \times \{0,1\} \times \mathcal{T}_{\mathrm{TM}}$, with a finite TM alphabet $\mathcal{T}_{\mathrm{TM}} = \mathcal{A}_{\mathrm{tape}} \times (Q_{\mathrm{TM}} \cup \{\text{no head}\})$ for the fixed one-tape half-line program in Section 7. All TM cells start blank without a head. The transition $\mathtt{f} \to \mathtt{z}$ initializes its head in the start state at that cell. Thereafter a centre carrying (a,p) writes the symbol prescribed by $\delta(p,a,u)$ and retains a head only for a stationary move. A head moving right (left) is received by the right (left) neighbour, whose old tape symbol is retained. A centre receiving no head retains its symbol without a head. Invalid multiple incoming heads may be sent to a fixed error state; they never occur from this initialization. The left boundary # is read as a tape end marker and no transition moves left of it. The frozen front track is read-only input for this TM. A halting head writes a designated halt symbol `H` in the combined alphabet and keeps it. These prescriptions give a finite radius-one lookup table effectively from the finite TM transition table; all subsequent TM transitions ignore the pulse. Denote this finite product alphabet, with the halt convention, by Γ_{CA}. In the combined alphabet, `s` and `b` abbreviate the start and fully blank product states. Collapse any product state with a halting head to the single absorbing symbol `H`.

Blank quiescence holds before a front or head arrives. The `s` rule ignores the time-zero pulse and seeds `f` at time one via its right neighbour. The mark is permanent and later pulses cannot restart a head. Finally, all disturbances start at the origin, or at the marked front within its radius-one cone, so no nonblank cell can occur at $x > y$. These checks give all four conventions and the blank right ghost used below.

Let $(P_c)_{c\in\Gamma_{\mathrm{CA}}}$ be a centralizer PVM and P_{ret} a centralizer projection, with $P_{\mathrm{ret},0} = I - P_{\mathrm{ret}}$, $P_{\mathrm{ret},1} = P_{\mathrm{ret}}$. Require

$$P_{\mathtt{s}}P_xP_y = P_xP_y, \qquad P_{\mathtt{b}}(I-P_x)P_y = (I-P_x)P_y, \qquad P_{\mathtt{H}} = 0. \tag{38}$$

Define centralizer polynomials

$$L_{\#} = P_x, \qquad L_c = U_xP_cU_x^*(I-P_x),$$
$$N_r = U_x^*P_rU_x(I-P_{x,\mathrm{last}}) + \mathbf{1}_{\{r=\mathtt{b}\}}P_{x,\mathrm{last}}.$$

For every $a \in \Gamma_{\mathrm{CA}}$ impose

$$\left[U_y^*P_aU_y - \sum_{\Phi(\ell,c,r,\varepsilon)=a} L_\ell P_c N_r P_{\mathrm{ret},\varepsilon}\right](I-P_{y,\mathrm{last}}) = 0. \tag{39}$$

The factors have the displayed order. They need not commute globally. The factor $I - P_{y,\mathrm{last}}$ removes a spurious transition from the final row to row zero. The ghost to the right of the last column is blank.

Lemma 5.1 (Definite computation)**.** *Suppose a root vector ξ belongs to a sector where the clock pulse on every coordinate has a specified scalar value. Then every P_c acts on $A_w\xi$ as the indicator of the genuine classical computation at $(x(w), y(w))$.*

Proof. Work at a fixed depth m and induct on time, for every spatial coordinate at once. Equation (38) gives row zero. On an induction vector every factor $L_\ell, P_c, N_r, P_{\mathrm{ret},\varepsilon}$ acts by its classical zero or one. Their ordered product therefore selects exactly one update. Equation (39) gives $U_y^*P_aU_y$ with the correct scalar on that vector. Multiplying by U_y gives the next-row assertion.

The half-line and the square agree throughout this induction. At time y no disturbance has travelled beyond coordinate y. In particular the ghost cell 2^m is blank for all times $y \le 2^m - 1$, by radius one and quiescence under either pulse. The left ghost is always #. This proves the genuine-prefix assertion. A halting event at finite coordinates and time occurs in every sufficiently large square and contradicts $P_{\mathtt{H}} = 0$. □

The tableaux at distinct depths need not intertwine with the creation operators. Determinism forces each of them to the same finite prefix; soundness needs only one sufficiently large square. Every depth has positive normal weight by (32).

5.4 A worked front and mask example

Table 4 is rendered by `spatial/review/generate_toy.py` from `tableau(3,3)[3]`. Its `d` denotes a passed cell and `m` a frozen mark. These persistent annotations differ from the minimal front layer above, whose passed cells are blank and whose mark is `z`; both put the mark at $(3, 4)$. The pulse at time zero is ignored by `s`. The pulse at $y = 3$ produces the mark in row 4, and the later pulse at $y = 6$ has no effect.

y	pulse	0	1	2	3	4	5	6	7	update mask
0	1	s	b	b	b	b	b	b	b	1
1	0	d	f	b	b	b	b	b	b	1
2	0	d	d	f	b	b	b	b	b	1
3	1	d	d	d	f	b	b	b	b	1
4	0	d	d	d	m	b	b	b	b	1
5	0	d	d	d	m	b	b	b	b	1
6	1	d	d	d	m	b	b	b	b	1
7	0	d	d	d	m	b	b	b	b	0

Table 4: Depth $m = 3$, period $q = 3$: the decoder's 8×8 toy tableau. P_x marks column 0, $P_{x,\mathrm{last}}$ column 7, P_y row 0 and $P_{y,\mathrm{last}}$ row 7. The last-column neighbour is replaced by blank; the last-row mask suppresses the toroidal $7 \to 0$ update.

6 A finite-order clock and its period reader

6.1 A determinant supplies a finite witness

Add centralizer unitaries $J_{\mathrm{W}}, K_{\mathrm{W}}, Z_{\mathrm{clk}}$ with

$$\begin{gathered} J_{\mathrm{W}} K_{\mathrm{W}} = Z_{\mathrm{clk}} K_{\mathrm{W}} J_{\mathrm{W}}, \qquad [Z_{\mathrm{clk}}, J_{\mathrm{W}}] = [Z_{\mathrm{clk}}, K_{\mathrm{W}}] = 0, \\ [J_{\mathrm{W}}, A_{ab}] = [K_{\mathrm{W}}, A_{ab}] = [Z_{\mathrm{clk}}, A_{ab}] = 0. \end{gathered} \tag{40}$$

A unitary commuting with an isometry also commutes with its adjoint: its inverse commutes with the isometry, and taking adjoints gives the assertion. Thus these labels reduce $\mathcal{K} = P_\circ \mathcal{H}$ and commute with ρ_{spec}. Their action on every word sector is their root action.

Every positive eigenspace of ρ_{spec} is finite-dimensional. It reduces $J_{\mathrm{W}}, K_{\mathrm{W}}, Z_{\mathrm{clk}}$ and decomposes into Z_{clk} eigenspaces. On a nonzero such eigenspace of dimension d, with phase ζ, the Weyl relation gives

$$\det(J_{\mathrm{W}} K_{\mathrm{W}}) = \zeta^d \det(K_{\mathrm{W}} J_{\mathrm{W}}), \qquad \zeta^d = 1. \tag{41}$$

Since the positive eigenspaces exhaust $\mathcal{K}$, Z_{clk} has pure point spectrum there and every phase is a root of unity. At least one sector is nonzero. Conversely a d_{clk}-dimensional Weyl pair realizes every primitive d_{clk}th root, with density $I_{d_{\mathrm{clk}}}/d_{\mathrm{clk}}$.

6.2 The clock and a counter without a positive lower bound

Take a centralizer unitary W and a positive centralizer contraction C_{ctr} with

$$WP_\circ = P_\circ, \qquad WA_{ab} = A_{ab}Z_{\mathrm{clk}}^b W^2, \tag{42}$$

$$C_{\mathrm{ctr}}P_\circ = P_\circ, \qquad C_{\mathrm{ctr}}A_{ab} = 2^{-b}A_{ab}C_{\mathrm{ctr}}^2. \tag{43}$$

Also require Z_{clk} to commute with $U_x, U_y, P_x, P_y, W, C_{\mathrm{ctr}}, P_{\mathrm{ret}}, T_{\mathrm{inv}}$ and all P_c. Induction on word length, using the coordinate recurrence, gives

$$W|_{(x,y)} = Z_{\mathrm{clk}}^y, \qquad C_{\mathrm{ctr}}|_{(x,y)} = 2^{-y}I. \tag{44}$$

In particular C_{ctr} is injective even though its spectrum accumulates at zero.

For a centralizer projection P_{ret} and contraction T_{inv}, impose

$$(W-I)P_{\mathrm{ret}} = 0, \qquad T_{\mathrm{inv}}(W-I) = \tfrac{1}{2}C_{\mathrm{ctr}}(I-P_{\mathrm{ret}}). \tag{45}$$

The first equation puts $\operatorname{ran} P_{\mathrm{ret}}$ in $\ker(W-I)$. For ξ in that kernel the second gives $C_{\mathrm{ctr}}(I - P_{\mathrm{ret}})\xi = 0$. Injectivity of C_{ctr} gives $P_{\mathrm{ret}}\xi = \xi$. Thus

$$P_{\mathrm{ret}} = \operatorname{proj}\ker(W-I). \tag{46}$$

On a root phase sector of order d_{clk}, the pulse is exactly $\mathbf{1}_{\{d_{\mathrm{clk}}|y\}}$ at every spatial coordinate. It cannot vanish spuriously or postpone the first return.

6.3 A bounded inverse on every prime clock

Lemma 6.1 (A good primitive phase)**.** *For every prime d_{clk} there is a primitive d_{clk}th root ζ such that*

$$|\zeta^t - 1| \geq 2^{-t-1} \quad (t \geq 1,\ d_{\mathrm{clk}} \nmid t).$$

Proof. For $1 \leq r < d_{\mathrm{clk}}$,

$$|e^{2\pi i r/d_{\mathrm{clk}}} - 1| \geq 4\min(r, d_{\mathrm{clk}} - r)/d_{\mathrm{clk}}.$$

Fix $1 \leq t < d_{\mathrm{clk}}$. Multiplication by t permutes nonzero residues. A phase $e^{2\pi i k/d_{\mathrm{clk}}}$ violating the required inequality must satisfy $\min(tk \bmod d_{\mathrm{clk}}, d_{\mathrm{clk}} - (tk \bmod d_{\mathrm{clk}})) < d_{\mathrm{clk}}2^{-t-3}$. There are at most $d_{\mathrm{clk}}2^{-t-2}$ such residues. Summing over t excludes fewer than $d_{\mathrm{clk}}/4$ choices, while $d_{\mathrm{clk}}/4 < d_{\mathrm{clk}} - 1$. One of the $d_{\mathrm{clk}} - 1$ nontrivial phases works. For $t \geq d_{\mathrm{clk}}$ reduce to a nonzero residue $r < d_{\mathrm{clk}}$; its lower bound 2^{-r-1} is at least 2^{-t-1}. □

For this phase set

$$T_{\mathrm{inv}}|_{(x,y)} = \begin{cases} 2^{-y-1}(\zeta^y - 1)^{-1}I, & d_{\mathrm{clk}} \nmid y, \\ 0, & d_{\mathrm{clk}} \mid y. \end{cases} \tag{47}$$

Lemma 6.1 gives $\|T_{\mathrm{inv}}\| \leq 1$. Equations (44) and (46) then verify (45). The game coefficients remain rational: the phase belongs to the strategy, not the predicate.

7 Spatial hardness: the reduction and value one

Fix any decidable predicate $\mathcal{R}_{\mathrm{dec}}(e, j, t)$ and the set $\mathcal{A}_{\mathrm{dec}} = \{e : \exists j\, \forall t\ \mathcal{R}_{\mathrm{dec}}(e, j, t)\}$. We effectively choose the cellular automaton in Section 5 to run the following program, with e hardwired.

Read a positive-time return. At time y a waiting front is at position y. Ignore the time-zero pulse. At the first positive-time pulse, freeze the front and retain its coordinate as unary input d_{clk}. With the transition convention above the mark is written into row $d_{\mathrm{clk}}+1$ at position d_{clk}. Test d_{clk}. If $d_{\mathrm{clk}}=1$ or d_{clk} is composite, halt. Otherwise compute the unique $j\geq 0$ with $d_{\mathrm{clk}}=p_{j+1}$, the $(j+1)$st prime. Then evaluate $\mathcal{R}_{\mathrm{dec}}(e,j,0),\mathcal{R}_{\mathrm{dec}}(e,j,1),\dots$ and halt at the first false answer. All later computation states ignore the external pulse.

For a Σ_2^0-complete choice of $\mathcal{A}_{\mathrm{dec}}$, there is no computable bound on a sufficient clock order as a function of the input game: such a bound would leave only finitely many possible j in this reduction, making $\exists j\,\forall t\ \mathcal{R}_{\mathrm{dec}}(e,j,t)$ a Π_1^0 predicate. Indeed one tests, for every finite time cutoff, whether some one of those finitely many candidates survives all checks so far; if none survives forever, all have failed by a common finite cutoff. This contradicts Σ_2^0-hardness.

This finite program has an effective finite radius-one simulation. The initial-front rule, frozen unary track and ordinary tape/head tracks give the stated boundary conventions. We do not require a bound on its running time. Each completed false check causes the halt symbol after finitely many updates.

The clock presentation and its labels. Include the Fock, coordinate, clock, reader and tableau equations above, all typing relations, and the stated Z_{clk} commutators. They form an admissible finite presentation with $n=4$. In the four-bridge compiler of Lemma 4.5 the transported typed list is exactly

$$P_\circ,\ U_x,\ U_y,\ P_x,\ P_y,\ W,\ C_{\mathrm{ctr}},\ Z_{\mathrm{clk}},\ J_{\mathrm{W}},\ K_{\mathrm{W}}. \tag{48}$$

Table 5 records the label types. Here $P_\circ, P_x, P_y$ are projections, C_{ctr} is positive, and the other six are unitary. Thus one unitary encoding per typed label suffices, so $|\Lambda|=10$. There are fifty support equations, forty block equations and sixteen refinements per bridge per party. Source-only labels $P_{\mathrm{ret}}, T_{\mathrm{inv}}, P_c$ do not occur at left endpoints of an A_α relation and need no transport. For example the transported polynomials include

$$W_4 a_{ab} = a_{ab} Z_{\mathrm{clk},0}^{b} W_0^2, \qquad C_{\mathrm{ctr},4} a_{ab} = 2^{-b} a_{ab} C_{\mathrm{ctr},0}^2, \qquad J_{\mathrm{W},4} a_{ab} = a_{ab} J_{\mathrm{W},0}.$$

The source-only return equations and tableau equations are centralizer polynomials. Lemma 4.5 therefore computes a finite rational Boolean game G_e^{clk}.

Label	Type	Transported?	Use
$P_\circ$	projection	yes	Fock root
U_x, U_y	unitary	yes	coordinate increments
P_x, P_y	projection	yes	boundaries
W	unitary	yes	clock
C_{ctr}	positive contraction	yes	counter
$Z_{\mathrm{clk}}, J_{\mathrm{W}}, K_{\mathrm{W}}$	unitary	yes	Weyl clock
P_{ret}	projection	no	pulse
T_{inv}	general contraction	no	reader inverse
P_c	projection	no	tableau
a_α	general contraction	scaffold only	lifted creation

Table 5: Typed application labels of the clock presentation. Transported labels are copied through all bridges for use at a left endpoint.

Theorem 7.1. *The map $e\mapsto G_e^{\mathrm{clk}}$ satisfies*

$$G_e^{\mathrm{clk}}\in \mathrm{PERF}_{\mathrm{qs}} \iff G_e^{\mathrm{clk}}\in \mathrm{PERF}_{\mathrm{finS}} \iff \exists j\,\forall t\ \mathcal{R}_{\mathrm{dec}}(e,j,t).$$

No G_e^{clk} has a perfect finite-dimensional strategy.

$$\begin{gathered}\text{Presentation} \xrightarrow{\text{Lemma 4.5}} \text{finite game } G_e^{\mathrm{clk}}\\ \text{Perfect normal strategy} \longrightarrow \text{normal realization}\\ \longrightarrow \text{Fock tree} \longrightarrow \text{finite clock sector}\\ \longrightarrow \text{deterministic tableau} \longrightarrow \forall t\ \mathcal{R}_{\mathrm{dec}}(e,j,t)\\ \text{Witness } j \longrightarrow \text{Fock/Weyl model}\\ \longrightarrow \text{scaffold, frame, row ancilla} \longrightarrow \text{perfect strategy}\end{gathered}$$

Figure 3: Dependencies of the spatial hardness proof. Compactness and Bell attainment have separate upper-bound arguments.

Proof. A perfect normal strategy extracts a nonzero faithful root realization. The determinant argument supplies a nonzero root phase sector of finite order d_{clk}. By (40) its word descendants persist at all depths. Equations (42)–(46) force the exact return sequence, with its first positive return at d_{clk}. Lemma 5.1 needs only that the initial root vector lie in this Z_{clk}-eigenspace: the clock pulse is then scalar on all its word descendants. The imposed $[Z_{\mathrm{clk}}, P_c] = 0$ relations are redundant for this induction. An invalid d_{clk} halts. For valid $d_{\mathrm{clk}} = p_{j+1}$ any false $\mathcal{R}_{\mathrm{dec}}(e,j,t)$ also halts. A sufficiently large square would contain that halt, contrary to $P_{\mathtt{H}} = 0$. Hence the extracted j witnesses membership in $\mathcal{A}_{\mathrm{dec}}$.

Conversely take a witness j and set $d_{\mathrm{clk}} = p_{j+1}$. Use (32) with $\rho_{\mathrm{spec}} = I_{d_{\mathrm{clk}}}/d_{\mathrm{clk}}$, a good primitive phase and its Weyl pair, the canonical coordinate permutations, (44), (46), (47), and the classical nonhalting tableau. Every required operator is bounded and commutes with the density when so specified. This is a simultaneous realization. On $\mathbb{C}^{d_{\mathrm{clk}}}$ one may take $J_{\mathrm{W}}|i\rangle = \zeta^i|i\rangle$, $K_{\mathrm{W}}|i\rangle = |i+1 \bmod d_{\mathrm{clk}}\rangle$, and $Z_{\mathrm{clk}} = \zeta I$. Its root entropy and largest Schmidt probability are

$$S(h) = \log_2 d_{\mathrm{clk}} + \log_2(4/3) + 4/3, \qquad p_{\max} = 3/(4d_{\mathrm{clk}}). \tag{49}$$

The compiler adds finite factors, giving a perfect finite-entropy strategy.

The tensor-product witness can be written without an implicit extension. Let $\mathcal{H}_F = \ell^2(\mathcal{W})$ and let $\mathcal{H}_{\mathrm{sc}} = \mathbb{C}^{17}$ index the scaffold corners. Take

$$\mathcal{H}_A = \mathcal{H}_B = \mathcal{H}_{\mathrm{sc}} \otimes \mathcal{H}_F \otimes \mathbb{C}^{d_{\mathrm{clk}}} \otimes \mathbb{C}^3 \otimes \mathbb{C}^{d_{\max}}.$$

With $\mu_4 = (1,2,2,2,2,4,4,4,4,8,8,8,8,16,16,16,16)/121$, use the pure vector

$$|\psi\rangle = |\sqrt{\mu_4}\rangle\!\rangle \otimes |\sqrt{h_F}\rangle\!\rangle \otimes |\Phi_{d_{\mathrm{clk}}}\rangle \otimes |\Phi_3\rangle \otimes |\Phi_{d_{\max}}\rangle, \quad |\sqrt{\mu_4}\rangle\!\rangle = \sum_i \sqrt{\mu_{4,i}}|i,i\rangle.$$

The Fock factor is $\sum_w \sqrt{(3/4)16^{-|w|}}|w,w\rangle$. Alice uses the scaffold FMS effects, actual graph projections, refinements and column partitions of Lemma 4.5; Bob uses the prescribed opposite FMS effects and the transposed transferred centralizer products. The row factor is the finite maximally entangled ancilla of Lemma 4.3. These simultaneous effects were checked in the compiler proof. The marginal entropy is $H(\mu_4) + S(h_F) + \log_2 d_{\mathrm{clk}} + \log_2 3 + \log_2 d_{\max} < \infty$.

Finally finite-dimensional perfect play would extract four isometries with mutually orthogonal ranges on one nonzero finite-dimensional root space. Their range dimensions would sum to four times the dimension of that space, which is impossible. The argument applies to every input, whether or not it has a witness. □

Together with Section 3, this proves both completeness statements in Theorem A. The value-one promise is established next. For the promise version, the reduction maps every input into that promise; the unrestricted Σ_2^0 membership predicate restricts to it, giving promise-problem completeness. The construction can be made syntactically nonsynchronous

and non-imitation on every input by adding a fresh binary question to each side, accepting every pair involving it. Either answer then has the same nonempty allowed-answer support against every opposite question, which violates the imitation separation condition. This addition does not change perfection or the value-one property.

Size of the finite game. Here is a coarse order-of-magnitude bound, independent of the witness d_{clk}. Let $|\Gamma_{\mathrm{CA}}|$ be the alphabet size, let N_e^{proj} count the named graph projections, rows and word prefixes after endpoint translation, and let b_e bound the bit length of their rational coefficients. The lookup table has at most $2(|\Gamma_{\mathrm{CA}}|+1)|\Gamma_{\mathrm{CA}}|^2$ entries, so unshared explicit polynomials give $N_e^{\mathrm{proj}} = \mathrm{poly}(|\Gamma_{\mathrm{CA}}|)$ for this fixed list of operator relations. Denominator clearing and signed repetitions give row dimension $d_{\max} \le 2^{O(N_e^{\mathrm{proj}}(b_e+\log(N_e^{\mathrm{proj}}+1)))}$. The scalar contexts, column questions and marginal bindings are polynomial in $N_e^{\mathrm{proj}}, d_{\max}$, so the padded question count Q_e and maximum answer count obey the same coarse exponential bound. An explicit predicate table is polynomial in those two counts and has the same exponential order. Thus the effective compilation depends on the hardwired program, never on its chosen prime or running time. For scale, the bundled minimal one-isometry example already has 748 rows and 27 991 questions per party; it is not a size claim for the full reduction. A dry count of the full ten-label instance $e = 0$ gives 6 191 638 087 questions per party for literal expanded tableau polynomials. Shared bounded-circuit gates for the eight update rows give 8 516 580 questions per party. The dry count uses the explicit eight-symbol automaton for $e = 0$, with alphabet $\mathtt{s}, \mathtt{b}, \mathtt{f}, \mathtt{d}, \mathtt{z0}, \mathtt{z}, \mathtt{h}, \mathtt{H}$, which halts unless the clock order is two. Its tables are `verification/full-compile/*/automaton.json`; it is a concrete scan automaton, not the generic product alphabet of Section 5.3. These are two encodings of that specified instance; neither full Boolean game was materialized. The latter adds source-only gates and preserves the canonical soundness and completeness translation. The counts are reproducible dry runs, not a claim that arbitrary verifier families have those sizes.

7.1 Every constructed game has value one

The finite-dimensional approximations do not need a witness for e. Truncate the Fock tree to depths $0, \dots, m$ and choose a prime $d_{\mathrm{clk}} > 2^m$ with a good phase (Lemma 6.1). Use the normalized density

$$h^{[m]} = \frac{\sum_{|w|\le m} 16^{-|w|}|w\rangle\langle w|}{\sum_{i=0}^{m} 4^{-i}} \otimes \frac{I_{d_{\mathrm{clk}}}}{d_{\mathrm{clk}}}.$$

Every represented time is less than d_{clk}. The decoder therefore remains in the waiting stage in every square and never halts. Let P_m denote the top-depth projection. Its mass is

$$t_m = \mathrm{Tr}(h^{[m]}P_m) = \frac{3}{4^{m+1}-1}. \tag{50}$$

Every endpoint identity $L_{\mathrm{end}}A_\alpha = A_\beta R_{\mathrm{end}}$ remains exact. Below top depth it is the canonical identity; on top depth both sides vanish because the endpoint expression R_{end} preserves depth. The modular equations, the range-sum identity and all off-diagonal $A_\alpha^* A_\beta = 0$ also remain exact. The only failed original relations are the four diagonal isometry equations:

$$A_\alpha^* A_\alpha - I = -P_m \quad (\alpha = 00, 01, 10, 11). \tag{51}$$

All graph and triangle projections are defined from the actual products. Every structural graph, synchrony, binding, FMS, refinement and column test stays exact.

Here is the full row accounting. For each α let ℓ_α be the denominator-clearing integer and d_α the column dimension of the row obtained by translating $a_\alpha^* a_\alpha - e_0 = 0$ into graph projections. These depend on the finite input presentation, not on m or d_{clk}. In the scalar

qutrit frame its defect S_α has off-diagonal blocks $-(P_m)_0$ and adjoint and zero diagonal blocks. The frame density is $I_3/3$ and the source scaffold mass is $1/121$, so

$$\mathrm{Tr}(h_{\mathrm{old}} S_\alpha^2) = \frac{2}{3}\frac{1}{121} t_m. \tag{52}$$

The same calculation applies separately to each of the four rows. Their row-zero events have distinct column questions and occur in no other failed rule. If the game has Q_e equally padded questions per party, (19) gives

$$1 - \omega(G_e^{\mathrm{clk}}, \mathcal{S}_m) \le C_e \frac{3}{4^{m+1} - 1}, \tag{53}$$

$$C_e = \frac{1}{Q_e^2}\frac{2}{363} \sum_{\alpha \in \{00,01,10,11\}} \frac{4\ell_\alpha^2}{d_\alpha^3}. \tag{54}$$

If the four row constructions have common ℓ, d, this is $C_e = 32\ell^2/(363 Q_e^2 d^3)$. We include one oriented row-zero event per row. The opposite column questions are retained for identification without a second row-zero test. This fixes the convention in the constant. Duplicating the four row-zero tests in the opposite orientation would multiply the constant by two.

Since the density factor $I_{d_{\mathrm{clk}}}/d_{\mathrm{clk}}$ has trace one, no factor of d_{clk} enters the loss. Thus $\omega_{\mathrm{q}}(G_e^{\mathrm{clk}}) = 1$ for every e, proving the promise in Theorem A. This calculation uses four rows, rather than suppressing them inside an unspecified continuity estimate.

8 Exact Bell attainment

The same compactness decides when a rational Bell functional attains a value exactly. A Bell functional generalizes a game: its coefficients may be any rationals, and attainment of its supremum replaces perfection.

Theorem 8.1 (Bell attainment)**.** *For rational Bell functionals, each of the following is Σ_2^0-complete in the normal spatial model: attainment of a supplied rational global upper bound; attainment of the supremum, with no bound supplied; and attainment of a supplied rational scalar value. In the first problem, validity of the supplied bound can either be promised or required as part of the decision problem.*

Let $L(p) = \sum_{xyab} c_{xyab} p(a, b \mid x, y)$ have rational coefficients and let $C_{\mathrm{abs}} = \sum |c_{xyab}|$. Its value on any strategy lies in $[-C_{\mathrm{abs}}, C_{\mathrm{abs}}]$. Finite Schmidt truncation shows that its supremum over spatial strategies equals its supremum over finite-dimensional strategies.

8.1 A specified global bound

Assume first that $\beta \in \mathbb{Q}$ is a global upper bound. For fixed measurements the Bell operator

$$B(E, F) = \sum c_{xyab} E_{a|x} \otimes F_{b|y}$$

satisfies $B \le \beta I$. A normal state attains β only if every positive-weight pure component does so. For a pure vector this is equivalent to $B\psi = \beta\psi$, because $\beta I - B$ is positive. Compress to its Schmidt supports and normalize the largest amplitude as before. Vectorization yields the homogeneous equation

$$\sum c_{xyab} E_{a|x} D_{\mathrm{Sch}} F_{b|y}^{\mathsf{T}} = \beta D_{\mathrm{Sch}}. \tag{55}$$

At stage n use the variables of (3) and require every squared entry of the residual in (55) to be at most $C_{\mathrm{abs}}^2 \kappa/(n+1)$. Compression of a true witness satisfies this bound: only the

compact tails of the sandwiched terms contribute, with coefficient sum C_{abs}. Conversely the compact-sandwich lemma gives an exact nonzero limit solution. Its vectorization is an attaining strategy. Thus attainment is $\exists\kappa\,\forall n$ of a decidable predicate.

If validity is not promised, impose also $\omega_q(L) \le \beta$. This is Π_1^0: for each finite dimension, the universal assertion that all states and POVMs give value at most β is decidable by real quantifier elimination. Finite-dimensional density states suffice for this validity test. The conjunction remains Σ_2^0.

8.2 No bound supplied

The quantifier pattern is explicitly $\exists\kappa \ge 1\,\forall n \ge 1\; F_{\mathrm{Bell}}(L,\kappa,n)$, with F_{Bell} decidable as follows. At stage n introduce also a real variable $\beta \in [-C_{\mathrm{abs}}, C_{\mathrm{abs}}]$, use the same residual bound, and require $\beta \ge \omega_n(L)$, where ω_n is the maximum in local dimension n. This latter requirement is a universal semialgebraic formula in β, not a numerical oracle. Full real quantifier elimination decides the stage formula with its existential and universal real variables.

An attaining strategy supplies all stages. Conversely pass to a subsequence with $\beta_n \to \beta_*$, $D_{\mathrm{Sch},n} \to D_{\mathrm{Sch}}$ in norm and effects weakly convergent. For every fixed dimension d, all sufficiently late stages have $\beta_n \ge \omega_d(L)$. Hence $\beta_* \ge \sup_d \omega_d(L)$. The limit equation (55) gives a unit vector attaining β_*. Its expectation cannot exceed that supremum, so equality holds. This proves Σ_2^0 membership for attainment of the unspecified supremum.

8.3 A specified scalar value

Let $t \in \mathbb{Q}$ be supplied without a bound promise. A value strictly between the infimum and supremum has finite-dimensional values on either side, by equality of the finite and spatial extrema. A finite direct-sum mixture then attains t. Existence of a finite-dimensional strategy with value exactly t is Σ_1^0, by dimension search and quantifier elimination. Every remaining attainable value is an endpoint. Upper endpoints are handled above; lower endpoints are upper endpoints for $-L$. This finite union is Σ_2^0.

For hardness in all three cases use the winning-probability functional of G_e^{clk}. Its coefficients are rational, its global bound and finite-dimensional supremum are both one, and attaining the specified value one is precisely perfect spatial play. Theorem 7.1 and Section 7.1 therefore prove Theorem 8.1. In its promised-bound version every reduction output has valid bound 1, and the upper predicate restricts to the valid-bound domain, so the same map establishes promise-problem completeness.

9 The entanglement spectrum and its decision problems

A finite entropy bound controls the tail of the Schmidt distribution. A positive moment gives stronger control, and varying the moment order reveals a strict ladder of existence languages. We first prove compactness and the upper predicates, then give the spectral separators. The common hardness family and the realization of all possible critical indices follow in Section 10, using the nested compiler proved in Appendix A.

For $0 < \alpha < 1$ let $\mathcal{P}_\alpha$ require a perfect pure spatial strategy with $\operatorname{Tr} h^\alpha < \infty$. Put $\mathcal{P}_0 = \mathbf{q}$, $\mathcal{P}_1 = \mathbf{E}$, and $\mathcal{P}_{\mathrm{normal}} = \mathbf{I}$. For $\alpha > 1$, finiteness of Rényi entropy imposes no restriction: $0 < \operatorname{Tr} h^\alpha \le 1$ for every density, and its largest eigenvalue is positive, so the order-infinity entropy is finite too. Thus all those languages equal $\mathcal{P}_{\mathrm{normal}}$. The notation $\mathcal{P}_{\mathrm{normal}}$ is a language label, not a moment of infinite order. Mixed witnesses can be purified on Bob's side preserving Alice's marginal. Pure witnesses therefore suffice for these existence languages.

9.1 Moment budgets are compact and effective

For sorted probabilities with $\sum_i p_i^\alpha \le K$, one has

$$t_D := \sum_{i>D} p_i \le K^{1/\alpha}(D+1)^{-(1-\alpha)/\alpha}. \tag{56}$$

Indeed $p_{D+1}^\alpha \le K/(D+1)$ and $t_D \le p_{D+1}^{1-\alpha}\sum_{i>D} p_i^\alpha$. This gives uniform tails. Coordinate limits preserve total mass and converge in ℓ^1, while Fatou preserves the moment bound. The Schmidt vectors converge in norm, since $\|\sqrt{p}-\sqrt{q}\|_2^2 \le \|p-q\|_1$. Bounded POVMs have a common weakly convergent subsequence on fixed copies of ℓ^2. Their tensor products converge weakly on finite simple tensors and then on all vectors by uniform boundedness. Thus all game probabilities converge. The maximum winning probability $v_G^\alpha(K)$ under the supplied budget is attained. The same proof uses (5) at $\alpha = 1$. Trace one alone supplies no uniform tail, as I_N/N demonstrates.

Write $v_{G,D}^\alpha(K)$ for the maximum winning probability with local dimension at most D and moment budget K. Normalize the first D probabilities by their retained mass. This truncation majorizes the full probability vector, so does not increase the concave moment sum or entropy. Compression of the POVMs therefore gives

$$0 \le v_G^\alpha(K) - v_{G,D}^\alpha(K) \le \sqrt{t_D}.$$

For rational α, the finite-dimensional budget constraints are semialgebraic using nonnegative root variables. For a fixed computable irrational order, approximate x^α uniformly on $[0,1]$ by rational polynomials. Compact outer approximations give upper semicomputability. For lower approximation, move a nonpure boundary spectrum slightly towards $(1,0,\dots,0)$: the resource strictly decreases and probabilities change continuously. Rational spectra and feasible POVMs with strict slack are dense. At moment budget $K = 1$, only a pure marginal is possible and the value is the finite classical value. The analogous entropy exception is budget zero. These arguments give computable budget values for computable orders, and in particular the upper semicomputability needed here. Equivalently, compact rational outer feasibility tests, with vanishing rejection slack and uniform constraint error, give

$$G \in \mathcal{P}_\alpha \iff \exists K \ge 1\, \forall D \ge 1\ R_\alpha(G,K,D)$$

with R_α decidable. Explicitly, compute rational polynomials q_D with $\sup_{x\in[0,1]} |q_D(x) - x^\alpha| \le D^{-2}$. Choose rational bounds $u_D \ge K^{1/\alpha}(D+1)^{-(1-\alpha)/\alpha}$ tending to zero and rational $\epsilon_D \ge 4\sqrt{u_D}$ tending to zero. If $u_D \ge 1$, let the stage be true. Otherwise require ordered $p_i = s_i^2$, $s_i \ge 0$, two D-dimensional POVM assignments, and

$$1 - u_D \le \sum_{i=1}^D p_i \le 1, \qquad \sum_{i=1}^D q_D(p_i) \le K + D^{-1}, \qquad \mathrm{loss}_G(s,E,F) \le \epsilon_D.$$

These are rational semialgebraic constraints. A perfect budget-K witness supplies every stage by compression. Conversely, the true moment of each stage is at most $K + 2/D$; the same uniform-tail compactness argument and Fatou give a normalized perfect limit with moment at most K. The rational-order version is written explicitly in Section 10. No unrelativized effectiveness claim is made for arbitrary noncomputable orders.

9.2 Tree spectra separate the rungs

For b orthogonal isometries of common ratio ν and faithful normal density h, the word-sector argument of Section 5 and its b-ary version in Appendix B gives

$$h = \tau_{b,\nu} \otimes \rho_{\mathrm{spec}}, \qquad \tau_{b,\nu} = (1-b\nu)\sum_w \nu^{|w|}|w\rangle\langle w|, \quad b\nu < 1.$$

One can also see exhaustion by pulling a positive density eigenvector back through adjoint edges: every pullback multiplies its eigenvalue by ν^{-1}, so it must terminate before exceeding $\|h\|$. Consequently

$$\operatorname{Tr} h^{\alpha} = \frac{(1-b\nu)^{\alpha}}{1-b\nu^{\alpha}} \operatorname{Tr} \rho_{\mathrm{spec}}^{\alpha} \tag{57}$$

when $b\nu^{\alpha} < 1$, and the moment diverges otherwise. The bare-tree critical index is $\log_2 b/n$ at $\nu = 2^{-n}$. Its finite compiler is the n-bridge construction with a finite list of b edge labels, as specified in Appendix B. For four branches the normality requirement is $n > 2$; for $0 < \alpha \leq 1$, $n = \lfloor 2/\alpha \rfloor + 1$ ensures both normality and moment summability. For larger orders the requirement $n > 2$ remains.

For any $0 < a < b < 1$, choose integers n, j with $a < \log_2 j/n < b$. Such choices exist by taking n large and an integer between the corresponding exponential bounds. The bare j-ary tree has finite b-moment, infinite a-moment in every realization, and finite entropy in its canonical realization. A comb with critical index one and finite entropy, constructed next, separates every order below one from entropy. The Catalan game of Section 14 separates entropy from arbitrary normal play. The uniform family of Section 10 separates finite rank from every positive rung. Thus

$$\mathcal{P}_0 \subsetneq \mathcal{P}_a \subsetneq \mathcal{P}_b \subsetneq \mathcal{P}_1 \subsetneq \mathcal{P}_{\mathrm{normal}} \quad (0 < a < b < 1). \tag{58}$$

Compression to the positive source corner preserves finite moments; finite compiler factors preserve them in completeness. These separators therefore concern every perfect strategy, not just a displayed model.

9.3 Complexity, differences, and critical indices

The construction in Section 10 gives one computable family H_e such that

$$e \in \mathcal{A}_{\mathrm{dec}} \iff H_e \in \mathbf{I} \iff H_e \text{ has one perfect state with every positive moment finite.}$$

Every instance has $\omega_q = 1$ and no finite-dimensional perfection. Together with the upper predicates this proves Σ_2^0-completeness for every fixed positive computable rung, including entropy and normal play. The finite-rank rung is Σ_1^0-complete.

Exact conjunction and disjunction preserve these resources. For conjunction tensor the witnesses; moment sums multiply. For disjunction, the globally matched branch projections reduce the marginal and give $h = \bigoplus_j p_j h_j$. Hence $\operatorname{Tr} h^{\alpha} = \sum_j p_j^{\alpha} \operatorname{Tr} h_j^{\alpha}$ and $S(h) = H(p) + \sum_j p_j S(h_j)$; each positive branch inherits the resource. Use the full-support branch agreement proved in Section 14. If $\mathcal{P} \subsetneq \mathcal{Q}$ are distinct positive rungs, fix $G_{\mathrm{sep}} \in \mathcal{Q} \setminus \mathcal{P}$. Then

$$H_e \wedge (G_{\mathrm{sep}} \vee H_f) \in \mathcal{Q} \setminus \mathcal{P} \iff e \in \mathcal{A}_{\mathrm{dec}} \text{ and } f \notin \mathcal{A}_{\mathrm{dec}}.$$

All nonempty such differences are $D_2(\Sigma_2^0)$-complete; a positive rung minus finite rank is Σ_2^0-complete. Reverse differences and differences between collapsed orders above one are empty. The assertion that NO instances have no normal perfect strategy belongs to H_e, not to every NO instance of the difference family. Both hardness promises persist under these operations.

For a density put $\alpha_c(h) = \inf\{\alpha > 0 : \operatorname{Tr} h^{\alpha} < \infty\}$, and for a spatially perfect game put

$$c(G) = \inf_{\psi \text{ perfect}} \alpha_c(h_A^{\psi}).$$

Unused entanglement can increase an individual strategy's index; the invariant is this infimum. Section 10 proves that its possible values are exactly the right-c.e.-in-$0'$ reals in $[0, 1]$. Every such value has a realizing game with a finite-entropy witness attaining the minimum. The computability characterization uses *limsup* of computable rational sequences, following Becher, Bugeaud and Slaman [5, Lemma 4].

9.4 Approximate play: a dimension upper bound

Let $D_G(\varepsilon)$ be the least local dimension of a strategy with loss at most ε. For a fixed perfect witness, let $D_{\mathrm{trunc}}(\varepsilon)$ be the least Schmidt cutoff whose discarded probability mass is at most ε. If a perfect witness has $K = \operatorname{Tr} h^\alpha < \infty$, $0 < \alpha < 1$, its loss effect satisfies $0 \le L_G \le I - |\psi\rangle\langle\psi|$. Its normalized Schmidt truncation therefore loses at most the discarded mass t_D, rather than just its square root. Equation (56) gives

$$D_G(\varepsilon) \le \left\lceil K^{1/(1-\alpha)} \varepsilon^{-\alpha/(1-\alpha)} \right\rceil .$$

For a bare tree with critical index $r < 1$, sorting by depth gives $D_{\mathrm{trunc}}(\varepsilon) \asymp \varepsilon^{-r/(1-r)}$. This is a sharp spectral truncation law. A lower bound for every near-perfect strategy of the finite game remains open; exact factor forcing alone does not prove it.

10 A uniform family and programmable critical indices

The construction separates the spectrum from the computation that verifies it. Two commuting one-sided shifts first force quarter-plane coordinates and a geometric density. A finite clock and a deterministic tableau then read an integer witness. For programmable critical indices, a comb supplies the desired spectral threshold while an independent quarter-plane register verifies its branching rule. The nested compiler translates the quadratic shift relations into finite game tests.

10.1 The quarter-plane register and its finite presentation

Use two modular contractions S, T and impose

$$S^*S = T^*T = I, \quad ST = TS, \quad S^*T = TS^*. \tag{59}$$

Name projections P_x, P_t with $SS^* = I - P_x$, $TT^* = I - P_t$. Name positive contraction counters C_x, C_t and impose their commutation, $C_xP_x = P_x$, $C_tP_t = P_t$, and

$$C_xS = \tfrac{1}{2}SC_x, \; C_xT = TC_x, \quad C_tS = SC_t, \; C_tT = \tfrac{1}{2}TC_t. \tag{60}$$

All clock and computation labels commute with both counters. For the clock use centralizer unitaries Z, J, K, W with

$$JK = ZKJ, \; [Z, J] = [Z, K] = 0,$$

Z, J, K constant along both shifts, and

$$WS = SZW, \; WT = TW, \; WP_x = P_x.$$

For a projection E_{ret} and a contraction R_{ret} use

$$(W - I)E_{\mathrm{ret}} = 0, \qquad R_{\mathrm{ret}}(W - I) = \tfrac{1}{2}C_x(I - E_{\mathrm{ret}}). \tag{61}$$

Every row is either centralizer polynomial, degree-one endpoint, or one of the quadratic rows of the translation table in Appendix A. In particular the mixed row $S^*T = TS^*$ of (59) is submitted as $a_S'^* a_T' = t a_T a_S^* t^*$, in the notation of that appendix; no modular transport is synchronized.

To specify a verifier, supply a finite radius-one deterministic one-sided cellular automaton, with alphabet $\mathcal{A}$, interior rule $F : \mathcal{A}^3 \to \mathcal{A}$, and left-boundary rule $G : \mathcal{A}^2 \to \mathcal{A}$. For every d

name four projection families P_d, L_d, R_d, N_d, each summing to identity and commuting with C_x, C_t. Their shift equations are

$$L_d S = S P_d, \qquad P_d S = S R_d, \qquad P_d T = T N_d. \tag{62}$$

Thus $L_d(x,t) = P_d(x-1,t)$ for $x > 0$, $R_d(x,t) = P_d(x+1,t)$, and $N_d(x,t) = P_d(x,t+1)$. The unused $L_d(0,t)$ is arbitrary. Submit the exact finite update polynomials

$$N_d = (I - P_x) \sum_{F(a,b,c)=d} L_a P_b R_c + P_x \sum_{G(b,c)=d} P_b R_c. \tag{63}$$

Specify every initial track on P_t: the return marker from E_{ret}, the head from P_x, and the blank work tracks. If ι maps these finitely many input bits to an alphabet symbol, the initial row for d is $P_t P_d = P_t \sum_{\iota(b)=d} \prod_j Q_j^{b_j}$, where $Q_j^1 = Q_j$, $Q_j^0 = I - Q_j$. The input projections commute: they are scalar coordinate markers in the normal representation proved next. Equivalently one may add their finite commutator rows. For each rejecting symbol impose $P_d = 0$. Equations (62) have modular degree one; (63) has modular degree zero, regardless of its centralizer polynomial degree. This is the precise admissibility distinction needed for the tableau.

Use the Fibonacci convention $F_1 = F_2 = 1$, $F_{k+2} = F_{k+1} + F_k$. For a total computable predicate $R(e,n,s)$, a finite machine first scans for the first positive return marker, verifies that its period is F_{n+3}, decodes n, and searches for s with $\neg R(e,n,s)$. Its ordinary finite transition table gives such a cellular automaton: a symbol records a tape letter and optionally a head with its control state. A head writes its new letter, and its indicated neighbor receives the new head at the next step; a boundary rule handles the missing left neighbor. Invalid head collisions can be assigned a rejecting symbol. Multiple tapes can be interleaved with finitely many tracks and finite control, with extra time steps as needed. Only the unique initialized one-head computation is used by the soundness induction. This construction is effective in a supplied program for R and in e. Concrete word and row counts depend on that table. Appendix D supplies explicit validation programs; the reduction family is parametric.

10.2 Normal classification, the clock, and all positive moments

Cyclicity gives $\mathrm{Tr}(h_0 S^k S^{*k}) = 2^{-k}$ and the same for T. Hence normality removes both unitary Wold parts. Double commutation makes the two wandering decompositions compatible and gives

$$\mathcal{H}_0 = \ell^2(\mathbb{N}^2) \otimes \mathcal{K}, \quad h_0 = \rho_{\mathrm{reg}} \otimes \rho_{\mathrm{reg}} \otimes \rho_{\mathrm{spec}}, \quad \rho_{\mathrm{reg}} = \sum_{x \geq 0} 2^{-x-1} |x\rangle\langle x|. \tag{64}$$

Indeed the mutually orthogonal spaces $S^x T^t P_x P_t \mathcal{H}_0$ exhaust the support; their density blocks are 2^{-x-t} times the joint root block. Normalization makes that block $\rho_{\mathrm{spec}}/4$. Equation (60) then forces $C_x = 2^{-x}$ and $C_t = 2^{-t}$, by induction from each boundary. Commutation with these counters prevents labels from mixing coordinates of equal total density weight.

The constant Z, J, K act on $\mathcal{K}$ and commute with the compact ρ_{spec}. Pick a positive finite-dimensional ρ_{spec} eigenspace and then a nonzero $Z = zI$ subspace. It is invariant under J, K, and determinants in $JK = zKJ$ give $z^d = 1$. Let its finite order be d_{clk}. The equations for W force $W(x,t) = Z^x$. Equation (61) forces $E_{\mathrm{ret}}(x,t) = I$ exactly when d_{clk} divides x on this sector: its range lies in $\ker(W - I)$; on that kernel the second equation and $C_x > 0$ force it to be identity. On each chosen clock sector the initialized tableau labels are scalars. Equations (62)–(63) force each next row to be the unique classical successor; the induction also proves invariance of the sector for those projections. Thus rejecting symbols cannot be absent unless $\forall s\, R(e,n,s)$ for the decoded integer.

Conversely a true witness uses $d_{\mathrm{clk}} = F_{n+3}$, $a = F_{n+2}$, $z = \exp(2\pi i a/d_{\mathrm{clk}})$, $Z = zI_{d_{\mathrm{clk}}}$, and Weyl matrices on $\mathbb{C}^{d_{\mathrm{clk}}}$. With $\rho_{\mathrm{spec}} = I_{d_{\mathrm{clk}}}/d_{\mathrm{clk}}$ the honest initialized tableau is diagonal and

all the relations hold. Before the finite compiler factors,

$$\mathrm{Tr}(h_0^{\alpha}) = 4^{-\alpha} {d_{\mathrm{clk}}}^{1-\alpha}(1-2^{-\alpha})^{-2} \quad (\alpha > 0), \qquad S(h_0) = 4 + \log_2 d_{\mathrm{clk}}. \tag{65}$$

This is one common witness for all positive moments.

A bounded reader at every nonreturn. Put $\theta = (\sqrt{5}-1)/2$. If a is a nearest integer to $x\theta$, then $\delta = x\theta - a$ has nonzero integer norm $\delta(x\theta' - a)$, where $\theta' = -(1+\sqrt{5})/2$. The second factor has magnitude at most $\sqrt{5}x + 1/2 < 3x$, giving $\|x\theta\| > 1/(3x)$. Also $F_k\theta - F_{k-1} = (-1)^{k-1}\theta^k$ and $F_k\theta^k < 1$, so the Fibonacci approximation error is less than F_k^{-2}. These elementary bounds give, when $x \le d_{\mathrm{clk}}/3$,

$$\|xa/{d_{\mathrm{clk}}}\| > 1/(3x) - x/{d_{\mathrm{clk}}}^2 \ge 2/(9x).$$

For every nonreturn $x \ge 2d_{\mathrm{clk}}/9$, the coprime residue bound gives $\|xa/d_{\mathrm{clk}}\| \ge 1/d_{\mathrm{clk}} \ge 2/(9x)$. These ranges overlap and cover every positive nonreturn, including $x > d_{\mathrm{clk}}$. Concavity of sine on $[0, \pi/2]$ gives $|z^x - 1| \ge 4\|xa/d_{\mathrm{clk}}\| \ge 8/(9x)$. Consequently the canonical reader

$$R_{\mathrm{ret}}(x) = \begin{cases} 2^{-x-1}/(z^x-1), & d_{\mathrm{clk}} \nmid x, \\ 0, & d_{\mathrm{clk}} \mid x \end{cases}$$

has norm at most $\sup_{x\ge 1} 9x/2^{x+4} = 9/32$. Returns, including $x = 0$, are checked separately and give zero on both sides of (61).

Value one. For any e, take arbitrarily large Fibonacci periods and the genuine computation even if it eventually rejects. All equations except the forbidden-reject row remain exact. The first scan prevents rejection before time d_{clk}. Its root probability is at most $\sum_{t\ge d_{\mathrm{clk}}} 2^{-t-1} = 2^{-d_{\mathrm{clk}}}$, so (131) gives *loss at most* $C_{\mathrm{nest}} 2^{-d_{\mathrm{clk}}}$; the constant contains the compilation factors of the row tests. Since C_{q} and C_{qs} have the same closure (Section 2), the finite-dimensional value is one. Finite-dimensional perfection is impossible: $S^*S = I$ would make S unitary, whereas $h_0 S = S h_0/2$ and trace cyclicity would give $1 = 1/2$.

10.3 The comb and the independent checker

The spectral threshold will come from the number of branches attached at each spine position. A separate geometric register verifies the branching guard. Keeping its computation time in that separate factor allows arbitrarily slow total programs without changing the programmed critical index. First normality establishes coordinates; only then does the deterministic verifier force the guard on each coordinate fibre.

Let Q, P, E_{guard} be centralizer projections, $P \le Q$, and let C_{sp}, L be commuting positive contractions, commuting with Q, P. Use modular contractions $V_{\mathrm{sp}}, B_0, B_1$ with

$$\begin{aligned} {V_{\mathrm{sp}}}^* V_{\mathrm{sp}} &= Q, & V_{\mathrm{sp}} {V_{\mathrm{sp}}}^* &= Q - P, \\ B_i^* B_j &= \delta_{ij} E_{\mathrm{guard}}, & B_0 B_0^* + B_1 B_1^* &= I - Q, \\ C_{\mathrm{sp}} P &= P/2, & C_{\mathrm{sp}} V_{\mathrm{sp}} &= V_{\mathrm{sp}} C_{\mathrm{sp}}/2, \\ C_{\mathrm{sp}} B_i &= B_i C_{\mathrm{sp}}, & LQ &= Q, \qquad LB_i = B_i L/2. \end{aligned}$$

All modular symbols have ratio $1/2$. These are inner-product, range, and endpoint rows of the translation table. In particular a range sum is compiled as a sum $t a_i a_i^* t^*$, together with the transported range projection. No XX^* is submitted as a root centralizer label without its defining translated row.

Add an independent quarter-plane pair S, T satisfying (59)–(60). For every comb modular symbol $X \in \{V_{\mathrm{sp}}, B_0, B_1\}$ impose both $XS = SX$, $X^*S = SX^*$ and the corresponding equations with T. For Q, P, C_{sp}, L impose shift commutation by endpoint rows. These relations give the tensor decomposition (64) with the entire comb algebra acting on its multiplicity space. The checker has its own C_x, C_t, P_x, P_t and a separate computation time.

The guard preserves the coordinate fibres. Since $E_{\mathrm{guard}} = B_0^* B_0$, the counter relations and their adjoints give

$$E_{\mathrm{guard}} C_{\mathrm{sp}} = B_0^* C_{\mathrm{sp}} B_0 = C_{\mathrm{sp}} E_{\mathrm{guard}}, \qquad E_{\mathrm{guard}} L = 2 B_0^* L B_0 = L E_{\mathrm{guard}}. \tag{66}$$

Ordinary and double commutation with checker shifts similarly give commutation of E_{guard} with those shifts and adjoints. Adding these redundant commutator rows is harmless and is done in the generated validation instance.

The four orthogonal projections $P, V_{\mathrm{sp}} {V_{\mathrm{sp}}}^*, B_0 B_0^*, B_1 B_1^*$ sum to identity. Starting from an h eigenvector of eigenvalue $p > 0$, each nonzero pullback through ${V_{\mathrm{sp}}}^*, B_i^*$ doubles its eigenvalue. After finitely many steps it would exceed $\|h\|$ and must vanish. Thus descendants of root vectors span the support. Since $V_{\mathrm{sp}} B_i = 0$, surviving words have form $B_w {V_{\mathrm{sp}}}^{m-1} P$, with $m \geq 1$. The counter relations force $C_{\mathrm{sp}} = 2^{-m}$ and $L = 2^{-|w|}$ on their ranges. Their span is dense, so the joint dyadic eigenspaces $\mathcal{H}_{m,\ell}$ exhaust the support. Equation (66) shows E_{guard} preserves each such eigenspace.

Name equality projections $E_{C_{\mathrm{sp}}}, E_L$ and contraction readers $F_{C_{\mathrm{sp}}}, F_L$. Impose

$$(C_{\mathrm{sp}} - C_x) E_{C_{\mathrm{sp}}} = 0, \quad F_{C_{\mathrm{sp}}} (C_{\mathrm{sp}} - C_x) = \tfrac{1}{2} C_{\mathrm{sp}} C_x (I - E_{C_{\mathrm{sp}}}), \tag{67}$$

and the analogous rows for L. For two unequal dyadic values $u, v \in (0, 1]$, $|uv/[2(u - v)]| \leq 1$: the larger is at least twice the smaller. On equal fibers the second equation forces the equality projection to be identity; on unequal fibers the first forces it to be zero. The canonical reader is $uv/[2(u - v)]$, with value zero at equality. Thus the initial checker tracks mark exactly $x = m$ and $x = \ell$, and receive the bit E_{guard}. Require every checker label to commute with $C_{\mathrm{sp}}, L, E_{\mathrm{guard}}, C_x, C_t$, and initialize all tracks and the head. Its total program computes k_m and rejects exactly when the bit is different from $\mathbf{1}_{\{\ell < k_m\}}$. Apply the same finite (62)–(63) presentation and forbid rejection.

On a joint $(m, \ell, E_{\mathrm{guard}})$ fiber the initial symbols are scalars. The deterministic tableau induction therefore forces the actual finite computation of this total program. A wrong guard fiber would produce a nonzero rejecting projection, contradicting faithfulness and the forbidden row. Hence

$$E_{\mathrm{guard}} | \mathcal{H}_{m,\ell} = \mathbf{1}_{\{\ell < k_m\}} I. \tag{68}$$

Now, and only now, legal word maps are isometries on the root with mutually orthogonal ranges. They exhaust the support by the previous pullback argument, and yield

$$\begin{aligned} h_{\mathrm{comb}} &\cong \rho_{\mathrm{comb}} \otimes \rho_{\mathrm{spec}}, \\ \rho_{\mathrm{comb}} &= Z_{\mathrm{comb}}^{-1} \sum_{m \geq 1,\ |w| \leq k_m} 2^{-(m+|w|)} |m, w\rangle\langle m, w|, \\ Z_{\mathrm{comb}} &= \sum_{m \geq 1} (k_m + 1) 2^{-m}. \end{aligned} \tag{69}$$

The complete checker adds $\rho_{\mathrm{reg}} \otimes \rho_{\mathrm{reg}}$, independently of how long the program takes to verify a particular fiber.

10.4 Programming and classifying the minimum index

For a computable rational sequence a_m with $\limsup a_m = r \in [0,1]$, put

$$u_m = \min\!\left(\max(a_m, 0), \frac{m}{m+1}\right), \qquad k_m = \left\lfloor \frac{m u_m}{1-u_m} \right\rfloor.$$

Then $0 \le k_m \le m^2$, and rounding changes $k_m/(m+k_m)$ from u_m by less than $1/m$. Thus its limsup is r, and $1 \le Z_{\mathrm{comb}} \le 7$. For $0 < \alpha < 1$ the moment of (69) converges precisely when

$$\sum_m 2^{(1-\alpha)k_m - \alpha m} < \infty.$$

The exponent equals $(k_m/(m+k_m) - \alpha)(m+k_m)$. If $\alpha > r$ the tail decays geometrically; if $\alpha < r$ a subsequence of terms grows. Therefore $\alpha_c(\rho_{\mathrm{comb}}) = r$, without a claim about convergence at the endpoint $\alpha = r$. The entropy is bounded by

$$\log_2 7 + \sum_{m \ge 1} 2^{-m} \left[(m^2+1)m + \frac{m^2(m^2+1)}{2} \right] = 106 + \log_2 7.$$

Every realization contains the comb factor, so extra multiplicity cannot lower its critical index. Root multiplicity one gives the canonical witness attaining minimum index r, with finite entropy.

For necessity, restrict the moment order to rational $0 < \beta < 1$. For sorted Schmidt probabilities with $\sum_i p_i^\beta \le K$, use the tail bound (56) with $\alpha = \beta$ and $D = d_{\mathrm{cut}}$. Choose computable rational upper bounds $t^{\mathrm{bd}}_{d_{\mathrm{cut}}}$ for this tail, tending to zero, and rational $\epsilon_{d_{\mathrm{cut}}} \ge 4\sqrt{t^{\mathrm{bd}}_{d_{\mathrm{cut}}}}$, also tending to zero. For fixed $G, \beta, K, d_{\mathrm{cut}}$ let $\mathcal{R}(G, \beta, K, d_{\mathrm{cut}})$ say there exist two local d_{cut}-dimensional POVM assignments and a subnormalized Schmidt vector with ordered probabilities $p_1, \ldots, p_{d_{\mathrm{cut}}}$ such that

$$1 - t^{\mathrm{bd}}_{d_{\mathrm{cut}}} \le \sum_{i \le d_{\mathrm{cut}}} p_i \le 1, \qquad \sum_{i \le d_{\mathrm{cut}}} p_i^\beta \le K, \qquad \Pr(\text{reject}) \le \epsilon_{d_{\mathrm{cut}}}.$$

When $t^{\mathrm{bd}}_{d_{\mathrm{cut}}} \ge 1$ take this stage to be true. At the other stages this is a decidable semialgebraic feasibility predicate: rational powers are encoded by nonnegative root variables, and POVM positivity by principal minors. A perfect witness compresses to these assignments; its truncated vector differs in norm by at most $\sqrt{t^{\mathrm{bd}}_{d_{\mathrm{cut}}}}$, giving the displayed slack. Conversely the uniform tail bound makes these ordered vectors compact in ℓ^2. A diagonal subsequence of the bounded POVM matrix entries gives positive limiting POVMs with the same finite outcome sums. Finite coordinate tests followed by the uniform vector tail pass their tensor probabilities to the limit. The limiting vector has norm one, zero rejection and β-moment at most K by Fatou's lemma. Consequently the quantifiers are explicitly

$$G \in \mathcal{P}_\beta \quad \Longleftrightarrow \quad \exists K \in \mathbb{N}_{\ge 1}\ \forall d_{\mathrm{cut}} \in \mathbb{N}_{\ge 1}\ \mathcal{R}(G, \beta, K, d_{\mathrm{cut}}).$$

For rational $0 < a \le 1$, $c(G) < a$ is equivalent to this condition for some rational $0 < \beta < a$, when $c(G)$ denotes the infimum of critical indices of perfect witnesses and is finite in $[0,1]$. Cuts above 1 are automatic on this domain. The rational order and integer budget can be paired in one existential integer, leaving the same $\exists K\, \forall d_{\mathrm{cut}}$ form. Consequently the strict upper cut is c.e. in $0'$. Together with the characterization of Becher, Bugeaud and Slaman [5, Lemma 4] by limsups of computable rationals, this gives exactly the right-c.e.-in-$0'$ indices in $[0,1]$. The input program computes k_m, and contains no oracle.

11 Semifinite hosts: a critical tree selects a finite depth

In a diffuse semifinite algebra the Weyl determinant of Section 6 is unavailable, so the spatial witness cannot be used. A trace still forces a finite object. A critical binary tree with modular ratio $1/2$ must have a nonzero leaf sector at some finite depth, and a group word chosen by that depth carries the Σ_2^0 witness. This section builds the resulting reduction and proves the semifinite half of Theorem B. It does not use the Weyl-clock reduction.

11.1 The critical-tree selector

Work on a faithful normal support. Fix integers $d \geq 2$ and a real $0 < \gamma < 1$. Let P, Q be centralizer projections, $0 \leq C \leq I$ a centralizer contraction, and let $B_1, \ldots, B_d$ satisfy

$$B_i^* B_j = \delta_{ij}(I - P), \qquad \sum_i B_i B_i^* = I - Q, \tag{70}$$

$$\sigma_t^{\varphi}(B_i) = d^{-it} B_i, \tag{71}$$

$$CQ = Q, \qquad CB_i = \gamma B_i C, \qquad [C, P] = 0. \tag{72}$$

These labels $P, Q, C, B_1, \ldots, B_d$ belong to the selector presentation only.

Lemma 11.1 (Critical-tree selector)**.** *For branching d, modular ratio $1/d$ and depth ratio γ, in a normal semifinite realization, $P \neq 0$, $\ker C = 0$, and $E_n = 1_{\{\gamma^n\}}(C)$ satisfy $\sum_{n \geq 0} E_n = I$ strongly. Some finite n has $PE_n \neq 0$.*

Proof. The modular eigenoperators are entire analytic. KMS gives $\varphi(B_i B_i^*) = \varphi(B_i^* B_i)/d$. Summing gives

$$\varphi(P) = \varphi(Q). \tag{73}$$

If $P = 0$, faithfulness makes $Q = 0$, giving a full Cuntz family. Write $\varphi(x) = \tau(hx)$, h injective, $\tau(h) = 1$. The covariance is $h^{it} B_i = B_i (h/d)^{it}$, hence $f(h) B_i = B_i f(h/d)$ for bounded Borel f. For bounded $g \geq 0$ with compact support in $(0, \infty)$, $hg(h)$ is positive integrable, and

$$\varphi(g(h)) = \sum_i \tau(B_i^* h g(h) B_i) = \tfrac{1}{d} \sum_i \tau(h g(h/d)) = \varphi(g(h/d)).$$

The trace exchange is $\tau(T^{1/2} v v^* T^{1/2}) = \tau(v^* T v)$ for positive integrable T. Use indicators of $(d^k, d^{k+1}]$, or monotone approximations to them. All these windows have equal φ-mass and partition $(0, \infty)$, whose mass is one. Countable additivity is contradicted. Thus $P \neq 0$.

Let $z = 1_{\{0\}}(C)$. Covariance and its adjoint give $zB_i = B_i z$ by functional calculus at zero. Also $Qz = 0$ and $[P, z] = 0$. Since z is a centralizer projection, zMz is modularly invariant. The KMS calculation there gives $\varphi(Pz) = 0$, hence $Pz = 0$. If $z \neq 0$, the normalized restriction of φ is faithful normal on a semifinite corner with a full critical Cuntz family. The same window argument excludes this. Hence $z = 0$.

Put $\mathcal{T}(X) = \sum_i B_i X B_i^*$. Telescoping gives

$$I = \sum_{j=0}^{m-1} \mathcal{T}^j(Q) + \mathcal{T}^m(I). \tag{74}$$

For a length-m word v, $CB_v = \gamma^m B_v C$. Bounded spectral calculus puts its range in $1_{[0, \gamma^m]}(C)\mathcal{H}$. Thus

$$0 \leq \mathcal{T}^m(I) \leq 1_{[0, \gamma^m]}(C) \longrightarrow 0$$

strongly by injectivity of C. Each $\mathcal{T}^j(Q)$ is supported in the γ^j eigenspace of C. These positive terms sum to I and have mutually orthogonal spectral supports. Compressing the sum to each eigenspace gives

$$E_j = \mathcal{T}^j(Q), \qquad \sum_{j\geq 0} E_j = I. \tag{75}$$

There is no residual spectral subspace since the sum is I. As $[P, C] = 0$ and $P \neq 0$, some PE_n is nonzero. In particular $Q = E_0$. □

The selector of this section uses $d = 2$ and $\gamma = 1/2$, with edges reindexed B_0, B_1; the coupled construction uses $d = 16$ and the same depth ratio.

11.2 Finite models and a small example

For words w of length at most n, let B_i append i unless w is a leaf, when it vanishes. Let Q select the empty word and P the depth-n words. Put

$$C|w\rangle = 2^{-|w|}|w\rangle, \qquad h_n|w\rangle = \frac{2^{-|w|}}{n+1}|w\rangle. \tag{76}$$

Each nonleaf has two distinct children and each nonroot a unique parent. This proves (70), including the cross term. The child-to-parent density ratio is $1/2$, proving (71); the same ratio for C proves (72). Each depth has state mass $1/(n+1)$, whence

$$\varphi_n(P) = \varphi_n(Q) = \frac{1}{n+1}, \qquad S(h_n) = \log_2(n+1) + \frac{n}{2}. \tag{77}$$

For $n = 2$, the seven word densities are $1/3$, two copies of $1/6$, and four copies of $1/12$. Their C labels are $1, 1/2, 1/4$. The leaf rank is four but its state mass is $1/3$. A finite host's normalized trace gives $\tau(P) \geq 1/2$ from (70); this does not bound $\varphi(P)$ away from zero uniformly over states.

11.3 A finite presentation verifies the depth

Fix a total recursive predicate $\mathcal{R}_{\mathrm{dec}}(e, n, t)$ and $\mathcal{A}_{\mathrm{dec}} = \{e : \exists n\, \forall t\, \mathcal{R}_{\mathrm{dec}}(e, n, t)\}$. With $[x, y] = xyx^{-1}y^{-1}$ set

$$\begin{aligned} \Gamma_e = \langle a, b, c \mid {r^{\mathrm{grp}}}_n^2 = 1 \ (n \geq 0), \\ {r^{\mathrm{grp}}}_n = 1 \text{ if } \exists t\, \neg\mathcal{R}_{\mathrm{dec}}(e, n, t)\rangle, \qquad {r^{\mathrm{grp}}}_n = [b, a^n c a^{-n}]. \end{aligned} \tag{78}$$

Lemma 11.2. *${r^{\mathrm{grp}}}_n$ is nontrivial in Γ_e exactly when $\forall t\, \mathcal{R}_{\mathrm{dec}}(e, n, t)$; a nontrivial ${r^{\mathrm{grp}}}_n$ has order two. Uniformly there is a finite presentation $\mathcal{G}_e$ and explicit words for embedded a, b, c preserving this property.*

Proof. Let $\mathcal{D}_{\mathrm{good}} \subseteq \mathbb{N}$ be the good integers. Take two copies of $\mathbb{V} = \bigoplus_{m\in\mathbb{Z}} \mathbb{F}_2\varepsilon_m$ and $\mathbb{W} = \bigoplus_{k\in\mathcal{D}_{\mathrm{good}}} \mathbb{F}_2 z_k$. Define $\mathfrak{b}(\varepsilon_m, \varepsilon_\ell) = z_{\ell-m}$ if $\ell - m \in \mathcal{D}_{\mathrm{good}}$, zero otherwise. On $\mathbb{V} \oplus \mathbb{V} \oplus \mathbb{W}$ put

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \mathfrak{b}(x, y')). \tag{79}$$

Bilinearity makes the third coordinate in either association $z + z' + z'' + \mathfrak{b}(x, y') + \mathfrak{b}(x, y'') + \mathfrak{b}(x', y'')$. The identity is $(0, 0, 0)$; the inverse of (x, y, z) is $(x, y, z + \mathfrak{b}(x, y))$. Simultaneously shifting both $\mathbb{V}$ indices preserves $\mathfrak{b}$; adjoin this automorphism as a. Put $b = (\varepsilon_0, 0, 0)$ and $c = (0, \varepsilon_0, 0)$. Multiplication gives ${r^{\mathrm{grp}}}_n = (0, 0, z_n)$ for $n \in \mathcal{D}_{\mathrm{good}}$, identity otherwise. All relators hold, and every good word has a nontrivial order-two image. Bad words are killed by listed relators, proving the assertion.

For precision we give a uniform Craig conversion [16] of the c.e. relator list to a decidable literal-word list. Chiodo's convention already permits recursive enumeration, so this extra normalization is optional for the cited embedding theorem. Here is a Craig conversion that also preserves the old generator words. Adjoin d with relator $d = 1$. For an enumerated relator r at time t, replace that occurrence by the literal word rd^k, where distinct increasing codes k are chosen effectively with $k \geq t$. Given a word of length ℓ, membership in the resulting literal relator set is decidable by simulating the first ℓ enumeration steps, constructing those coded words of length at most ℓ, and comparing literally. Use freely reduced old words r, which contain no d; the final d^k therefore survives free reduction. Include d itself as a relator. Modulo $d = 1$, the coded relations are exactly the original r. The group is therefore unchanged and the old generator images remain explicit. Higman embedding [30], in the effective uniform formulation of [9, Theorem 2.2], gives a finite presentation with an explicit injective embedding. Injectivity preserves the nontrivial ${r^{\mathrm{grp}}}_n$. The model (79) need not itself be computable. □

Represent the generators of $\mathcal{G}_e$ by centralizer unitaries satisfying its finite relators. Add centralizer unitaries T, L. Require every group generator and T to commute with B_0, B_1, C. Impose

$$LQ = cQ, \quad LB_i = B_i a L a^*, \quad [L, C] = [L, P] = 0, \tag{80}$$

and the centralizer polynomial

$$P\big(T[b, L]T^* + [b, L]\big) = 0. \tag{81}$$

Lemma 11.3 (Level reader)**.** *In every semifinite realization $LE_n = a^n c a^{-n} E_n$. The terminal equation then implies that some n is good.*

Proof. Group generators commute with C, hence with $Q = E_0$. In particular $a^*Q = Qa^*$; this is needed in the induction. For a word v of length n, the endpoint equation and commutation of a with the B_i give $LB_vQ = B_v a^n L a^{-n} Q = B_v a^n c a^{-n} Q$. By (75) these ranges span $E_n\mathcal{H}$, proving the level identity. A unitary commuting with B_i also commutes with B_i^*: commute its inverse, then take adjoints. It therefore commutes with P. Thus PE_n reduces the group generators, T and L. Choose a nonzero such sector by Lemma 11.1. On it (81) becomes $T{r^{\mathrm{grp}}}_n T^* = -{r^{\mathrm{grp}}}_n$. A bad n has relator ${r^{\mathrm{grp}}}_n = I$, giving $I = -I$ on a nonzero space. This contradiction proves goodness. □

Lemma 11.4 (Finite realization)**.** *A good n gives a finite-algebra realization of all relations.*

Proof. In $\mathcal{G}_e \times \langle j \mid j^2 = 1\rangle$, the subgroup $\langle {r^{\mathrm{grp}}}_n, j\rangle$ is C_2^2. The map ${r^{\mathrm{grp}}}_n \mapsto j{r^{\mathrm{grp}}}_n$, $j \mapsto j$ is its automorphism. Form the HNN extension Γ_n^{HNN} with stable letter T:

$$T{r^{\mathrm{grp}}}_n T^{-1} = j{r^{\mathrm{grp}}}_n, \qquad TjT^{-1} = j. \tag{82}$$

The HNN normal-form theorem [31] embeds the base group. Consequently $j \neq 1$, and j commutes with the base and with T, so is central in Γ_n^{HNN}. This is the group-extension mechanism used in the nonlocal-game setting of [53]. In $L(\Gamma_n^{\mathrm{HNN}})$, $p_- = (1 - \lambda(j))/2$ is central and has trace $1/2$. The corner $M_{\mathrm{HNN}} = p_- L(\Gamma_n^{\mathrm{HNN}})$ with trace 2τ is finite. Compressed group unitaries are unitaries of M_{HNN}, and (82) becomes $T{r^{\mathrm{grp}}}_n T^* = -{r^{\mathrm{grp}}}_n$.

Tensor M_{HNN} with the depth-n tree, with density $h_n \otimes 1$ relative to matrix trace times the normalized trace of M_{HNN}. Group generators and T are constant across vertices. Set $L = a^\ell c a^{-\ell}$ on depth ℓ. It is a unitary centralizer label commuting with P, C. On a nonleaf, $LB_i = B_i a L a^*$ because depth increases by one; both sides vanish on a leaf. At the root $LQ = cQ$. The terminal equation is the sign-flip identity on the leaves and is vacuous elsewhere. The tree relations hold by (76). Tensor with R_{hyp} and select a factor fibre to obtain a II_1 host. The presentation itself never needs to find the good n. □

For a small example, at good depth $n = 1$ the tree is $\{\emptyset, 0, 1\}$ and the blocks of L are c, aca^{-1}, aca^{-1}. The terminal word is $r^{\mathrm{grp}}{}_1 = [b, aca^{-1}]$. The model (79) with good set $\{1\}$ sends it to $z_1 \neq 1$, while $r^{\mathrm{grp}}{}_0, r^{\mathrm{grp}}{}_2$ vanish. The HNN step flips the chosen involution after compression to $j = -1$. One T is not required to flip all good words.

11.4 One bridge compiles the selector presentation

Use five orthogonal synchronized corners $e_0, m_{00}, m_{01}, m_{10}, m_{11}$ covering I, with one FMS copy whose flag is e_0 and magic support is their other four corners. Put $e_1 = m_{00}$. The actual bridge $v = \mathsf{U}_1 : e_0 \to e_1$ obeys (14). KMS and the cover give $\varphi(e_0) = 1/9$ and $\varphi(m_{ab}) = 2/9$; here $9 = \mathfrak{c}_1$. These masses are independent of the root model.

$$\underbrace{e_0}_{1/9} \xrightarrow[\sigma_t^{\varphi}(v)=2^{it}v]{v} \boxed{\underbrace{m_{00} = e_1}_{2/9} \quad m_{01} \quad m_{10} \quad m_{11}} \qquad \text{each magic corner has mass } 2/9.$$

Figure 4: One calibrated bridge. All copies of transported labels use this same v; the box is the four-dimensional magic support.

Bind the five corners and the two positive ZI, IX parameters to the qutrit frame. Introduce the two axis unitaries $\mathfrak{x}, \mathfrak{z}$ with the explicit formulas of Section 4.1. A *typed label* is an operator name together with its required class (projection, positive contraction or unitary). The transported typed label list is exactly

$$Q,\ C,\ T,\ L,\ \{U_g : g \text{ is a generator of } \mathcal{G}_e\}. \tag{83}$$

Encode a projection by a reflection, a positive contraction by $u_C = 2C - I + 2i\sqrt{C(I-C)}$, and a unitary directly. A *unitary name* is a formal generator for the chosen unitary encoding, including separate names for separate summands when needed. For each resulting unitary name ℓ introduce copies $u_{r,\ell}$, $r = 0, 1$, with $(u_{r,\ell} - I)(I - e_r) = 0$. The *decoded copies* are the actual corner operators recovered from these encodings by the following formulas, respectively: $e_r(I + u_{r,Q})/2$, $e_r(2I + u_{r,C} + u_{r,C}^*)/4$, and $e_r u_{r,\ell}$. Require $u_{r,Q}^2 = I$. Introduce the block unitary $V_{1,\ell}$ of (25); since the cover is the whole space it reads

$$V_{1,\ell} = e_0 u_{0,\ell} + \sum_{a,b} \mathfrak{x}^a \mathfrak{z}^b e_1 u_{1,\ell} \mathfrak{z}^{-b} \mathfrak{x}^{-a}.$$

Refine the retained FMS signs separately for the six frame projections

$$F_1, \quad F_2, \quad F_3, \quad G_{12}(1), \quad G_{13}(1), \quad G_{23}(1),$$

and for every $G_{12}(V_{1,\ell})$. Lemma 4.2 first makes v preserve the frame, so $v = I_3 \otimes \widetilde{v}$. The graph-label commutators in the target–source block give

$$v u_{0,\ell} = u_{1,\ell} v. \tag{84}$$

Thus the same actual calibrated transport works for every label. Transport of decoded adjoints and polynomials follows too. P is root-only, encoded by $P_0 = e_0(I + u_P)/2$ with $u_P^2 = I$ and $(u_P - I)(I - e_0) = 0$.

Introduce two centralizer contractions $b_i = e_1 b_i e_0$. For compilation encode each as an average of two global unitaries. Extract $B_i = v^* b_i$ on $e_0 M e_0$. Their modular eigenvalue is $1/2$ because that of b_i is one. The complete endpoint translation is

Root relation	Centralizer polynomial
$B_i^* B_j = \delta_{ij}(I - P)$	$b_i^* b_j = \delta_{ij}(e_0 - P_0)$
$\sum_i B_i B_i^* = I - Q$	$\sum_i b_i b_i^* = e_1 - Q_1$
$CB_i = \frac{1}{2} B_i C$	$C_1 b_i = \frac{1}{2} b_i C_0$
$U_g B_i = B_i U_g$	$(U_g)_1 b_i = b_i (U_g)_0$
$TB_i = B_i T$	$T_1 b_i = b_i T_0$
$LB_i = B_i a L a^*$	$L_1 b_i = b_i a^{[0]} L_0 (a^{[0]})^*$

Every remaining relation is a root centralizer polynomial: the finite relators of $\mathcal{G}_e$, $C_0 Q_0 = Q_0$, $[C_0, P_0] = 0$, $[U_g, C_0] = [T_0, C_0] = 0$, $L_0 Q_0 = c^{[0]} Q_0$, $[L_0, C_0] = [L_0, P_0] = 0$, and the terminal equation. Expand its root projection as $P = (I + s_P)/2$. Here $a^{[0]}, c^{[0]}$ mean the source words for the embedded a, c. If $W_{\mathrm{ter}} = [b, L]$, its non-self-adjoint polynomial is explicitly

$$p = \tfrac{1}{2}(T W_{\mathrm{ter}} T^* + W_{\mathrm{ter}} + s_P T W_{\mathrm{ter}} T^* + s_P W_{\mathrm{ter}}).$$

Use (17) for these words and their prefixes. In global scaffold coordinates multiply by e_0 and expand $e_0 = (I + s_{e_0})/2$ as well. The two off-diagonal blocks impose $p = 0$ and $p^* = 0$. Expand all other typed polynomials in the same manner, apply graphs and triangles, then Lemma 4.3. Retain all FMS and refinement questions. Sample all padded question pairs uniformly, reject the listed events, and accept unspecified pairs. This is an effective finite rational Boolean game, denoted G_e^{sel}.

Soundness and simultaneous completeness. Soundness uses the nonzero root corner with normalized state $9\varphi|_{e_0 M e_0}$. Equations (84) and the table give all root relations as bounded operator identities. For example $C_0 v^* = v^* C_1$ gives $C_0 B_i = v^* C_1 b_i = B_i C_0/2$. Conjugating the range equation by v^* uses $v^* Q_1 v = Q_0$. The corner of a semifinite host is semifinite, so Lemmas 11.1 and 11.3 prove soundness.

Conversely take any faithful normal root realization (M, φ). On $\mathrm{Mat}_5 \bar{\otimes} M$ use the state

$$\varphi_{\mathrm{sc}} = \mathrm{diag}(1, 2, 2, 2, 2)/9 \ \otimes \varphi. \tag{85}$$

Use the identity corner maps $v = |e_1\rangle\langle e_0| \otimes I$, the ideal real FMS effects, and $b_i = |e_1\rangle\langle e_0| \otimes B_i$. The product modular group makes each b_i centralizer because the two modular eigenvalues 2 and $1/2$ cancel. The whole centralizer is finite, with its faithful trace φ_{sc}. For any contraction x in a finite von Neumann algebra, the polar partial isometry extends to a unitary u: equivalent initial and final projections have equivalent complements in a finite algebra. Then

$$u_\pm = u(|x| \pm i\sqrt{I - |x|^2}), \qquad x = (u_+ + u_-)/2.$$

This supplies the exact encodings of both b_i at once. Use $2Q - I, 2P - I$ and the displayed functional-calculus encoding of C, copying the same choices to each required corner. Amplify by the scalar qutrit frame. All block labels commute with the ideal FMS signs, so use their actual joint products for refinement. Separate labels use separate questions. Finally apply the standard-form construction in Lemma 4.3, with Bob's effects defined by its conjugation $\mathcal{J}_{\mathrm{new}}$. Formula (85) and the ideal real FMS effects give exactly the five-dimensional FMS probabilities, including the nonsynchronous signs. All graph and endpoint identities hold simultaneously. A finite-algebra root stays finite; after tensoring with R_{hyp} a II_1 host exists. A $\mathrm{III}_{1/2}$ root stays $\mathrm{III}_{1/2}$ under the finite amplifications.

11.5 Both uniform promises and the hardness theorem

Theorem 11.5 (Uniform semifinite reduction)**.** *Uniformly from a total recursive $\mathcal{R}_{\mathrm{dec}}$ and an input e, the compiler gives a finite rational Boolean game G_e^{sel} such that*

$$G_e^{\mathrm{sel}} \in \mathrm{PERF}(\mathrm{II}_1) \iff G_e^{\mathrm{sel}} \in \mathrm{PERF}_{\mathrm{sf}} \iff \exists n \, \forall t \, \mathcal{R}_{\mathrm{dec}}(e, n, t).$$

For every input e, G_e^{sel} has a perfect $\mathrm{III}_{1/2}$ strategy and $\omega_q(G_e^{\mathrm{sel}}) = 1$.

Proof. For the equivalence, soundness follows from the compiler and the selector and level-reader lemmas. A good n gives a finite model by Lemma 11.4, which the compiler turns into perfect finite-host play and hence a II_1 host.

For the first uniform promise take the uniform KMS representation of $\mathcal{O}_2$ with generators s_i satisfying $s_i^* s_j = \delta_{ij} I$, $\sum_i s_i s_i^* = I$ and $\sigma_t(s_i) = 2^{-it} s_i$. Its weak closure is $R_{1/2}$; see the Cuntz/KMS identification in [45] for the gauge KMS state and [29, Theorem 4.9 and Example 4.13] for this factor identification, also used in [20, Section 3.1]. Set $B_i = s_i$, $P = Q = C = 0$, and take all group generators, T, L equal to I. Every root equation holds: $LQ = cQ$ and the terminal row are vacuous. In the scaffold (85), $b_i = |e_1\rangle\langle e_0| \otimes s_i$ is centralizer. Its exact two-unitary completion exists in the finite centralizer, as proved above. The encoding of $C = 0$ is $u_C = -I$ on each corner; the zero projections use reflection $-I$ there as well. Their transported block labels commute with the ideal signs. Thus every compiler gadget has a simultaneous perfect realization. The host, including all finite ancillary matrices, is $\mathrm{III}_{1/2}$.

For the second promise take the finite-tree strategy $\mathcal{V}_N$ with the depth-N finite tree and trivial group labels. Every root equation is exact except the terminal row. Its residual is

$$R_N^{\mathrm{ter}} = 2P, \qquad \varphi_N(R_N^{\mathrm{ter}*} R_N^{\mathrm{ter}}) = 4\varphi_N(P) = \frac{4}{N+1}. \tag{86}$$

Build all graph, prefix, joint and column measurements by their exact formulas even for this nonzero application row. All their structural tests remain exact. Only its final row-zero event can lose. For an honest row, (19) gives its loss.

More explicitly, if the lifted root residual is $p = e_0 R_N^{\mathrm{ter}}$, its graph row has off-diagonal blocks p, p^*. The qutrit trace and root mass give

$$\varphi_{\mathrm{old}}(S_{\mathrm{graph}}^2) = \tfrac{1}{3}\varphi_{\mathrm{sc}}(p^*p + pp^*) = \tfrac{2}{27}\varphi_N(R_N^{\mathrm{ter}*} R_N^{\mathrm{ter}}).$$

The last equality uses traciality on the centralizer. The *padded question count* is the common size of the two question sets after adding always-accepted fresh questions. An *orientation* is a choice of which party receives the row question and which receives the scalar reference question. Let $m_e \geq 1$ be the finite number of retained terminal-test copies in the chosen encoding, determined solely by e. If Q_e is the padded question count and ℓ_e, d_e are the row denominator and slot dimension, then, for each retained terminal-test copy, union bounding its orientations gives a loss of at most $64\ell_e^2/(27 d_e^3 Q_e^2 (N+1))$. Thus the total loss satisfies

$$1 - \omega(G_e^{\mathrm{sel}}, \mathcal{V}_N) \leq m_e \frac{64\ell_e^2}{27 d_e^3 Q_e^2} \frac{1}{N+1} = \frac{C_e^{\mathrm{sel}}}{N+1}.$$

Take

$$C_e^{\mathrm{sel}} = m_e \frac{64\ell_e^2}{27 d_e^3 Q_e^2}.$$

Thus C_e^{sel} depends only on the finite compiled specification, denominators, row dimensions and the fixed root mass $1/9$. It is independent of N, the density spectrum and the finite tree dimension. Letting $N \to \infty$ proves $\omega_q(G_e^{\mathrm{sel}}) = 1$. □

Choose $\mathcal{A}_{\mathrm{dec}}$ to be Σ_2^0-complete. Theorem 11.5 and (10), together with Theorem 3.9, prove Σ_2^0-completeness of every claimed semifinite existence language in Theorem B. Furthermore every G_e^{sel} is commuting perfect and has quantum value one, and

$$G_e^{\mathrm{sel}} \in \text{III-ONLY} \iff e \notin \mathcal{A}_{\mathrm{dec}}, \qquad \text{III-ONLY} := \mathrm{PERF}_{\mathrm{co}} \setminus \mathrm{PERF}_{\mathrm{sf}}.$$

This proves Π_2^0-hardness within that promise, a stronger statement than hardness of intersection with the value-one language. Membership follows from $\mathrm{PERF}_{\mathrm{co}} \in \Pi_1^0$ and $\mathrm{PERF}_{\mathrm{sf}}^c \in \Pi_2^0$: their intersection is Π_2^0.

12 Each type-III subtype has the commuting language

A type-III factor has no nonzero finite projection, so the corner of Section 3.5 does not exist. This section shows that nothing replaces it: every fixed subtype hosts exactly the commuting-perfect games. The state and host may both be chosen. This quantifier is essential.

Background lemma (Expected-inclusion transfer; [54, 20])**.** *If $E : N \to M$ is a faithful normal conditional expectation, every faithful standard-form correlation on M is realized on N.*

Proof. For the state $\varphi_N = \varphi_M \circ E$, put $\mathcal{H}_{\mathrm{inc}} = \overline{M\xi_{\varphi_N}} \subset L^2(N)$. The map $x\xi_{\varphi_M} \mapsto x\xi_{\varphi_N}$ is an isometry. Takesaki's expectation theorem [54] makes M invariant under the modular group. The Tomita operators agree on the common core $M\xi$, and their conjugations restrict as $\mathcal{J}_N|_{\mathcal{H}_{\mathrm{inc}}} = \mathcal{J}_M$. Thus for left effects A and right effects $\mathcal{J}_M B \mathcal{J}_M$,

$$\langle \xi_{\varphi_N}, A\mathcal{J}_N B \mathcal{J}_N \xi_{\varphi_N}\rangle = \langle \xi_{\varphi_M}, A\mathcal{J}_M B \mathcal{J}_M \xi_{\varphi_M}\rangle.$$

Positivity, sums and all probabilities are preserved. Support reduction handles nonfaithful states. This is also [20, Lemma 3.1]. □

Lemma 12.1. *For any factor N_{in} with separable predual, $N_{\mathrm{in}}\bar{\otimes}R_\infty$ is III_1, and contains N_{in} with a faithful normal expectation.*

Proof. The expectation is $\mathrm{id}\otimes\rho$ for a faithful normal state ρ on R_∞. We use the proof, rather than the stronger hyperfinite conclusion, of [27, Lemma 3.8]. For each $0 < \lambda < 1$, choose $0 < \mu < 1$ with $\log\lambda/\log\mu$ irrational. The injective factors satisfy $R_\infty \cong R_\lambda\bar{\otimes}R_\mu$ and $R_\infty\bar{\otimes}R_\lambda \cong R_\infty$. Consequently $(N_{\mathrm{in}}\bar{\otimes}R_\infty)\bar{\otimes}R_\lambda \cong N_{\mathrm{in}}\bar{\otimes}R_\infty$. The asymptotic-ratio argument of Connes [13, Section 3.6], used in that proof, gives $\lambda \in r_\infty(N_{\mathrm{in}}\bar{\otimes}R_\infty)$. Connes's implication [13, Theorem 3.6.1] gives $r_\infty \setminus \{0\} \subseteq S$. Every $0 < \lambda < 1$ therefore belongs to the S-invariant, and the factor is III_1. These steps do not assume hyperfiniteness of N_{in}. Only the last identification with the unique injective III_1 factor in Haagerup–Musat needs hyperfiniteness, and we do not make that identification here. □

Theorem 12.2. *For each fixed $\lambda \in [0,1]$, $\mathrm{PERF}(\mathrm{III}_\lambda) = \mathrm{PERF}_{\mathrm{co}}$.*

Proof. The inclusion into commuting perfection is immediate. For the converse, standardize a commuting-perfect strategy and choose a factor host N_{in} by central decomposition as in Section 2. Put $N_{\mathrm{abs}} = N_{\mathrm{in}}\bar{\otimes}R_\infty$. By expected-inclusion transfer and Lemma 12.1, this III_1 factor hosts the game. Retain it for $\lambda = 1$.

Fix a faithful normal state φ on N_{abs}. For a countable subgroup $\Upsilon \subset \mathbb{R}$, given the discrete topology, form $N_\Upsilon = N_{\mathrm{abs}} \rtimes_{\sigma^\varphi|_\Upsilon} \Upsilon$. The Fourier expectation onto N_{abs} is faithful normal, so transfers the game. It is a factor: $\mathrm{T}(N_{\mathrm{abs}}) = \{0\}$ for a III_1 factor, hence every nonzero modular time is outer. If a central element has Fourier coefficient x_g, commutation with N_{abs} makes x_g an intertwiner between id and σ_g^φ. A nonzero such coefficient for $g \neq 0$ would, by polar decomposition in a factor, implement that automorphism innerly. Thus only $g = 0$ remains, and x_0 is scalar. Faithfulness of Fourier coefficients proves factoriality.

The continuous core is the crossed product by the modular $\mathbb{R}$-action; its dual action restricted to the center is the *flow of weights*. For factors this flow distinguishes the type-III subtypes [15]. Here is the core computation, with the dual flow specified. The modular group of the dual state fixes the canonical unitaries u_g of the discrete crossed product. Writing $\lambda(t)$ for the continuous-core implementing unitaries, it follows that

$$\begin{aligned} c(N_\Upsilon) &\cong c(N_{\mathrm{abs}}) \rtimes_{\mathrm{Ad}\,\lambda(g)} \Upsilon \\ &\cong c(N_{\mathrm{abs}})\bar{\otimes}L(\Upsilon). \end{aligned} \tag{87}$$

For the second isomorphism the unitaries $y_g = \lambda(g)^* u_g$ commute with $c(N_{\mathrm{abs}})$ and satisfy the group law of Υ; the inverse sends u_g to $\lambda(g) \otimes y_g$. Since N_{abs} is III_1, $c(N_{\mathrm{abs}})$ is a factor, so the center is $L(\Upsilon) = L^\infty(\widehat{\Upsilon})$. The dual action obeys $\theta_s(\lambda(t)) = e^{-ist}\lambda(t)$ and $\theta_s(u_g) = u_g$; therefore $\theta_s(y_g) = e^{isg} y_g$. Its action on the center is translation by the character $\chi_s(g) = e^{isg}$.

For $0 < \lambda < 1$, take $\Upsilon = t_\lambda \mathbb{Z}$, $t_\lambda = 2\pi/(-\log \lambda)$. The resulting flow is rotation on the circle with least period $-\log \lambda$, which is the flow of a III_λ factor [15]. For $\lambda = 0$ take $\Upsilon = \mathbb{Z} + \sqrt{2}\mathbb{Z}$. Its dual is the two-torus, and the flow is $s \mapsto (e^{is}, e^{i\sqrt{2}s})$. Fourier characters show ergodicity: an invariant character indexed by (m, n) requires $m + n\sqrt{2} = 0$, hence $m = n = 0$. Every orbit is a dense one-dimensional subgroup coset of Haar measure zero (its bounded segments have zero area). Thus the flow is not transitive. The Connes–Takesaki flow classification [15] gives subtype III_0. This is the same modular crossed-product mechanism as [27, Example 3.9], without requiring injectivity of N_{abs}.

For clarity non-semifiniteness is a separate conclusion. The core flow of a semifinite factor is translation on $L^\infty(\mathbb{R})$; it is transitive and admits no invariant normal probability state. Each of our circle or torus flows preserves Haar probability, so cannot be that semifinite flow. For N_{abs} itself non-semifiniteness was already proved by its S-invariant. Hence none of the subtype conclusions silently includes a semifinite alternative. □

Slofstra's commuting classification [53], also proved in [42, Theorem 6.4], now proves Π_1^0-completeness for every fixed subtype, including zero. With Section 11 this completes the proof of Theorem B.

13 One selector for spatial and semifinite perfection

The critical selector can choose the same witness in spatial and semifinite models. The construction uses two edge families on one scaffold: the selector is critical, while the four-letter computation tree is subcritical. The key point is that a nonzero selected terminal sector has positive intersection with the computation root. This supplies a finite witness without using a density eigenvector in a diffuse host.

Lemma 13.1 (Several edge families on a common chain)**.** *Fix $n \geq 1$ and finitely many edge families $X_i^{(r)}$. Each family has its own Gram equations, initial defect and range equation, and its own within-family endpoint equations with typed centralizer coefficients. There are no prescribed cross-family Gram equations. If all edges have modular ratio 2^{-n}, one finite rational Boolean game compiles these systems on a common nonzero source corner. It extracts them from every perfect normal commuting strategy, and every faithful normal realization extends simultaneously, preserving host type under finite amplification and finite entropy for spatial realizations.*

Proof. Use the corners, activities, bound axes, and common chain $v = \mathsf{U}_n \cdots \mathsf{U}_1$ of Lemma 4.6. Its source mass is $\mathfrak{c}_n^{-1}$. For each edge, independently introduce a supported centralizer contraction $a_i^{(r)} = e_n a_i^{(r)} e_0$. A family with initial defect D_r and range R_r is submitted as

$$a_i^{(r)*} a_j^{(r)} = \delta_{ij}(e_0 - (D_r)_0), \qquad \sum_i a_i^{(r)} a_i^{(r)*} = (R_r)_n.$$

Only that family's own equations are submitted. Transport the union of all labels appearing at left endpoints or in range equations, with one separate graph refinement per encoding and bridge. The common original signs force $L_n v = v L_0$ for every label in this union. Decode $X_i^{(r)} = v^* a_i^{(r)}$. Multiplication by v, v^* proves the Gram, range and endpoint translations in both directions; for example $L_n a_i^{(r)} = a_j^{(r)} B_0$ becomes $L_0 X_i^{(r)} = X_j^{(r)} B_0$. The centralizer

condition and the modular eigenvalue of v give ratio 2^{-n} for every extracted edge. All equations concern the same faithful corner and state.

Conversely use $\mathrm{diag}(\mu_n) \otimes \varphi_0$ and $a_i^{(r)} = |e_n\rangle\langle e_0| \otimes X_i^{(r)}$ for every family. Every lifted contraction lies in the finite centralizer and has its own two-unitary encoding. Choose the root encodings once and copy them through the identity corner maps. The resulting block labels commute with their bridge's canonical signs. Actual joint products give all separate refinements, followed by the common qutrit and row extensions. There is one state and one assignment for all families. The construction adds only finite factors and imposes no mixed orthogonality. □

13.1 The coupled root presentation

Use four edges A_{ab} satisfying (30) at modular ratio 1/16, with root $P_\circ$, and sixteen different edges S_i with

$$S_i^* S_j = \delta_{ij}(I - P), \quad \sum_{i=1}^{16} S_i S_i^* = I - Q, \quad \sigma_t^{\varphi}(S_i) = 16^{-it} S_i.$$

For a positive centralizer contraction C_{sel} impose

$$C_{\mathrm{sel}} Q = Q, \quad C_{\mathrm{sel}} S_i = \tfrac{1}{2} S_i C_{\mathrm{sel}}, \quad [C_{\mathrm{sel}}, P] = 0, \quad [P, A_{ab}] = [C_{\mathrm{sel}}, A_{ab}] = 0.$$

Lemma 11.1 above uses modular ratio 1/16 and depth ratio 1/2. Retain the coordinate and boundary equations (34)–(37) and the counter equations (43), with $[C_{\mathrm{ctr}}, C_{\mathrm{sel}}] = 0$. Replace the clock by

$$(2C_{\mathrm{ctr}} - C_{\mathrm{sel}})P_{\mathrm{ret}} = 0, \qquad T_{\mathrm{inv}}(2C_{\mathrm{ctr}} - C_{\mathrm{sel}}) = \tfrac{1}{8} C_{\mathrm{ctr}} C_{\mathrm{sel}}(I - P_{\mathrm{ret}}). \tag{88}$$

Here T_{inv} is a general contraction. The tableau has the same initial and update equations, but its halt prohibition is $P_{\mathtt{H}} P = 0$. Its machine reads the first positive pulse d_{clk}, sets $n = d_{\mathrm{clk}} - 1$, and tests $\mathcal{R}_{\mathrm{dec}}(e, n, 0), \mathcal{R}_{\mathrm{dec}}(e, n, 1), \ldots$, halting at the first false answer. No prime test is used.

Lemma 13.1 with $n = 4$ compiles the two separate Gram systems. The transported typed list is exactly

$$P_\circ, U_x, U_y, P_x, P_y, C_{\mathrm{ctr}}, P, C_{\mathrm{sel}}, Q.$$

The return projection, its inverse and the tableau projections are source-only. In particular no relation $A_{ab}^* S_i = 0$ is imposed. Denote this finite game by F_e.

13.2 The terminal sector meets the root

Lemma 13.2. *Let orthogonal isometries $A_1, \ldots, A_r$ have modular ratio ν on a faithful normal support. If a centralizer projection p commutes with them and $r\nu < 1$, then*

$$\varphi\left(p\left(I - \sum_i A_i A_i^*\right)\right) = (1 - r\nu)\varphi(p).$$

Proof. Commutation with a self-adjoint projection also gives commutation with the adjoints. KMS gives $\varphi(A_i p A_i^*) = \nu\varphi(p A_i^* A_i) = \nu\varphi(p)$. Subtract the sum of these r terms from $\varphi(p)$. □

In a semifinite realization, Lemma 11.1 with $d = 16$ gives some nonzero $p_n = P 1_{\{2^{-n}\}}(C_{\mathrm{sel}})$. It commutes with all A_{ab} and their adjoints. Hence

$$\varphi(p_n P_\circ) = \tfrac{3}{4}\varphi(p_n) > 0. \tag{89}$$

Choose $0 \neq \xi \in p_n P_\circ \mathcal{H}$. Its word descendants remain in $p_n \mathcal{H}$, are nonzero, and are orthogonal at every fixed word length. The coordinate inductions identify every finite square on these vectors.

For self-adjoint X, Y, with Y injective, the two equations $(2X-Y)R = 0$ and $T(2X-Y) = \kappa XY(I-R)$, $\kappa \neq 0$, force $R = 1_{\{0\}}(2X-Y)$. Indeed $R \leq K = 1_{\{0\}}(2X-Y)$. For $\zeta \in K\mathcal{H}$ put $\eta = (I-R)\zeta \in K\mathcal{H}$. The second equation gives $XY\eta = 0$, and $Y\eta = 2X\eta$ gives $2X^2\eta = 0$. Self-adjointness gives $X\eta = 0$, then $Y\eta = 0$, then $\eta = 0$. Apply this to (88), using injectivity of C_{sel} from the selector lemma. On the chosen descendants, $C_{\mathrm{ctr}} = 2^{-y}$ and $C_{\mathrm{sel}} = 2^{-n}$. Thus the pulse is exactly $y = n+1$, including no time-zero pulse.

Lemma 13.3 (Definite computation on a terminal sector)**.** *Suppose $0 \neq \xi \in P_\circ \mathcal{H}$, every $A_w \xi$ belongs to $P\mathcal{H}$, and P_{ret} has the specified classical scalar on each $A_w \xi$. With the coordinate, initial and update equations above, every symbol projection acts on these descendants as the genuine classical symbol. If the presentation imposes $P_{\mathtt{H}} P = 0$, that classical computation cannot halt.*

Proof. At each fixed depth, the coordinate recursions give cyclic shifts and boundary masks on the word descendants. Row zero follows from (38) without its global halt equation. Induct on time at all spatial coordinates simultaneously. Each factor in the displayed ordered update product acts as its classical zero or one on the induction vector: left and right factors refer to the preceding row, and the pulse is scalar by hypothesis. Exactly the prescribed successor survives. Multiplication by U_y gives the next-row assertion. The last-row mask removes wraparound; radius one and blank quiescence make the right ghost blank throughout the required prefix. This proves the half-line computation on every sufficiently large square. If a halt occurred, one nonzero descendant $\eta \in P\mathcal{H}$ would satisfy $P_{\mathtt{H}}\eta = \eta$, contradicting $P_{\mathtt{H}}\eta = P_{\mathtt{H}} P \eta = 0$. The conclusion uses the restricted prohibition, not a claim that $P_{\mathtt{H}} = 0$ on the entire space. □

Theorem 13.4 (A shared family)**.** *The family F_e satisfies*

$$F_e \in \mathbf{E} \iff F_e \in \mathbf{I} \iff F_e \in \mathbf{U} \iff \exists n \, \forall t \, \mathcal{R}_{\mathrm{dec}}(e, n, t), \qquad F_e \notin \mathbf{J}.$$

Proof. Semifinite soundness follows from the selected sector, the exact pulse and Lemma 13.3. A false check at that selected n would halt, so n is a witness.

For completeness let n be a witness. On the finite sixteen-letter tree of words v of length at most n, use prefix maps S_i, zero on leaves, with P the leaves, Q the root and

$$h_{\mathrm{sel}}|v\rangle = (n+1)^{-1} 16^{-|v|} |v\rangle, \qquad C_{\mathrm{sel}}|v\rangle = 2^{-|v|}|v\rangle.$$

Tensor it with the four-letter Fock density h_F of (32). At selector depth ℓ, put $P_{\mathrm{ret}} = 1_{\{y=\ell+1\}}$ and

$$T_{\mathrm{inv}}(y,\ell) = \begin{cases} [8(2^{\ell+1} - 2^y)]^{-1}, & y \neq \ell+1, \\ 0, & y = \ell+1. \end{cases}$$

The nonzero denominator before the factor eight is an integer. The norm is exactly $1/8$, attained at $\ell = 0, y = 0$. Negative values occur. Use the genuine classical tableau for input ℓ on each such fibre; all symbol projections are its diagonal indicator projections. Coordinate permutations preserve Fock depth, and every other centralizer label is diagonal. Both edge families have ratio $1/16$. Computations at depths $\ell < n$ may halt; the terminal equation excludes only depth n. This gives every label and equation simultaneously, also at $n = 0$, where $S_i = 0$, $P = Q = C_{\mathrm{sel}} = I$.

The selector entropy is $\log_2(n+1) + 2n$, so the root product entropy is

$$\log_2(n+1) + 2n + \log_2(4/3) + 4/3 < \infty.$$

The compiler adds finite entropy. The inclusions $\mathbf{E} \subseteq \mathbf{I} \subseteq \mathbf{U}$ finish the equivalence. In a finite host the four extracted isometries would have orthogonal range traces summing to four under a normalized faithful trace, but their sum is at most the identity. Thus every instance is outside $\mathbf{J}$. □

14 Comparing the type languages

Equal arithmetical complexity does not mean equal languages. This section shows that the type-I and type-II_1 languages are incomparable and classifies the differences "perfect in X but not in Y" among six models.

14.1 A fixed one-bridge separator

Apply the complete one-bridge assembly of Section 11.4 to the fixed root presentation

$$A^*A = I, \qquad AA^* = I - P_\circ, \qquad \sigma_t^{\varphi}(A) = 2^{-it}A, \tag{90}$$

where $P_\circ$ is a centralizer projection. Call the resulting finite rational Boolean game G_{sh}. Here the transported label list contains just $P_\circ$. Encode its copies by reflections, introduce a centralizer contraction $a = e_1 a e_0$, and impose exactly

$$a^*a = e_0, \qquad aa^* = e_1 - (P_\circ)_1.$$

Use the six frame refinements and the graph-label refinement for $P_\circ$ from Section 11.4; apply the same graphs, triangles and rational rows. Extraction is $A = v^*a$, and $P = (P_\circ)_0$. Thus $A^*A = e_0$, $AA^* = e_0 - P$, and the modular ratio is $1/2$. Conversely every root model of (90) extends by that assembly, with $a = |e_1\rangle\langle e_0| \otimes A$. This specifies the game without a chained-bridge hypothesis.

For its type-I model take $\ell^2(\mathbb{N}_0)$, $A|m\rangle = |m+1\rangle$, $P_\circ = |0\rangle\langle 0|$, and

$$h|m\rangle = 2^{-m-1}|m\rangle, \qquad S(h) = \sum_{m\geq 0}(m+1)2^{-m-1} = 2 \text{ bits}.$$

The relation $hA = Ah/2$ verifies modular covariance. The scaffold and the finite maximally entangled compiler factors add finite entropy, so G_{sh} is perfect spatially with finite entropy.

In any perfect finite-host strategy, extraction takes place in the nonzero finite corner $e_0 M e_0$. Its isometry A must be unitary: a faithful finite trace gives $\tau(e_0 - AA^*) = 0$. Consequently $AA^* = e_0$ forces $P = 0$. On the other hand KMS for the original faithful state gives

$$\varphi(AA^*) = \tfrac{1}{2}\varphi(A^*A) = \tfrac{1}{2}\varphi(e_0), \qquad \varphi(P) = \tfrac{1}{2}\varphi(e_0) > 0,$$

a contradiction. Therefore G_{sh} has no finite host and witnesses $\mathrm{PERF(I)} \not\subseteq \mathrm{PERF}(\mathrm{II}_1)$. In particular it has no finite-dimensional perfect strategy.

14.2 What happens to the Weyl-clock family

The spatial reduction of Sections 5–7 uses the four Fock isometries A_{ab} of (30), with h_F of (32), the coordinate unitaries and boundary projections of (34)–(37), the Weyl relation (40), the clock and counter values (44), and the reader equations (45). Spatial soundness uses finite positive density eigenspaces and a determinant to force a finite-order phase. This determinant is unavailable in a diffuse finite factor.

Here is a normal II_∞ model of *every* input of that family. Put $\vartheta = (\sqrt{5}-1)/2$, $\zeta = e^{2\pi i\vartheta}$. In the finite rotation factor $L^\infty(\mathbb{T}) \rtimes_\vartheta \mathbb{Z}$, take unitaries $J_{\mathrm{W}}, K_{\mathrm{W}}$ with commutator ζI and

put $Z_{\mathrm{clk}} = \zeta I$. The invariant integral and Fourier coefficient at zero give a faithful finite trace. Irrational rotation is free and ergodic, so a central Fourier coefficient vanishes except at zero and its zero coefficient is constant; the algebra is a factor. It is infinite-dimensional and hence II_1. It is the hyperfinite factor R_{hyp} by amenability of this crossed product and injective-factor classification [14]. The construction itself only needs that it is finite.

Use the tensor density $h_F \otimes 1$ on $B(\ell^2(\mathcal{W}))\bar{\otimes}R_{\mathrm{hyp}}$. Set $W = \zeta^y$, $C_{\mathrm{ctr}} = 2^{-y}$, $P_{\mathrm{ret}} = 1_{\{y=0\}}$, and

$$T_{\mathrm{inv}}|_{(x,y)} = \begin{cases} 0, & y = 0, \\ 2^{-y-1}(\zeta^y - 1)^{-1}, & y > 0. \end{cases}$$

The norm bound follows from the quadratic irrationality estimate. For an integer p nearest to $y\vartheta$, the nonzero integer $p^2 + py - y^2$ factors as $(p - y\vartheta)(p - y\vartheta')$, $\vartheta' = -(1+\sqrt{5})/2$. Since $|p - y\vartheta| \le 1/2$,

$$\|y\vartheta\| \ge \frac{1}{\sqrt{5}\,y + 1/2}, \qquad |\zeta^y - 1| = 2\sin(\pi\|y\vartheta\|) \ge \frac{4}{\sqrt{5}\,y + 1/2} > 2^{-y-1}.$$

The sine inequality uses concavity, $\sin(\pi t) \ge 2t$ for $0 \le t \le 1/2$. In particular it supplies the explicit all-integer bound $|\sin(\pi y\vartheta)| \ge 8/(11y)$, since $\sqrt{5} < 9/4$ and $y \ge 1$. For the last inequality it suffices to replace $\sqrt{5}$ by $9/4$: $2^{y+3} > 9y/4 + 1/2$ for $y = 1$, and doubling proves it inductively. Thus T_{inv} is a contraction and (45) holds. The binary coordinate recurrences $y \mapsto 2y + b$ verify the clock and counter endpoint equations.

In the tableau of Sections 5 and 7 a waiting front advances one cell per time step, at $x = y$ at time y. The first positive-time return pulse P_{ret} freezes it, and the frozen coordinate becomes the unary input of the check of the hardwired predicate; the time-zero pulse is ignored. Here P_{ret} selects $y = 0$ only, so no positive-time pulse occurs. The front never freezes, and the decoder never starts a checking computation, irrespective of the hardwired input. Its finite deterministic tableau relations are satisfied by that classical nonhalting tableau in every finite square. The coordinate, boundary and tableau projections are diagonal on word sectors and commute with $h_F \otimes 1$. All endpoint labels have the required centralizer type. Lemma 4.6 with $n = 4$ applies to exactly the conditional copies and transported labels of Lemma 4.5 and to the endpoint rows of Sections 5–7. Its left-endpoint labels are $P_\circ, U_x, U_y, P_x, P_y, W, C_{\mathrm{ctr}}, Z_{\mathrm{clk}}, J_{\mathrm{W}}, K_{\mathrm{W}}$; source-only labels are $P_{\mathrm{ret}}, T_{\mathrm{inv}}$ and the tableau projections. The activity corners and separate refinements coincide with that compiler. Its root mass is $1/121$. The resulting ambient host is II_∞ for every input. The supplementary exact margin certificate is in Appendix C.1.

Lemma 4.6, with $n = 4$, also extracts the four isometries of every Weyl-clock game in any normal commuting host. Two isometries with orthogonal ranges cannot lie in a nonzero finite corner. Hence every Weyl-clock game has no finite host. This additional fact is independent of the separator G_{sh} above.

14.3 A tracial game with no spatial perfection

For the other direction use a linear-system game with a perfect commuting strategy and no perfect finite-dimensional strategy, provided by [52]. The solution-group criterion [10, Theorems 4–5] says its distinguished central involution j is nontrivial. The central corner $(1 - \lambda(j))L(\Gamma_{\mathrm{sol}})/2$ of the solution group Γ_{sol} has normalized trace and a perfect left/right regular strategy: equation observables commute in each constraint, square to I, and their product is the specified sign because $j = -I$ in the corner. Use their joint spectral projections on the left and the commutant mirrors of variable observables on the right. Consistency and each equation are then exact. A factor fibre is a finite factor host; tensoring with R_{hyp} gives II_1. This is the tracial mechanism in [20, Proposition 7.4]. Call this game G_{lin}.

Cleve–Mittal [11] and [10, Theorem 5] characterize the finite-dimensional case. Spatial perfect play also implies finite-dimensional perfect play for linear-system games; Slofstra [52] uses this implication in the nonclosure construction. Here is the normal spatial argument. Each variable appearing in a constraint has an Alice involution with a self-adjoint Bob mirror on the state vector, by perfect consistency. Equality of those vectors makes Alice's trace-class marginal commute with every such involution. The within-constraint products and parity equations hold on the marginal support, by the zero-loss equations and support separation. A positive density eigenvalue has a nonzero finite-dimensional eigenspace reducing all variable involutions. Restriction to it preserves the products and all constraint equations. Its maximally entangled vector with transposed Bob variables is a finite-dimensional perfect strategy, a contradiction. For a mixed spatial state first purify; the same Alice eigenprojection argument applies. Type-I hosting decomposes into spatial factor fibres, so it is also excluded. Thus $\mathrm{PERF}(\mathrm{II}_1) \not\subseteq \mathrm{PERF}(\mathrm{I})$.

Every type-I factor host tensored with R_{hyp} has type II_1 in finite dimension and type II_∞ in infinite dimension; a further $B(\ell^2)$ amplification handles the former case. Every II_1 host similarly amplifies to II_∞. The two witnesses just constructed prove

$$\mathrm{PERF}(\mathrm{I}) \text{ and } \mathrm{PERF}(\mathrm{II}_1) \text{ are incomparable}, \qquad \mathrm{PERF}(\mathrm{I}),\ \mathrm{PERF}(\mathrm{II}_1) \subsetneq \mathrm{PERF}(\mathrm{II}_\infty).$$

Equal arithmetical completeness does not imply equal languages.

14.4 Exact conjunction and disjunction

We now compare the six languages $\mathbf{q}, \mathbf{E}, \mathbf{I}, \mathbf{J}, \mathbf{U}, \mathbf{C}$ of Section 2. Equation (10) identifies $\mathbf{U}$ with both PERF(II) and $\mathrm{PERF}(\mathrm{II}_\infty)$; Theorem 12.2 identifies $\mathbf{C}$ with each fixed type-III language.

Background lemma (Exact game operations)**.** *Uniform effective operations $G \wedge G'$ and $G \vee G'$ satisfy, for every $\mathbf{X} \in \{\mathbf{q}, \mathbf{E}, \mathbf{I}, \mathbf{J}, \mathbf{U}, \mathbf{C}\}$,*

$$G \wedge G' \in \mathbf{X} \iff G, G' \in \mathbf{X}, \qquad G \vee G' \in \mathbf{X} \iff G \in \mathbf{X} \text{ or } G' \in \mathbf{X}.$$

The same statements hold for each of the type languages in this paper.

Proof. First pad question and answer alphabets to common finite alphabets. For an original zero-weight question pair accept everything; preserve its original predicate on positive-weight pairs. Give every padded question pair positive rational weight, accept all pairs involving fresh questions, and reject a padded answer on an original question. This changes no perfect-play language. It permits fixed common question alphabets with full support; no implication below relies on an untested pair.

For conjunction, send both players the same tag and questions for that tagged game, and test its predicate. A perfect strategy restricts to either tag in the same ambient algebra and state. Conversely, tensor two perfect strategies and on a tagged question measure the appropriate factor. Type I is closed under tensor products, as are finite factors, type II_∞ factors, and semifinite algebras. Amplification gives the stated II existence languages when needed. For $\mathbf{E}$ use pure spatial witnesses: marginal entropies add, so their sum is finite. Finite-dimensional witnesses stay finite. Commuting strategies tensor as well. For III_λ use its language equality with $\mathbf{C}$ from Theorem 12.2; no closure of that subtype under direct tensoring is asserted.

For disjunction, ask the common questions and require each answer to name a branch $s \in \{G, G'\}$ as well as an answer in that branch. Reject mismatched branches and otherwise test the named predicate. A strategy for either game wins by always naming that branch. For soundness let $A^x_{s,a}, B^y_{s,b}$ be the effects and put $P^x_s = \sum_a A^x_{s,a}$, $Q^y_s = \sum_b B^y_{s,b}$. Work first on the

faithful standard-form support. The matching argument of Section 2.4 makes each branch marginal a projection, with $P_s^x\xi = Q_s^y\xi$ for every x, y. Fixing y and using separation makes $P_s^x = P_s$ independent of x; likewise $Q_s^y = Q_s$. These are opposite mirrors, $P_s\xi = Q_s\xi =: \xi_s$. At least one ξ_s is nonzero. Since $0 \le A_{s,a}^x \le P_s$, we have $A_{s,a}^x P_s = P_s A_{s,a}^x = A_{s,a}^x$, and the same holds on Bob's side. Complete each branch POVM to the identity by adding $I - P_s$ to one Alice answer and $I - Q_s$ to one Bob answer. These added effects annihilate ξ_s. A losing branch product annihilates ξ and, by the displayed support identities, also ξ_s. Normalizing ξ_s therefore wins branch s. All effects and the vector remain on the same Hilbert space, in the same ambient M, M'; the selected vector need not be faithful. Thus its host type is preserved. This also handles mixed strategies by purification and standardization.

For finite entropy one can work in the original spatial purification. Orthogonality of Bob's branch projections gives $\rho_A = \sum_s p_s \rho_{A,s}$, $p_s = \|\xi_s\|^2$. Entropy concavity, valid also by finite-rank truncation for infinite entropy, gives $\sum_s p_s S(\rho_{A,s}) \le S(\rho_A) < \infty$. Every positive-weight branch has finite entropy. This proves the last model-specific assertion and completes both directions. □

14.5 Four fixed separators and two uniform families

Choose both reductions G_e^{clk} and G_e^{sel} for the same predicate, $e \in \mathrm{FIN}$, and write $\mathrm{Fin}(e)$ for its truth value. Theorems 7.1 and 11.5, including Lemma 11.4, give

$$G_e^{\mathrm{clk}} \in \mathbf{E} \iff G_e^{\mathrm{clk}} \in \mathbf{I} \iff \mathrm{Fin}(e), \tag{91}$$

$$G_e^{\mathrm{sel}} \in \mathbf{J} \iff G_e^{\mathrm{sel}} \in \mathbf{U} \iff \mathrm{Fin}(e), \qquad G_e^{\mathrm{sel}} \in \mathbf{C}. \tag{92}$$

Every G_e^{clk} has no finite-dimensional perfect strategy; every G_e^{sel} is commuting perfect with $\omega_q = 1$. The chained lemma additionally excludes G_e^{clk} from $\mathbf{J}$ and places it in $\mathbf{U}$ (Section 14.2). The reductions below use its displayed spatial coordinates and use G_{sh} for finite-host exclusion.

Take the fixed one-bridge game G_{sh} of Section 14.1. Its shift model has root entropy 2 bits and its finite compiler factors preserve finite entropy. The finite-corner KMS contradiction there proves that it has no finite host. Thus its complete profile uses only the one-bridge assembly.

The tracial solution-group game G_{lin} is in $\mathbf{J} \setminus \mathbf{I}$; spatial perfection would imply finite perfection. Let $G_{\mathrm{Cat}}^{(2)}$ be the compiled Catalan game of [23, Theorem C and Sections 3 and 6]. Its perfect spatial strategies have infinite marginal entropy, so it is in $\mathbf{I} \setminus \mathbf{E}$. Lemma 4.6 with $n = 2$ applies to its nine-corner assembly: activity $e_{g-1} + m_g$, root mass $1/25$, transported encodings b_+, b_-, p, q, c (names of [23, Section 6]), and the source-only polynomials of [23, Section 6]. Thus its extracted isometry A has modular ratio $1/4$ also on a commuting normal host. In a finite root corner A would be unitary, and KMS would give $1 = 1/4$; hence it is outside $\mathbf{J}$ as well. Finally choose a fixed NO instance of G^{sel} and call it G_{III}. Theorem 11.5 gives $G_{\mathrm{III}} \in \mathbf{C} \setminus \mathbf{U}$ and $\omega_q(G_{\mathrm{III}}) = 1$. Their complete profiles are

	q	**E**	**I**	**J**	**U**	**C**
G_{sh}	0	1	1	0	1	1
G_{lin}	0	0	0	1	1	1
$G_{\mathrm{Cat}}^{(2)}$	0	0	1	0	1	1
G_{III}	0	0	0	0	0	1

In particular $G_{\mathrm{sh}} \wedge G_{\mathrm{lin}}$ is in $\mathbf{U}$ but in neither $\mathbf{I}$ nor $\mathbf{J}$, so even $\mathbf{I} \cup \mathbf{J} \subsetneq \mathbf{U}$. The remaining inclusions asserted above are strict: G_{sh} separates $\mathbf{q}$ from $\mathbf{E}$, $G_{\mathrm{Cat}}^{(2)}$ separates $\mathbf{E}$ from $\mathbf{I}$, G_{lin} separates $\mathbf{I}$ from $\mathbf{U}$, and G_{III} separates $\mathbf{U}$ from $\mathbf{C}$. Both $\mathbf{E}$ and $\mathbf{I}$ are incomparable with $\mathbf{J}$.

14.6 The table, with reductions for every nonempty entry

Theorem 14.1 (Directional differences)**.** *The following table gives row minus column. Each suffix c means computable many-one completeness for the displayed pointclass. Here D_2 abbreviates $D_2(\Sigma_2^0)$. Every nonempty cell is complete for its displayed class.*

	q	**E**	**I**	**J**	**U**	**C**
q	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
E	Σ_2^0-c	$\varnothing$	$\varnothing$	D_2-c	$\varnothing$	$\varnothing$
I	Σ_2^0-c	D_2-c	$\varnothing$	D_2-c	$\varnothing$	$\varnothing$
J	Σ_2^0-c	D_2-c	D_2-c	$\varnothing$	$\varnothing$	$\varnothing$
U	Σ_2^0-c	D_2-c	D_2-c	D_2-c	$\varnothing$	$\varnothing$
C	Π_1^0-c	Π_2^0-c	Π_2^0-c	Π_2^0-c	Π_2^0-c	$\varnothing$

Proof. All empty entries follow from the stated inclusions. For upper bounds, $\mathbf{q} \in \Sigma_1^0$, $\mathbf{E}, \mathbf{I}, \mathbf{J}, \mathbf{U} \in \Sigma_2^0$, and $\mathbf{C} \in \Pi_1^0$. A difference between two middle languages is therefore in D_2. For a middle $\mathbf{X}$, $\mathbf{X} \setminus \mathbf{q} \in \Sigma_2^0$: write its predicate as $\exists k \forall m$ for $\mathbf{X}$ and conjoin the Π_1^0 predicate for nonmembership in $\mathbf{q}$. Also $\mathbf{C} \setminus \mathbf{q} \in \Pi_1^0$ and $\mathbf{C} \setminus \mathbf{X} \in \Pi_2^0$. It remains to prove the matching lower bounds.

For the four incomparable entries use $G_e^{\mathrm{clk}} \wedge G_{\mathrm{sh}}$ as the **E**- or **I**-side family, and $G_f^{\mathrm{sel}} \wedge G_{\mathrm{lin}}$ as the **J**-side family. Their disjunction has **E** and **I** coordinates exactly $\mathrm{Fin}(e)$ and **J** coordinate exactly $\mathrm{Fin}(f)$. Thus its membership in either $\mathbf{E} \setminus \mathbf{J}$ or $\mathbf{I} \setminus \mathbf{J}$ is $\mathrm{Fin}(e) \wedge \neg\mathrm{Fin}(f)$. Swap e, f between the two branches for the reverse differences. The canonical D_2 problem reduces in all four cases. This settles I-ONLY, defined here as $\mathbf{I} \setminus \mathbf{J}$. It cannot mean absence of every type-II host, since $\mathbf{I} \subset \mathbf{U}$.

For the **q** column, use G_e^{clk} for **E** and **I**; use $G_e^{\mathrm{sel}} \wedge G_{\mathrm{lin}}$ for **J** and **U**. Their larger coordinate is $\mathrm{Fin}(e)$ and their finite-dimensional coordinate is identically zero. Each such entry is therefore Σ_2^0-hard.

For the four rightward entries of the **C** row, use $G_e^{\mathrm{clk}} \vee G_{\mathrm{III}}$ when $\mathbf{X} = \mathbf{E}, \mathbf{I}$, and $G_e^{\mathrm{sel}} \vee G_{\mathrm{III}}$ when $\mathbf{X} = \mathbf{J}, \mathbf{U}$. The commuting coordinate is always one and the **X** coordinate is exactly $\mathrm{Fin}(e)$, giving Π_2^0-hardness. These instances all have $\omega_q = 1$, since choosing branch G_{III} approximates perfection in finite dimension. For $\mathbf{C} \setminus \mathbf{q}$, take a Σ_1^0-complete finite-perfection reduction G_e^{MIP} with $G_e^{\mathrm{MIP}} \in \mathbf{q} \iff e$ halts, available by perfect completeness in [32, Theorem 12.6] and finite-dimensional feasibility. Then $G_e^{\mathrm{MIP}} \vee G_{\mathrm{Cat}}^{(2)}$ is always commuting perfect, has $\omega_q = 1$ because $G_{\mathrm{Cat}}^{(2)}$ is spatial perfect, and is in **q** exactly when e halts. Nonhalting reduces to the difference, proving Π_1^0-hardness even under the promise.

For the two remaining complete nested differences use a shared family. In general suppose $\mathbf{Y} \subset \mathbf{X}$ and a computable family $\mathcal{F}_e$ has both coordinates equal to $\mathrm{Fin}(e)$. Fix $G_{\mathrm{sep}} \in \mathbf{X} \setminus \mathbf{Y}$. For

$$\mathcal{D}(e, f) = \mathcal{F}_e \wedge (G_{\mathrm{sep}} \vee \mathcal{F}_f) \tag{93}$$

the **X** coordinate is $\mathrm{Fin}(e)$ and the **Y** coordinate is $\mathrm{Fin}(e) \wedge \mathrm{Fin}(f)$. The difference coordinate is therefore $\mathrm{Fin}(e) \wedge \neg\mathrm{Fin}(f)$, proving D_2-hardness. Apply this first with $(\mathbf{X}, \mathbf{Y}) = (\mathbf{I}, \mathbf{E})$, $\mathcal{F}_e = G_e^{\mathrm{clk}}$ and $G_{\mathrm{sep}} = G_{\mathrm{Cat}}^{(2)}$, using (91). Apply it again with $(\mathbf{X}, \mathbf{Y}) = (\mathbf{U}, \mathbf{J})$, $\mathcal{F}_e = G_e^{\mathrm{sel}}$ and $G_{\mathrm{sep}} = G_{\mathrm{sh}}$, using (92). Thus INF-ENTROPY-ONLY, defined as $\mathbf{I} \setminus \mathbf{E}$, and $\mathbf{U} \setminus \mathbf{J}$ are $D_2(\Sigma_2^0)$-complete. The required YES witness has finite entropy by Theorem 7.1.

For $\mathbf{U} \setminus \mathbf{I}$, $\mathbf{U} \setminus \mathbf{E}$, and the stronger language $\mathbf{U} \setminus (\mathbf{I} \cup \mathbf{J})$, take the shared family F_e of Theorem 13.4 for FIN and fix $H = G_{\mathrm{lin}} \in \mathbf{J} \setminus \mathbf{I}$. Set $\Lambda(e, f) = F_e \wedge (H \vee F_f)$. Its coordinates are

$$\mathbf{U}(\Lambda) = \mathrm{Fin}(e), \quad \mathbf{I}(\Lambda) = \mathbf{E}(\Lambda) = \mathrm{Fin}(e) \wedge \mathrm{Fin}(f), \quad \mathbf{J}(\Lambda) = 0.$$

Each difference therefore has predicate $\mathrm{Fin}(e) \wedge \neg\mathrm{Fin}(f)$. All three are in $D_2(\Sigma_2^0)$, completing the table and proving the additional simultaneous exclusion. Replace each F_e by $F_e \vee G_{\mathrm{III}}$ to

obtain commuting perfection and $\omega_q = 1$ on every input, with the same four lower coordinates. The two conjuncts then both have value one. □

Corollary 14.2. *The language of games with perfect spatial play but no finite-entropy perfect spatial strategy is $D_2(\Sigma_2^0)$-complete. In particular it belongs to neither Σ_2^0 nor Π_2^0.*

Proof. The identification is $\mathbf{I} \setminus \mathbf{E}$ in the theorem. A D_2-complete language is hard for both Σ_2^0 and Π_2^0 by fixing one of the canonical FIN indices to a YES or NO index. Membership in either pointclass would collapse these two pointclasses, contradicting strictness of the arithmetical hierarchy. □

The corollary concerns exact attainment; it asserts no rate at which entropy must diverge for near-optimal strategies.

For differences between middle models, value-one padding can be made explicit. Replacing G by $G \vee G_{\mathrm{III}}$ preserves all lower coordinates and sets the commuting coordinate to one, while forcing $\omega_q = 1$. To remove finite-dimensional perfect play as well, use $G \wedge (G_{\mathrm{sh}} \vee G_{\mathrm{lin}})$; the second factor has all five infinite-model coordinates equal to one and its **q** coordinate zero. In particular (93) for INF-ENTROPY-ONLY already has no finite-dimensional perfect instance; adding G_{III} also makes every instance value one. The **C**-row hardness families above already have value one, including the separate $\mathbf{C} \setminus \mathbf{q}$ construction.

Corollary 14.3 (A natural uniqueness problem)**.** *Deciding whether a graph on $\mathbb{N}$ with primitive recursive adjacency has exactly one finite connected component is $D_2(\Sigma_2^0)$-complete, and hence computably many-one equivalent to II-ONLY, $\mathbf{U} \setminus \mathbf{I}$.*

Proof. For fixed j, at least j finite components is Σ_2^0: existentially specify j disjoint nonempty finite connected sets and universally prohibit an edge leaving any one. Exactly one is at least one minus at least two.

For hardness, from c.e. sets W_a, W_b construct three stars with centres $0, 1, 2$. At stage s add vertices $3s+3, 3s+4, 3s+5$. Attach the first to centre zero if W_a gains a new distinct element, otherwise to two. Attach the second to centre one if either set gains a new element, otherwise to two. Always attach the third to two. Bounded enumeration simulation makes adjacency primitive recursive. The three components are finite precisely under the respective conditions W_a finite, both W_a, W_b finite, and never. Thus exactly one is finite precisely when $a \in$ FIN and $b \notin$ FIN. The natural statement still has a difference in its logical form. □

15 Zero-error codes and perfect play

A sender uses a finite noisy classical channel once. Beforehand, sender and receiver share an entangled state. The sender encodes one of m messages into the channel input, and the receiver must identify that message with certainty. We ask whether such a code exists. The channel and the message count remain finite; only the shared resource is unrestricted. Zero-error communication depends on exact zeros, rather than on how closely a protocol approaches perfect transmission. This makes attainment decisive, and the question becomes a question about perfect play.

For a finite rational stochastic matrix N and an integer $m \geq 1$, write $\mathrm{ZE}_t(N, m)$ for this feasibility problem. The model q uses finite-dimensional tensor products. The model qs uses a normal density operator on a spatial tensor product, with bounded finite-outcome POVMs. The model qc uses commuting local operator algebras and a state. The entropy restriction finS means that both resource marginals have finite von Neumann entropy. We use countably infinite Schmidt decompositions.

15.1 Channels and trace-class assemblages

For channel inputs x, x', write $x \sim x'$ when some output y has $N(y \mid x)N(y \mid x') > 0$. This is a reflexive relation: it includes $x = x'$. Its restriction to distinct inputs is the confusability graph. Every finite graph X has a rational realizing channel N_{X}: a nonisolated vertex outputs a uniformly random incident edge, and an isolated vertex outputs its own private symbol.

We first extend the finite-dimensional assemblage characterization of [18]. All supports below are closed range supports.

Lemma 15.1 (Normal assemblages)**.** *An m-message normal zero-error code exists if and only if there are positive trace-class operators $R_{i,x}$ and a density ρ such that*

$$\sum_x R_{i,x} = \rho \quad (1 \le i \le m), \qquad R_{i,x}R_{j,x'} = 0 \quad (i \ne j,\ x \sim x'). \tag{94}$$

The assemblage is realizable with the pure resource $|\sqrt{\rho}\rangle\!\rangle$. The equivalence includes mixed resources.

Proof. Given a mixed resource ω and Alice's encoding effect $E_{i,x}$, define

$$R_{i,x} = \mathrm{Tr}_A[(E_{i,x}^{1/2} \otimes I)\omega(E_{i,x}^{1/2} \otimes I)].$$

The sum is Bob's marginal, independently of i. For output y, Bob must distinguish the positive operators $T_i^y = \sum_x N(y \mid x)R_{i,x}$ perfectly. If F_i^y is a decoding POVM, perfect decoding gives $F_i^y = I$ on $\operatorname{supp} T_i^y$ and $F_j^y = 0$ there for $j \ne i$. Thus these supports are orthogonal. Positivity of the sum puts $\operatorname{supp} R_{i,x}$ in $\operatorname{supp} T_i^y$ whenever $N(y \mid x) > 0$. This proves the product condition, including equal inputs.

Conversely restrict to $\operatorname{supp}\rho$, so ρ is faithful. Each $0 \le R \le \rho$ has a unique bounded effect K with

$$R = \rho^{1/2}K\rho^{1/2}, \qquad 0 \le K \le I. \tag{95}$$

Indeed Douglas' factorization lemma [19] applied to $R \le \rho$ gives $R^{1/2} = \rho^{1/2}V_0$ for a contraction V_0, and $K = V_0V_0^*$. Uniqueness follows from the dense range of $\rho^{1/2}$. The sum identities imply $\sum_x K_{i,x} = I$, by the same dense-range argument. No unbounded inverse of ρ is used.

In a spectral basis for ρ, let Alice measure $K_{i,x}^{\mathsf{T}}$ on the purification $|\sqrt{\rho}\rangle\!\rangle$. Bob's conditional operator is $R_{i,x}$. Equation (94) makes the T_i^y have orthogonal supports, so their support projections, with any unused subspace assigned to one message, form a decoding POVM. This constructs a normal pure code.

Alternatively, any normal mixed perfect code decomposes spectrally into pure perfect codes with the same measurements: every forbidden probability is nonnegative, so it vanishes on every positive-weight component. This observation will also be used for entropy. □

15.2 Saturation of a clique partition

Beigi's Lovász-theta bound implies the finite-dimensional clique-cover bound [6]. The following trace-class saturation statement extends [37, Lemma 5].

Lemma 15.2. *If X is partitioned into k cliques $\mathcal{C}_c$, every normal code has at most k messages. If it has exactly k, put*

$$B_{i,c} = \sum_{x \in \mathcal{C}_c} R_{i,x}, \qquad P_{i,c} = \operatorname{supp} B_{i,c}.$$

On $\operatorname{supp}\rho$ one has

$$B_{i,c} = \rho P_{i,c} = P_{i,c}\rho, \qquad \sum_i P_{i,c} = I, \qquad \sum_c P_{i,c} = I. \tag{96}$$

Each row and column of $(P_{i,c})$ is an orthogonal projection partition.

Proof. For fixed c, distinct i give orthogonal $B_{i,c}$, because every pair of inputs in $\mathcal{C}_c$ is confusable, including identical inputs. Since $B_{i,c} \le \rho$,

$$\sum_i \operatorname{Tr} B_{i,c} \le \sum_i \operatorname{Tr}(P_{i,c}\rho) \le 1.$$

Summing over c gives $m \le k$. If $m = k$, all inequalities are equalities. Faithfulness of ρ gives $\sum_i P_{i,c} = I$. For each i, c, positivity and $\operatorname{Tr}[P_{i,c}(\rho - B_{i,c})] = 0$ imply $(\rho - B_{i,c})^{1/2} P_{i,c} = 0$. Hence $\rho P_{i,c} = B_{i,c} = P_{i,c}\rho$. Summing over c yields $\rho \sum_c P_{i,c} = \rho$; injectivity gives the row sum. A finite family of projections summing to identity is pairwise orthogonal: compress the sum by one member and use positivity of all other compressed members. All traces used here are finite traces of positive trace-class operators. □

A code is *saturating* for this partition if its message count equals the clique bound, $m = k$.

15.3 The normal correspondence

Let G have question sets $\mathcal{Q}_A, \mathcal{Q}_B$, finite nonempty answer sets, and a Boolean losing predicate on the question pairs of positive sampling probability. Its graph $\Gamma(G)$ has vertices (L, u, a) and (R, v, b). Distinct answers to the same tagged question form a clique; cross edges are precisely losing tuples. There are no other edges. Set $k = |\mathcal{Q}_A| + |\mathcal{Q}_B|$, the number of tagged local questions; Mančinska's question count q_{M} is our k.

For a pure strategy, use Schmidt form $|D\rangle\!\rangle$ with $D \ge 0$ and $\operatorname{Tr} D^2 = 1$. For bounded positive effects, zero probability is equivalent to $EDF^{\mathsf{T}} = 0$, as in Section 3: zero probability is the vanishing of the Hilbert–Schmidt norm of $\sqrt{E}D\sqrt{F^{\mathsf{T}}}$; multiplication by the square roots gives one implication, and equality of the closures of the ranges of an effect and its square root gives the other.

Theorem 15.3 (Trace-class correspondence)**.** *For every finite game G,*

$$\mathrm{ZE}_{\mathrm{qs}}(N_{\Gamma(G)}, k) \iff G \text{ has a perfect pure spatial strategy with } \operatorname{Tr} D < \infty.$$

For a specified code marginal ρ, the reverse strategy has $D = \rho/\|\rho\|_2$. For a specified game strategy D with finite trace, the forward pure code has marginal $\rho = D/\operatorname{Tr} D$. No synchrony hypothesis is required.

Proof. In the forward construction, store Bob's effects in transposed form, and denote all local effects by $E_{c,a}$ for tagged questions c. Put $\rho = D/\operatorname{Tr} D$ and

$$R_{c,(c,a)} = \rho^{1/2} E_{c,a} \rho^{1/2},$$

with zero operators off the designated clique. Each message family sums to ρ. For cross losing edges, the perfect-game sandwich $E_{c,a} D E_{c',b} = 0$ gives the required zero product. The remaining products between distinct messages vanish by their clique assignments. Lemma 15.1 realizes the code. The trace condition is exactly what normalizes its marginal.

For the converse, as in [37, Theorem 6], apply Lemma 15.2 to an arbitrary saturating code, not only a code in the displayed form. Aggregate

$$T_{c,a} = \sum_i R_{i,(c,a)}.$$

Each family sums to ρ. If $(c, a), (c', b)$ is a cross losing edge, every product $R_{i,(c,a)} R_{j,(c',b)}$ is zero. For $i \ne j$ this is a code condition. For $i = j$ its two factors are supported on the orthogonal row projections $P_{i,c}, P_{i,c'}$. Thus $T_{c,a} T_{c',b} = 0$.

Use (95) to write $T_{c,a} = \rho^{1/2} E_{c,a} \rho^{1/2}$ with bounded POVMs. The product equality becomes

$$\rho^{1/2} E_{c,a} \rho E_{c',b} \rho^{1/2} = 0.$$

The injectivity and dense range of $\rho^{1/2}$ cancel its two outside factors, giving $E_{c,a} \rho E_{c',b} = 0$. Use the left-tagged effects on Alice and the transposes of the right-tagged effects on Bob. With $D = \rho / \|\rho\|_2$ these are a perfect spatial strategy. Since $\operatorname{Tr} \rho = 1$, $\operatorname{Tr} D = 1/\|\rho\|_2 < \infty$.

Only the coarse clique projections must commute with ρ. The fine outcomes need not do so; this is why the argument retains nonsynchronous games. Restricting every step to finite dimension recovers [37, Theorem 6]. □

Remark. For the game marginal $h = D^2$, the condition $\operatorname{Tr} D < \infty$ is finite Rényi entropy of order $1/2$, namely $2 \log_2 \operatorname{Tr} \sqrt{h}$. It is stronger than finite von Neumann entropy. Replacing the forward marginal by D^2 generally destroys zero-error orthogonality; Appendix C includes an explicit check on the flagged magic square.

15.4 The resource correspondence at every order

Write $h = D^2$ for the game marginal, $Z = \operatorname{Tr} \sqrt{h}$, and $\rho_{\mathrm{code}} = \sqrt{h}/Z$ for the code marginal. A normal code first requires $Z < \infty$. Whenever it exists,

$$\operatorname{Tr} \rho_{\mathrm{code}}^{\beta} = Z^{-\beta} \operatorname{Tr} h^{\beta/2}, \qquad S(\rho_{\mathrm{code}}) = \log_2 Z + (2Z)^{-1} \operatorname{Tr}[\sqrt{h} \log_2(1/h)].$$

The resulting correspondence is

Code resource	Required game spectrum
Finite rank	finite rank
Finite Rényi entropy, $0 < \beta < 1$	$\operatorname{Tr} h^{\beta/2} < \infty$
Finite von Neumann entropy	$\operatorname{Tr}[\sqrt{h} \log_2(1/h)] < \infty$
Finite Rényi entropy, $1 < \beta \leq \infty$	$\operatorname{Tr} \sqrt{h} < \infty$
Arbitrary normal resource	$\operatorname{Tr} \sqrt{h} < \infty$

The logarithmic condition implies summability of the square roots: only finitely many eigenvalues exceed any fixed positive threshold, and below such a threshold the logarithm is bounded below. It cannot be replaced by the half-moment alone. The squared-Catalan channel of Section 19 has a summable half-moment and requires infinite code entropy. For $\beta > 1$, the four-bridge tree has finite $\beta/2$-moment but no normal saturating code. At the spectrum level $p_j \asymp j^{-2} (\log j)^{-4}$ has summable square roots and a divergent logarithmic half-moment. These examples explain both endpoints of the table.

16 Five bridges and the normal row

We apply the trace-class correspondence to the compiler. The four-bridge games of Section 7 lie exactly at the summability threshold: none of their channels admits a saturating normal code, even on YES inputs. Adding a fifth bridge changes the modular ratio and restores summability. The same five-bridge YES codes have finite marginal entropy. The threshold is visible in the four-letter tree density $h_{F,\nu} = (1 - 4\nu) \sum_w \nu^{|w|} |w\rangle\langle w|$ of Section 16.2:

$$\operatorname{Tr} \sqrt{h_{F,\nu}} = \sqrt{1 - 4\nu} \sum_{\ell=0}^{\infty} (4\sqrt{\nu})^{\ell}, \qquad \nu = 2^{-n}.$$

There are four branches per level, so summability requires $n \geq 5$.

16.1 An exact spectrum identity

One must be careful when removing the row factor in soundness: the FMS signs were not synchronized, so their transports need not commute with $\mathrm{Mat}_{d_{\max}}$. The following elementary cancellation supplies the needed reduction without a partial trace over an infinite space.

Finite-amplification cancellation. If two finite-dimensional tuples of unitaries become unitarily equivalent after $d_{\max}$ identical copies, the original tuples are unitarily equivalent. Indeed they are finite-dimensional representations of the universal $*$-algebra on the tuple. The image in their direct sum is a finite-dimensional C^*-algebra. Decompose its two representations into irreducibles; amplification multiplies every multiplicity by $d_{\max}$, so equality of amplified multiplicities cancels $d_{\max}$.

Lemma 16.1 (Exact spectrum identity)**.** *For every pure perfect strategy of $G^{(n)}$, with full marginal h and Schmidt operator $D = \sqrt{h}$, there is a normal realization with density h_0 such that, including multiplicities,*

$$\operatorname{spec}(h) = \operatorname{spec}\left(\operatorname{diag}(\mu_n) \otimes h_0 \otimes I_3/3 \otimes I_{d_{\max}}/d_{\max}\right). \tag{97}$$

Here h_0 is the reduced normalized source density after removing the qutrit and row factors; it differs from the source density on the full amplified Hilbert space in Lemma 4.5. Conversely every realization has a perfect strategy with this spectrum. In particular, in the extended nonnegative reals,

$$\operatorname{Tr} D = c_n^{\mathrm{amp}} \operatorname{Tr} \sqrt{h_0}, \qquad c_n^{\mathrm{amp}} = \sqrt{3 d_{\max}}\, \frac{1 + 4\sum_{g=1}^{n} 2^{g/2}}{\sqrt{\mathfrak{c}_n}}. \tag{98}$$

Proof. Restrict to the original marginal support. Synchronization makes the certified scalar matrix algebras commute with the density. The named old projections commute with $\mathrm{Mat}_{d_{\max}}$; the bound parameters and graph units also provide the scalar Mat_3 frame. These two scalar algebras commute, so their exact matrix-unit decompositions give a density $I_3/3 \otimes I_{d_{\max}}/d_{\max} \otimes h_{\mathrm{base}}$ and factorizations of every named old synchronized projection. This is a finite matrix-unit decomposition, not an uncontrolled partial trace.

The magic matrix units identify all four density blocks at each bridge. The U_g identify consecutive normalized selected density blocks, with ratio two. The cover therefore implies that the full density has the spectrum in (97), where h_0 is its normalized source block after removal of both scalar factors. It remains to verify that this h_0 is the density of a realization; simply compressing $v^* a_\alpha$ across $\mathrm{Mat}_{d_{\max}}$ would not prove that assertion.

The frame refinements make v act trivially on Mat_3. After removing that frame it is a unitary between the $d_{\max}$-amplifications of the source and far spaces, intertwining their normalized densities and every transported unitary label. In each positive density eigenspace these spaces are finite dimensional. Apply finite-amplification cancellation to their transported unitary tuples. This gives an unamplified unitary for each positive eigenvalue. Their Hilbert direct sum $\widetilde{v}$ intertwines h_0 with the normalized far density, and intertwines all transported labels simultaneously. The densities are faithful, so these eigenspaces exhaust both spaces.

All centralizer contractions a_α are polynomials in old labels and factor across both scalar algebras; write their reduced forms $\widetilde{a}_\alpha$. Define $\widetilde{A}_\alpha = \widetilde{v}^* \widetilde{a}_\alpha$. The reduced translated equations in Lemma 4.5 give every application identity. Intertwining of the normalized densities, together with the source/far weight 2^n, gives $h_0 \widetilde{A}_\alpha = 2^{-n} \widetilde{A}_\alpha h_0$. This proves the realization assertion. The new creations need not be restrictions of the original creations.

Completeness uses (27), followed by the two normalized maximally mixed factors, so has exactly the same spectrum. Taking the square root and the trace of this finite tensor product proves (98); nonnegative spectral sums justify it even when the answer is infinity. □

16.2 The tree at a general ratio

The tree part of Section 5 works for every $0 < \nu < 1/4$. For clarity we give the density calculation. For a word w in four letters, write A_w for the product of its creations and

$$Q_\ell = \sum_{|w|=\ell} A_w A_w^*.$$

The isometry relations and covariance $h_0 A_\alpha = \nu A_\alpha h_0$ imply

$$\operatorname{Tr}(h_0 Q_\ell) = (4\nu)^\ell, \qquad \operatorname{Tr}(h_0 P_\circ) = 1 - 4\nu.$$

The decreasing Q_ℓ have a strong limit of zero h_0-weight, hence zero by faithfulness. Their successive differences are $\sum_{|w|=\ell} A_w P_\circ A_w^*$. Thus, with $\mathcal{K} = P_\circ \mathcal{H} \neq 0$,

$$\mathcal{H} \simeq \ell^2(\mathcal{W}) \otimes \mathcal{K}, \quad h_0 = h_{F,\nu} \otimes \rho_{\mathrm{spec}}, \quad h_{F,\nu} = (1-4\nu) \sum_w \nu^{|w|} |w\rangle\langle w|, \tag{99}$$

where ρ_{spec} is a faithful density. The covariance makes every word-sector projection commute with h_0, so transporting the root block proves the equality, not only its diagonal entries. Depth ℓ has probability $(1-4\nu)(4\nu)^\ell$; $(4\nu)^\ell$ is the mass at depth at least ℓ. At $\nu = 1/16$ this is (32).

The coordinate, Weyl-clock, bounded period-reader and deterministic tableau equations of Sections 5–7 contain no Fock density ratio. Their proofs use only faithful normality, positive finite-dimensional density eigenspaces, the exact tree decomposition, and a positive weight at every finite depth. All hold in (99). Consequently their soundness and canonical completeness hold unchanged for every $\nu = 2^{-n} < 1/4$. In particular the Weyl determinant still forces a root phase of finite order: on a positive density eigenspace of dimension d on which $Z_{\mathrm{clk}} = \zeta$, the relation $J_{\mathrm{W}} K_{\mathrm{W}} = \zeta K_{\mathrm{W}} J_{\mathrm{W}}$ gives $\zeta^d = 1$. No clock relation is squared or otherwise changed.

16.3 Summability and its critical point

Equation (99) gives

$$\operatorname{Tr} \sqrt{h_{F,\nu}} = \sqrt{1-4\nu} \sum_{\ell=0}^{\infty} (4\sqrt{\nu})^\ell = \frac{\sqrt{1-4\nu}}{1-4\sqrt{\nu}} \tag{100}$$

when $\nu < 1/16$, and infinity when $1/16 \le \nu < 1/4$. For the canonical YES realization $\rho_{\mathrm{spec}} = I_{d_{\mathrm{clk}}}/d_{\mathrm{clk}}$ with clock order d_{clk}, Lemma 16.1 therefore gives

$$\operatorname{Tr} D = \sqrt{3 d_{\mathrm{clk}} d_{\max}}\, \frac{1 + 4\sum_{g=1}^{n} 2^{g/2}}{\sqrt{\mathfrak{c}_n}}\, \frac{\sqrt{1-2^{2-n}}}{1-2^{2-n/2}} \quad (n > 4). \tag{101}$$

For five bridges and the optional eight-bridge construction, the exact coefficients are

$$n = 5: \quad \operatorname{Tr} D = \frac{(81 + 53\sqrt{2})\sqrt{7}}{2\sqrt{83}} \sqrt{d_{\mathrm{clk}} d_{\max}}, \tag{102}$$

$$n = 8: \quad \operatorname{Tr} D = \frac{(121 + 60\sqrt{2})\sqrt{21}}{2\sqrt{2041}} \sqrt{d_{\mathrm{clk}} d_{\max}}. \tag{103}$$

They are approximately $22.65\sqrt{d_{\mathrm{clk}} d_{\max}}$ and $10.44\sqrt{d_{\mathrm{clk}} d_{\max}}$.

Remark (Four bridges are critical). At $n = 4$, every realization has $\operatorname{Tr} \sqrt{h_0} = \infty$, regardless of ρ_{spec}, because the per-depth amplitude sum in (100) is constant and nonzero. Lemma 16.1 then makes every perfect pure game strategy have infinite amplitude sum. Theorem 15.3 implies that the channel of every game G_e^{clk} of Theorem 7.1 has no k-message normal code, for every input index. Applying those games directly would therefore give an empty YES set, rather than the desired hardness reduction.

16.4 Upper membership

Convert a channel instance (N, m) to a finite game $\mathcal{G}_{\mathrm{code}}(N, m)$. Alice receives $i \in [m]$ and answers a channel input x; Bob receives a channel output y and answers $j \in [m]$. All question pairs have positive rational probability. A tuple loses exactly when $N(y \mid x) > 0$ and $j \neq i$. With the same resource and measurements,

$$\mathrm{ZE}_t(N, m) \iff \mathcal{G}_{\mathrm{code}}(N, m) \text{ has a perfect } t\text{-strategy.} \tag{104}$$

Indeed the error probability of the code is the finite sum of these nonnegative expectations weighted by $N(y \mid x)$. It vanishes exactly when every forbidden expectation vanishes. The equivalence preserves marginal entropy as well.

The compact-stratum predicate of Section 3 gives $\mathrm{PERF}_{\mathrm{qs}} = \bigcup_\kappa \mathrm{PERF}_{\mathrm{qs}}(\kappa)$ with each stratum Π^0_1, and mixed strategies reduce to pure perfect components, as in Lemma 15.1. Equation (104) therefore proves $\mathrm{ZE}_{\mathrm{qs}} \in \Sigma^0_2$. The entropy-budget strata of Section 3.4 prove $\mathrm{ZE}_{\mathrm{finS}} \in \Sigma^0_2$ in the same way. If a mixed perfect resource has finite entropy on one marginal, entropy concavity gives a positive-weight pure perfect component of finite entropy, and for a pure bipartite component the two marginal entropies agree. Hence using one or both finite marginal entropies leads to the same feasibility set.

16.5 Five bridges for normal hardness

Let $\mathcal{A}_{\mathrm{dec}} = \{e : \exists j\, \forall t\, \mathcal{R}_{\mathrm{dec}}(e, j, t)\}$ for any decidable predicate $\mathcal{R}_{\mathrm{dec}}$. Use the clock presentation of Sections 5–7, with its computation program hardwiring e, and change only its bridge count to five. Denote the resulting game by $G_e^{(5)}$. Its clock and computation mechanism is unchanged. At depth r the four-letter tree carries a 2^r-by-2^r coordinate square. A root Weyl pair commuting with the root density has a nonzero finite-order phase sector; its determinant on a positive density eigenspace forces this. On a phase of order d_{clk} the reader equations (45) force P_{ret} to be exactly the kernel projection of $W - I$, so the first positive return is precisely time d_{clk}. The tableau freezes a waiting front at that time, halts if d_{clk} is one or composite, and otherwise checks $\mathcal{R}_{\mathrm{dec}}(e, j, 0), \mathcal{R}_{\mathrm{dec}}(e, j, 1), \ldots$ for $d_{\mathrm{clk}} = p_{j+1}$, halting at the first failure. A halt is forbidden by an operator equation, and every depth has positive normal weight. These are exact identities, not assertions about high-probability truncations.

If the game is perfect, Lemma 4.5 extracts this mechanism and therefore a witness j. Conversely a witness j uses the canonical tree, a d_{clk}-dimensional Weyl pair for $d_{\mathrm{clk}} = p_{j+1}$, the good primitive phase and bounded reader of Section 6, and its nonhalting classical tableau. The reader norm estimate depends on the phase and the counter, not on ν. Its density is $h_{F,1/32} \otimes I_{d_{\mathrm{clk}}}/d_{\mathrm{clk}}$. This is the simultaneous canonical realization, so Lemma 4.5 supplies a perfect game strategy. Formula (102) makes its Schmidt amplitudes summable.

Now put

$$N_e = N_{\Gamma(G_e^{(5)})}, \qquad k_e = |\mathcal{Q}_{A,e}| + |\mathcal{Q}_{B,e}|,$$

where $\mathcal{Q}_{A,e}, \mathcal{Q}_{B,e}$ are the question sets of $G_e^{(5)}$. These data are finite and rational and are computed from e without knowing its witness. Theorem 15.3 and the preceding two implications give

$$\mathrm{ZE}_{\mathrm{qs}}(N_e, k_e) \iff e \in \mathcal{A}_{\mathrm{dec}}.$$

This proves Σ^0_2-hardness and the first part of Theorem C. No channel in this reduction has a finite-dimensional saturating code: otherwise the finite converse would extract four orthogonal isometries on a nonzero finite-dimensional root space, an impossibility. We make no claim here that a value-one promise is inherited from the source construction.

16.6 Finite entropy already at five bridges

Use exactly the same five-bridge reduction. The forward code marginal is the normalized square root of the canonical game marginal. Its tree factor is

$$\widehat{h}_5 = \frac{\sqrt{h_{F,1/32}}}{\operatorname{Tr}\sqrt{h_{F,1/32}}} = (1-2^{-1/2})\sum_w 2^{-5|w|/2}|w\rangle\langle w|.$$

At depth ℓ there are 4^ℓ words, so the depth distribution is geometric with ratio $2^{-1/2}$. Its mean is $2^{-1/2}/(1-2^{-1/2}) = 1+\sqrt{2}$. Each eigenvalue at depth ℓ has negative base-two logarithm $-\log_2(1-2^{-1/2}) + (5/2)\ell$. Taking its expectation proves

$$S(\widehat{h}_5) = -\log_2(1-2^{-1/2}) + \frac{5}{2}(1+\sqrt{2}) \approx 7.807 \text{ bits.} \tag{105}$$

The full canonical code marginal is

$$\operatorname{diag}(\hat{\mu}_5)\otimes\widehat{h}_5\otimes I_{d_{\mathrm{clk}}}/d_{\mathrm{clk}}\otimes I_3/3\otimes I_{d_{\max}}/d_{\max}, \qquad \hat{\mu}_{5,i} = \frac{\sqrt{\mu_{5,i}}}{\sum_{i'}\sqrt{\mu_{5,i'}}}.$$

Thus its entropy is $H(\hat{\mu}_5) + S(\widehat{h}_5) + \log_2(3d_{\mathrm{clk}}d_{\max})$, which is finite. The pure code has the same entropy on both sides. A NO input has no normal code at all, hence no finite-entropy code. The same map therefore proves the finite-entropy half of the first part of Theorem C.

Remark (Eight bridges). With $n = 8$, the normalized square root of $h_{F,1/256}$ is exactly $h_{F,1/16}$, with entropy $\log_2(4/3) + 4/3$. Its amplitude sum is (103). This is another finite-entropy construction; five bridges already suffice for hardness.

16.7 The finite-dimensional row, as an attributed corollary

For each fixed dimension, the channel code game in (104) has a rational semialgebraic perfect-strategy predicate: split matrix entries into real and imaginary parts, impose positivity, POVM sums, normalization and zero losing probabilities. Quantifier elimination decides it. Enumerating dimensions gives $\mathrm{ZE_q} \in \Sigma_1^0$.

For hardness, the perfect-completeness reduction of $\mathsf{MIP}^* = \mathsf{RE}$ gives, from a halting instance, a finite game with a perfect finite-dimensional strategy, and from a nonhalting instance a game of entangled value at most $1/2$ [32, Theorem 12.6]. Mančinska's Corollary 7 maps perfect finite-dimensional game feasibility to the saturating one-shot channel threshold [37]. Thus $\mathrm{ZE_q}$ is Σ_1^0-complete. This conclusion uses her finite theorem, not the new normal converse. Bhattacharyya, Mehta and Zhao leave undecidability of $C_{0,E}^{(1)}$ open in [7, Section 1.4]. This corollary settles its one-shot threshold problem: it is undecidable and Σ_1^0-complete already for classical channels. Here E denotes unrestricted finite-dimensional entanglement. The unrestricted asymptotic capacity question remains open.

17 The commuting row

With commuting resources one-shot zero-error feasibility drops to the commuting level. The code game (104) identifies $\mathrm{ZE_{qc}}$ with perfect commuting play for a particular finite game. Commuting correlations form a compact set, and the NPA hierarchy converges from above to their value [43]. Its rational moment relaxations can be decided by real quantifier elimination. A NO instance eventually has an infeasible perfect-value relaxation; conversely a feasible relaxation at every order gives a commuting strategy by compactness of the full moment system. Hence $\mathrm{ZE_{qc}} \in \Pi_1^0$.

We prove the reduction for the lower bound, including codes with arbitrary commuting resources. Slofstra proved Π_1^0-hardness for linear system games [53]; with the NPA bound this is Π_1^0-completeness, which Mousavi, Nezhadi and Yuen reprove for synchronous games [42, Theorem 6.4]. The source is perfect commuting feasibility for finite synchronous games with positive diagonal question probabilities; see [42, Theorem 6.4 and Section 1.3]. Their synchronous framework and tracial representation are stated in [42, Theorem 1.6]. Such a game has the same question and answer sets on both sides, and distinct answers on an equal-question pair are losing.

17.1 Normal functionals from a commuting code

Represent a code on a Hilbert space with vector ξ, encoding effects $E_{i,x}$ in an algebra commuting with Bob's von Neumann algebra $\mathcal{N}$, and decoder effects in $\mathcal{N}$. Define normal positive functionals

$$\phi(T) = \langle \xi, T\xi \rangle, \qquad \phi_{i,x}(T) = \langle \xi, E_{i,x} T \xi \rangle \quad (T \in \mathcal{N}).$$

The two operators commute, so these functionals are positive and $\sum_x \phi_{i,x} = \phi$. Compress $\mathcal{N}$ to the support of ϕ. This makes ϕ faithful. It preserves all the functionals, their sums, and decoder tests, since each $\phi_{i,x} \le \phi$ is supported on the same support. The compressed encoders still commute with the compressed Bob algebra.

When $i \neq j$ and $x \sim x'$, the functionals $\phi_{i,x}, \phi_{j,x'}$ have orthogonal supports in $\mathcal{N}$. To see this, for any common output y their positively weighted functionals are summands of the two distinct-message output functionals. Perfect decoding makes the support of one lie in the eigenvalue-one space of a decoder effect, and that of the other in its kernel. These spectral spaces are orthogonal. Support inclusion for positive summands proves the assertion.

Lemma 17.1 (Functional saturation)**.** *Suppose the input graph is partitioned into k cliques $\mathcal{C}_c$ and a commuting code sends k messages. For*

$$\psi_{i,c} = \sum_{x \in \mathcal{C}_c} \phi_{i,x}, \qquad s_{i,c} = \operatorname{supp} \psi_{i,c},$$

the $s_{i,c}$ lie in the centralizer $\mathcal{N}_\phi = \{s : \phi(sT) = \phi(Ts) \text{ for all } T \in \mathcal{N}\}$ and satisfy

$$\psi_{i,c}(T) = \phi(s_{i,c} T) = \phi(T s_{i,c}).$$

Their rows and columns are orthogonal projection partitions.

Proof. For each c the supports are orthogonal in i. Domination by ϕ gives

$$\sum_i \psi_{i,c}(1) \le \sum_i \phi(s_{i,c}) \le 1.$$

The sum over c is k, so every inequality is an equality. Faithfulness gives $\sum_i s_{i,c} = 1$. For each i, c, $\eta = \phi - \psi_{i,c}$ is positive and $\eta(s_{i,c}) = 0$. Cauchy–Schwarz for a positive functional gives $\eta(s_{i,c} T) = \eta(T s_{i,c}) = 0$. The support property of $\psi_{i,c}$ gives $\psi_{i,c}(s_{i,c} T) = \psi_{i,c}(T s_{i,c}) = \psi_{i,c}(T)$. Combining them proves the formula and centralizer membership.

Summing over c for fixed i gives $\phi((\sum_c s_{i,c} - 1)T) = 0$ for every T. Take $T = (\sum_c s_{i,c} - 1)^*$ and use faithfulness to obtain the row sum. A finite sum of projections equal to identity is orthogonal, as in Lemma 15.2. □

17.2 Synchrony makes the fine supports tracial

Apply the lemma to $\Gamma(G)$, with $k = 2|\mathcal{Q}_A|$, for a synchronous game G. Aggregate the fine functionals

$$\psi_{c,a} = \sum_i \phi_{i,(c,a)}.$$

They sum to ϕ for each tagged question c. Across a cross losing edge their supports are orthogonal: unequal message indices use the code condition, and equal message indices use distinct clique supports in one row of $(s_{i,c})$. Orthogonality passes to finite sums.

Fix an original question u. Write $\alpha_a = \psi_{(L,u),a}$, $\beta_a = \psi_{(R,u),a}$, and $p_a = \operatorname{supp} \alpha_a$. For $a \neq b$, synchrony gives $p_a \operatorname{supp} \beta_b = 0$. Hence

$$\alpha_a(1) \leq \phi(p_a) = \beta_a(p_a) \leq \beta_a(1).$$

Both outside families sum to one, so these inequalities are equalities for every a. The same Cauchy–Schwarz argument used in Lemma 17.1, now for $\phi - \alpha_a$ and $\phi - \beta_a$, gives

$$\alpha_a(T) = \beta_a(T) = \phi(p_a T) = \phi(T p_a).$$

Faithfulness implies that their supports are the same p_a, and that $\sum_a p_a = 1$. Thus the fine outcomes are projections in $\mathcal{N}_\phi$.

The restriction $\tau = \phi|_{\mathcal{N}_\phi}$ is a faithful normal tracial state. For any original losing tuple (u, a, v, b) the preceding support orthogonality gives $p_{u,a} p_{v,b} = 0$. The left/right multiplication construction on $L^2(\mathcal{N}_\phi, \tau)$ therefore gives a perfect commuting strategy for G. This argument starts from an arbitrary saturating code; it assumes neither projective encoders nor a tracial original resource.

Conversely a perfect commuting strategy for synchronous G has its usual tracial projection representation [47, Theorem 5.5]; see also [33, 42]. For each tagged question use its projection partition to encode its designated message in the left algebra of the tracial GNS space; use the right algebra to decode. The conditional functionals are $T \mapsto \tau(p_{c,a} T)$. For distinct designated messages and confusable vertices, their supports are orthogonal by the losing relation. Each output's distinct-message supports therefore give a decoding measurement. This is a commuting k-message code for $N_{\Gamma(G)}$.

Theorem 17.2. *The one-shot feasibility problem* $\mathrm{ZE}_{\mathrm{qc}}$ *for finite rational classical channels is* Π_1^0*-complete.*

Proof. For each finite synchronous game G with positive diagonal question probabilities, the two constructions give the effective equivalence

$$G \text{ is commuting-perfect} \iff \mathrm{ZE}_{\mathrm{qc}}(N_{\Gamma(G)}, 2|\mathcal{Q}_A|).$$

The source problem is Π_1^0-complete, so this proves hardness. The NPA upper predicate gives membership. □

18 The boundary between one-shot codes and capacity

A channel tensor power has a strong-product confusability graph. It is not the channel of ordinary AND repetition of its game: on all-Alice versus all-Bob vertices, strong-product adjacency requires every coordinate to lose, whereas AND repetition rejects when at least one coordinate loses. Mixed role strings also occur in the strong product. The exact results below separate finite-block saturation from an asymptotic rate claim.

Write $\alpha_t(X)$ for the largest one-shot message count for a graph X. The model t is finite dimensional (q), maximally entangled (ME), normal spatial (qs), or commuting (qc). Define

$$\Theta_t(X) = \lim_{n\to\infty} \alpha_t(X^{\boxtimes n})^{1/n}, \qquad C_{0,t}(X) = \log_2 \Theta_t(X).$$

Product codes and the finite input bound justify the limit by Fekete's lemma. The unassisted parameter is Θ, and the standard notation $\Theta_* = \Theta_q$ denotes arbitrary finite-dimensional assistance. It should not be confused with the quantum-homomorphism capacity.

18.1 Theta bounds normal and commuting codes

The following dimension-free vector argument is Beigi's technique [6], applied also to normal and commuting resources.

Lemma 18.1. *For every finite graph X, $\alpha_{\mathrm{qs}}(X), \alpha_{\mathrm{qc}}(X) \le \vartheta(X)$. If a finite game G is commuting-perfect, then $\vartheta(\Gamma(G)) = k$, its number of tagged questions.*

Proof. Dilate the finitely many encoder POVMs to projective measurements, using local ancillas; this preserves their commutation with the decoder algebra and all probabilities. Purify the state or use its GNS vector ξ. Put $\psi_{i,x} = E_{i,x}\xi$. For different messages and confusable inputs, perfect decoding gives orthogonality of these vectors: for a common output its decoder effect is identity on one vector and zero on the other. Within one message, distinct input projections are orthogonal. Thus $w_x = \sum_i \psi_{i,x}$ are orthogonal on every graph edge; at the same input the different messages are orthogonal as well. Consequently

$$\sum_x w_x = m\xi, \qquad \sum_x \|w_x\|^2 = m.$$

Their Gram matrix divided by m is feasible for the theta SDP, with objective m. Only finitely many vectors are used; the ambient Hilbert space dimension is immaterial.

For a commuting-perfect game, use vectors $E_{a|u}\xi$ and $F_{b|v}\xi$ after local projective dilation. They are orthogonal within each question clique and across every losing pair, since those opposite projections commute and have zero product on ξ. Each clique's vectors sum to ξ and their squared norms sum to one. The full Gram matrix divided by k has trace one and objective k. The k-clique partition gives the opposite theta bound. □

Multiplicativity of theta gives the same upper bound for each regularized rate, including normal and commuting codes.

Remark (No universal value bound from code density). An always-rejecting game with t Alice questions and one Bob question, with one answer per question, anchored independently with probability $1/2$ on each side, has value $3/4$. Its graph is $K_{t,1} \sqcup 2K_1$, with $k = t + 3$ question cliques. An independent set and a clique partition both have size $k-1$, so $\alpha_t(X^{\boxtimes n}) = (k-1)^n$ in every model bounded by theta. Its AND-repeated value is $(3/4)^n$. Choosing $k = 2n + 2$ keeps code density $((k-1)/k)^n > 1/2$ while that value vanishes. Since k is even, take a subcode of exactly $k^n/2$ messages. Any universal bound $\omega_q(G^{\wedge n}) \ge f(m/k^n)$ would give $f(1/2) \le (3/4)^n$ for every n, hence $f(1/2) = 0$. Thus no positive universal lower bound depending only on code density can apply across all such anchored games. This does not refute a bound for one fixed game.

18.2 Saturation cancels a tensor power

Theorem 18.2. *If X has a partition into k cliques, then for $t \in \{q, \mathrm{ME}, \mathrm{qs}\}$ and every $n \ge 1$,*

$$\alpha_t(X^{\boxtimes n}) = k^n \iff \alpha_t(X) = k.$$

A saturating block code yields a one-shot code with the same receiver density.

Proof. Let $R_{i,w}$ be its assemblage with common marginal ρ. Apply Lemma 15.2 to the product cliques $\mathcal{C}_{\mathbf{z}}$. It gives projections $P_{i,\mathbf{z}}$ with orthogonal rows and columns, and clique sums $\rho P_{i,\mathbf{z}}$. Set $T_w = \sum_i R_{i,w}$. Then $\sum_{w \in \mathcal{C}_{\mathbf{z}}} T_w = \rho$. If w, w' are confusable in distinct product cliques, $T_w T_{w'} = 0$: different message indices use zero error; equal indices use orthogonal row supports.

Fix $z_2, \dots, z_n$. For message z put

$$\widetilde{R}_{z,v} = \begin{cases} \displaystyle\sum_{v_2 \in \mathcal{C}_{z_2}, \dots, v_n \in \mathcal{C}_{z_n}} T_{(v,v_2,\dots,v_n)}, & v \in \mathcal{C}_z, \\ 0, & \text{otherwise.} \end{cases}$$

Each message sum is ρ. For different base messages and confusable first coordinates, all pairs in the remaining sums are confusable, since their other coordinates lie in fixed cliques. The required products therefore vanish. This is a one-shot code with marginal ρ. Finite dimension is preserved. If $\rho = I/d$, the pure code from Lemma 15.1 is maximally entangled. The reverse implication tensors codes. All sums are finite and no density inverse is used in the cancellation. □

For game channels this yields

$$\alpha_q(\Gamma(G)^{\boxtimes n}) = k^n \iff G \in \mathrm{PERF}_q.$$

For normal codes the condition is perfection with summable Schmidt amplitudes. Apply this to the squared-Catalan channel of Section 19. Its normal one-shot count and theta are both k, so $C_{0,\mathrm{qs}} = \log_2 k$, attained in one shot. Every finite-dimensional block count is at most $k^n - 1$. A strict inequality $C_{0,q} < \log_2 k$ remains open. Indeed $k^n - 1$ is itself a supermultiplicative numerical sequence with limiting rate $\log_2 k$.

18.3 Arithmetical upper bounds for rates

Let $P_t(N, n, m)$ mean a code for $N^{\otimes n}$. For rational b,

$$C_{0,t} > b \iff \exists n, m\, [P_t(N, n, m) \ \wedge \ m^q > 2^{np}], \quad b = p/q,$$

with the comparison interpreted by clearing negative powers. Also $C_{0,t} \geq b$ iff $C_{0,t} > b - 1/j$ for every $j \geq 1$. These give the following upper bounds; completeness is not asserted.

Model	$C > b$	$C \leq b$	$C \geq b$	$C < b$	$C = \log_2 k$
q, ME	Σ_1^0	Π_1^0	Π_2^0	Σ_2^0	Π_2^0
qs, qc	Σ_2^0	Π_2^0	Π_3^0	Σ_3^0	Π_3^0

The last column assumes a multiplicative bound $\alpha_t(N^{\otimes n}) \leq k^n$, for example a clique partition, and uses the integer condition $\forall j\, \exists n, m\, [P_t(N, n, m) \wedge m^j 2^n > k^{nj}]$. Finite-dimensional and ME one-shot predicates are existential dimension searches over semialgebraic feasibility. The other one-shot bounds were proved above. The strict lower cuts are uniformly c.e. for q and ME, and c.e. relative to $0'$ for normal and commuting assistance. All four positivity questions are decidable: $C_{0,t} > 0$ precisely when the confusability graph is not complete. A nonedge sends two classical messages; a complete graph and all its powers have the one-clique bound.

18.4 The exact rate identity and its limitations

Lemma 18.3. *For an m-message normal code for $X^{\boxtimes n}$, with a k-clique partition, let M be uniform on the messages, Q the input clique string, and B the receiver system. Its induced classical-quantum state Ω satisfies, for $\delta = \log_2 k - (\log_2 m)/n$,*

$$n\delta = D(\Omega_{QB} \| u_Q \otimes \rho) + H(Q \mid MB) = D(p_Q \| u_Q) + I(Q : B) + H(Q \mid MB). \tag{106}$$

Here D is Umegaki relative entropy with base-two logarithms, not Rényi relative entropy of order two.

Proof. The assemblage defines $\Omega = m^{-1}\sum_{i,j}|i,j\rangle\langle i,j| \otimes \sum_{w\in\mathcal{C}_j} R_{i,w}$. No signalling gives $\Omega_{MB} = I_M/m \otimes \rho$. For fixed j, different messages have orthogonal receiver supports, so $H(M \mid QB) = 0$. Thus $\log_2 m = I(M:Q \mid B) = H(Q \mid B) - H(Q \mid MB)$. Subtract this from $n\log_2 k$ and use the relative entropy chain rule. Every term displayed is nonnegative. For normal states with infinite $S(\rho)$, define conditional entropies of the finite classical registers using Holevo information; no subtraction of two infinite quantum entropies occurs. □

Three obstructions prevent this identity from supplying a capacity gap. First, even-parity binary strings have total relative entropy deficit one, but every coordinate is determined by all the others. Small deficit per coordinate supplies no nearly uniform all-other-coordinates slice. Second, a shared uniform key $R \in \mathbb{Z}_k$ can encode m as $j = m + R$. Its deficit is zero, but conditioning on j makes the receiver states for different messages orthogonal. Common sums are lost. Third, the reverse game map uses $\rho^2/\operatorname{Tr}\rho^2$. A sector of ρ-mass ϵ in $(1-\epsilon)I_d/d \oplus \epsilon|b\rangle\langle b|$ has squared-state mass $\epsilon^2/[\epsilon^2 + (1-\epsilon)^2/d]$, which tends to one. Trace-norm estimates on ρ do not control that normalization. Spectral compression also need not preserve a zero product: for $A = |0\rangle\langle 0|$, $B = |1\rangle\langle 1|$, and $P = |+\rangle\langle +|$, one has $AB = 0$ but $(PAP)(PBP) = P/4$.

18.5 Exact common sums from approximate ones

The remaining norm issue can be isolated without assuming an extraction theorem. Fix a question law $\mu(u,v)$ with marginals μ_U, μ_V. Let positive finite-dimensional families X_a^u, Y_b^v have sums S_u, T_v and put

$$\rho_{\mathrm{av}} = \tfrac{1}{2}\left(\sum_u \mu_U(u)S_u + \sum_v \mu_V(v)T_v\right) \neq 0, \quad R_u = S_u - \rho_{\mathrm{av}}, \quad R_v = T_v - \rho_{\mathrm{av}}.$$

Their average losing overlap is

$$L = \sum_{u,v}\mu(u,v) \sum_{a,b:\, V(a,b|u,v)=0} \operatorname{Tr}(X_a^u Y_b^v).$$

Put $\ell = L/\operatorname{Tr}\rho_{\mathrm{av}}^2$, and $\epsilon^2 = \sum_\eta \|R_\eta\|_2^2/\operatorname{Tr}\rho_{\mathrm{av}}^2$, where η runs over both question sets, without weights in this last sum.

Lemma 18.4. *These families extend in the same dimension to exact-common-sum assemblages inducing a game strategy with loss at most $\ell + 2\epsilon$.*

Proof. Put $K = (\sum_\eta R_\eta^2)^{1/2}$ and $W = \rho_{\mathrm{av}} + K$. Operator monotonicity of the square root gives $K \geq |R_\eta| \geq R_\eta$, so $W \geq S_u, T_v$. Add $W - S_u$ or $W - T_v$ to one designated answer of each question. The new families have common sum W. Their additional losing overlap is bounded by the additional overlap over all answer pairs. Averaging, using $\sum_u \mu_U R_u + \sum_v \mu_V R_v = 0$, bounds it by $2\operatorname{Tr}(KW)$; the subtracted term $\sum_{u,v}\mu(u,v)\operatorname{Tr}[(W-S_u)(W-T_v)]$ is nonnegative, since both factors in each trace are positive. Since $\|K\|_2 = \epsilon\|\rho_{\mathrm{av}}\|_2$ and $\|W\|_2 \geq \|\rho_{\mathrm{av}}\|_2$,

$$\widehat{L}/\operatorname{Tr}W^2 \leq \ell + 2\|K\|_2/\|W\|_2 \leq \ell + 2\epsilon.$$

On the support of W, factor each extended positive operator as $W^{1/2}EW^{1/2}$. These are bounded POVM effects. The vector $|W\rangle\!\rangle/\|W\|_2$, with transposed effects on the other side, has probabilities $\operatorname{Tr}(\widehat{X}\widehat{Y})/\operatorname{Tr}W^2$. Normalization by $\operatorname{Tr}W$ changes none of these quotients. □

If every strategy loses at least c, then

$$L + (2/c)\sum_{\eta} \|R_\eta\|_2^2 \geq (c/2)\,\mathrm{Tr}\,\rho_{\mathrm{av}}^2,$$

by $2\epsilon \leq c/2 + 2\epsilon^2/c$. This construction does not derive relative Hilbert–Schmidt control from a high-rate code.

18.6 A fourteen-vertex test of the theta conjecture

The 2×3 magic rectangle, studied by Adamson and Wallden [2], has Alice question $u \in \{0,1\}$, answer $a \in \{\pm 1\}^3$ with $a_1 a_2 a_3 = (-1)^u$, and Bob question $j \in \{1,2,3\}$, answer $b \in \{\pm 1\}$. It accepts $b = a_j$; the six question pairs are uniform. Its graph H has fourteen vertices, thirty-nine edges and five question cliques.

Take orthonormal h, e_1, e_2, e_3 and set

$$z_{u,a} = (h + \sum_j a_j e_j)/4, \qquad z_{j,b} = (h + b e_j)/2.$$

They are orthogonal on every edge. Each clique sums to h and its squared norms sum to one. Their Gram matrix divided by five is a trace-one theta witness of objective five; the clique cover proves $\vartheta(H) = 5$.

For arbitrary commuting strategies use projective dilation, coordinate reflections A_j^u, B_j, and $e_{u,j} = \|(A_j^u - B_j)\psi\|$. If ℓ is average loss, then $\sum_{u,j} e_{u,j}^2 = 24\ell$. Put $U = B_3 B_2 B_1$. Cross-party commutation and telescoping give

$$\|((-1)^u I - U)\psi\| \leq \sum_j e_{u,j},$$

because $A_1^u A_2^u A_3^u = (-1)^u I$. The sum of the two squared norms on the left is four. Cauchy–Schwarz gives $4 \leq 3\sum_{u,j} e_{u,j}^2 = 72\ell$. Thus every commuting strategy loses at least $1/18$, despite $\vartheta(H) = 5$.

For completeness we give an exact symmetry reduction for Schrijver's number, with convention

$$\vartheta^-(H) = \max\{\langle J, M\rangle : M \succeq 0, \mathrm{Tr}\, M = 1, M \geq 0 \text{ entrywise},\ M_{xy} = 0\ (xy \in E)\}.$$

Average over signed coordinate permutations. The nonzero entries are Alice diagonal a, cube distances one and three p, q, Bob diagonal b, different Bob questions r, and winning cross pairs s. All are nonnegative; $8a + 6b = 1$. The objective is $1 + 24p + 8q + 24r + 48s$. Cube Fourier characters and Bob coordinate contrasts split positivity into scalar blocks $a - p + q$ (multiplicity three), $a - 3p - q$ (one), $b - 2r$ (two), and

$$\begin{pmatrix} a + p - q & 2s \\ 2s & b \end{pmatrix} \quad \text{(three copies)}, \qquad \begin{pmatrix} a + 3p + q & 2\sqrt{3}s \\ 2\sqrt{3}s & b + 4r \end{pmatrix}.$$

Hence $24p + 8q \leq 8a$, $24r \leq 12b$, and $s \leq \sqrt{ab/3}$. With $y = 6b$ the objective is at most $2 + y + 4\sqrt{y(1-y)} \leq (5 + \sqrt{17})/2$. Equality is feasible at

$$y = (1 + 1/\sqrt{17})/2, \quad a = (1-y)/8, \quad b = y/6, \quad p = a/3,\ q = 0,\ r = b/2,\ s = \sqrt{ab/3}.$$

The three two-by-two blocks have determinant zero and nonnegative diagonal. The final block has determinant $2ab$; all remaining scalars are nonnegative. This proves exactly $\vartheta^-(H) = (5 + \sqrt{17})/2 < 5$. Cubitt and coauthors' bound $\alpha_q \leq \vartheta^-$ [17] gives $\alpha_q(H) \leq 4$. The

independent set consisting of $+++$ and Bob's three positive answers proves equality, and brute force independently gives $\alpha(H) = 4$.

Alice vertex $4b_1 + 2b_2 + b_3$ has question $b_1 + b_2 + b_3 \bmod 2$ and signs $(-1)^{b_j}$. Alice vertices are $0, \ldots, 7$. Label Bob vertices $8, \ldots, 13$ by $(j, c) = (1,0), (1,1), (2,0), (2,1), (3,0), (3,1)$, meaning question j with answer $b = (-1)^c$. An eighteen-message classical code in $H^{\boxtimes 2}$ is

$$\{(0,11), (2,1), (4,5), (4,13), (5,6), (6,7), (6,9), (7,10),\\ (9,3), (9,8), (9,13), (10,5), (10,13), (11,7), (11,9),\\ (12,4), (12,9), (12,12)\}.$$

Checking all 153 pairs proves

$$3\sqrt{2} \le \Theta(H) \le \Theta_*(H) \le 5.$$

18.7 What a capacity gap would have to prove

Li and Zuiddam record the folklore conjecture $\Theta_* = \vartheta$ [34, Conjecture 19, v3]. Their Corollary 27 shows it cannot hold simultaneously with their Conjecture 26 identifying the quantum and entanglement-assisted capacities. Neither conjecture is assumed here. The graph H is an explicit test of Conjecture 19: equality would require rates approaching $\log_2 5$ even though its game has a uniform commuting loss. Lemma 18.4 shows that extraction giving $\ell \le 1/72$ and $\epsilon \le 1/144$ would contradict the gap. Existence of a positive deficit threshold for such extraction is thus already a strict capacity-gap assertion, not an established consequence of the entropy identity.

A fixed example sharpening the value-gap route is G_{SEP} of [32, Theorem 12.10]: finite-dimensional value at most $1/2$ and commuting value one, attained. Give formerly untested pairs an accepting predicate, take the equal-weight mixture of the original distribution and uniform sampling, and anchor each party independently with probability $1/2$. Its finite-dimensional value is then at most $15/16$, while commuting perfection persists and Lemma 18.1 gives theta k. Anchored repetition supplies exponential decay of its AND-repeated finite-dimensional value [4, Theorem 1.3]. A proof that this one fixed graph has $\Theta_* < k$ would refute Conjecture 19 on it; finite-block nonsaturation alone does not prove that inequality. More generally, a uniform game-value capacity-soundness theorem for commuting-perfect NO instances must establish such a theta-capacity separation there. The Catalan channel, whose finite-dimensional game value is one as a supremum, cannot be separated by that value-gap route. These are obstructions and open questions, not capacity-undecidability results.

Remark (Fixed-game power laws). For one anchored game with a genuine finite-dimensional value gap, write $a_n = \alpha_q(\Gamma(G)^{\boxtimes n})$ and $\omega_n = \omega_q(G'^{\wedge n})$. A fixed product strategy and anchored repetition give $w^n \le \omega_n \le B2^{-\gamma n}$ for some positive w, B, γ. A strict capacity gap is equivalent to the existence of $A, p > 0$ with $\omega_n \ge A(a_n/k^n)^p$ for every n. One direction combines the two exponential bounds. For the other, if $\delta = \log_2 k - C_{0,q} > 0$, every block rate is at most capacity, so $a_n/k^n \le 2^{-\delta n}$; choose $p \ge \log_2(1/w)/\delta$ and $A = 1$. This equivalence supplies no effective constants or extraction theorem. It explains why such a per-game power law already contains the desired capacity separation.

The Rényi spectral points of Mghirbi and Douglas [41] dominate theta. They cannot separate H from five: its clique partition gives $H \le_* \overline{K_5}$, so every normalized $\le_*$-monotone spectral point is at most five there. A point also dominating theta is exactly five. A new separating point must be monotone for entanglement-assisted cohomomorphisms, not merely the weaker quantum preorder, and must avoid this domination obstruction.

19 A channel whose codes require infinite entropy

This section proves the second part of Theorem C. The channel comes from a four-bridge version of the Catalan assembly of [23]. Its game has finite-entropy perfect strategies, but the normalized square-root map of Theorem 15.3 turns every one of them into a code with infinite entropy.

19.1 The squared Catalan density

Let $\mathcal{W}_{\mathrm{Cat}}$ consist of the empty word and the legal chronological words in an up-step A and a down-step B. An up-step is always allowed; a down-step is allowed only at height at least two. Let $u(w)$ count up-steps. The number of words with j up-steps is the Catalan number Cat_j. On $\ell^2(\mathcal{W}_{\mathrm{Cat}})$ let A, B append the corresponding legal letter, with B zero when forbidden. Let C_{Cat} have eigenvalue $2^{-\mathrm{ht}(w)}$, let P project onto the empty word, and let Q project onto height one.

We require the local relations of [23, Section 3]:

$$A^*A = I, \qquad AA^* + BB^* + P = I, \qquad B^*B = I - P - Q, \tag{107}$$

$$C_{\mathrm{Cat}}P = P, \qquad C_{\mathrm{Cat}}Q = Q/2, \qquad 0 \le C_{\mathrm{Cat}} \le I, \tag{108}$$

$$C_{\mathrm{Cat}}A = AC_{\mathrm{Cat}}/2, \qquad C_{\mathrm{Cat}}B = 2BC_{\mathrm{Cat}}. \tag{109}$$

Here the modular ratio is squared:

$$h_0 A = A h_0/16, \qquad [h_0, B] = [h_0, C_{\mathrm{Cat}}] = [h_0, P] = [h_0, Q] = 0. \tag{110}$$

Here h_0 is a faithful normal density. In this section A, B, P, Q denote the Catalan labels just defined. The counter relation $C_{\mathrm{Cat}}A = AC_{\mathrm{Cat}}/2$ is unchanged.

Lemma 19.1 (Normal classification at ratio 1/16)**.** *Every normal realization of* (107)–(110) *is, up to a unitary,*

$$\mathcal{H} = \ell^2(\mathcal{W}_{\mathrm{Cat}}) \otimes \mathcal{K}_{\mathrm{Cat}}, \qquad h_0 = k_{\mathrm{Cat}} \otimes \rho_{\mathrm{spec}}, \qquad k_{\mathrm{Cat}} = \frac{1}{8 - 4\sqrt{3}} \sum_w 16^{-u(w)} |w\rangle\langle w|,$$

where ρ_{spec} is a faithful normal density. All five local operators act canonically on the first factor and as identity on the second.

Proof. We give the normality argument to check the ratio change in [23, Lemma 3.1]. The boundary equations put P, Q in the counter eigenspaces one and one half and give $PQ = 0$. The range projections AA^*, BB^*, P are orthogonal and sum to identity; B is a partial isometry with initial projection $I - P - Q$. All these projections commute with C_{Cat}.

There is no spectrum of C_{Cat} in $(1/2, 1)$, since there B is isometric and $C_{\mathrm{Cat}}B = 2BC_{\mathrm{Cat}}$ would transport it above one. The same reasoning at eigenvalue one gives $1_{\{1\}}(C_{\mathrm{Cat}}) = P$. At eigenvalue one half, B would transport the complement of Q into P, orthogonally to its own range, so $1_{\{1/2\}}(C_{\mathrm{Cat}}) = Q$. Iterating B excludes every interval $(2^{-j-1}, 2^{-j})$. Functional calculus for the bounded intertwining relation justifies these spectral transports.

The zero-counter space reduces the operators and h_0. On it $P = Q = 0$, so A, B are isometries with orthogonal ranges summing to identity. If its h_0-mass is t, cyclicity and (110) make the two range masses $t/16$ and t. Their sum is t, forcing $t = 0$. Faithfulness removes this space.

Apply legal words to $P\mathcal{H}$. Their sectors are orthogonal: different last letters have orthogonal ranges; equal last letters cancel with $A^*A = I$ or with $B^*B = I - P - Q$ on a legal predecessor;

and a nonempty sector is orthogonal to P. Denote the word isometry by V_w^{Cat}. Covariance gives

$$h_0 V_w^{\mathrm{Cat}} = 16^{-u(w)} V_w^{\mathrm{Cat}} h_0.$$

The word-sector projections commute with h_0, and their closed span reduces all the operators. If its orthogonal complement were nonzero, the compact faithful density there would have a largest positive eigenvalue, with nonzero finite-dimensional eigenspace L. The adjoint covariance gives $A^*L = 0$. Since $P = 0$ on this complement, $BB^* = I$ on L. Commutation with h_0 makes L reduce B, so $B|_L$ is unitary. It also reduces C_{Cat}. Tracing $C_{\mathrm{Cat}}B = 2BC_{\mathrm{Cat}}$ on L forces $\mathrm{Tr}_L\, C_{\mathrm{Cat}} = 0$, hence $C_{\mathrm{Cat}}|_L = 0$, contrary to the absence of zero-counter mass. Thus the word sectors exhaust the support.

Their density blocks are $16^{-u(w)} V_w^{\mathrm{Cat}} (P h_0 P)(V_w^{\mathrm{Cat}})^*$. The word count is Cat_j: for $j \geq 1$, words with j up-steps and b down-steps, $0 \leq b \leq j-1$, number

$$\binom{j+b-1}{b} - \binom{j+b-1}{b-1};$$

reflection of the first bad prefix gives this formula, and summing over b gives Cat_j. The empty word supplies $\mathrm{Cat}_0 = 1$. The generating function

$$F(z) = \sum_{j \geq 0} \mathrm{Cat}_j\, z^j = \frac{1 - \sqrt{1-4z}}{2z}$$

gives $F(1/16) = 8 - 4\sqrt{3}$. Normalization then gives the asserted tensor product. The same sector identifications fix the local operators. □

Define the critical Catalan density of [23] by

$$h_{\mathrm{Cat}} = \frac{1}{2} \sum_w 4^{-u(w)} |w\rangle\langle w|.$$

The generating function yields the exact identities

$$\mathrm{Tr}\, h_{\mathrm{Cat}}^2 = 2 - \sqrt{3}, \qquad k_{\mathrm{Cat}} = \frac{h_{\mathrm{Cat}}^2}{2-\sqrt{3}}, \qquad \mathrm{Tr}\, \sqrt{k_{\mathrm{Cat}}} = \frac{\sqrt{6}+\sqrt{2}}{2}, \qquad \frac{\sqrt{k_{\mathrm{Cat}}}}{\mathrm{Tr}\, \sqrt{k_{\mathrm{Cat}}}} = h_{\mathrm{Cat}}. \tag{111}$$

Moreover

$$S(h_{\mathrm{Cat}}) = \sum_{j \geq 0} \frac{\mathrm{Cat}_j}{2\, 4^j} (1 + 2j) = \infty, \tag{112}$$

since $\mathrm{Cat}_j\, 4^{-j} \sim (\sqrt{\pi} j^{3/2})^{-1}$. By contrast the squared Catalan factor has

$$S(k_{\mathrm{Cat}}) = \log_2(8 - 4\sqrt{3}) + \frac{4}{\sqrt{3}} - 2 \approx 0.4094 \text{ bits}.$$

Indeed, with F as above and $z = 1/16$, its mean up-step count is $zF'(z)/F(z) = 1/\sqrt{3} - 1/2$; averaging $\log_2 F(z) + 4u(w)$ gives the formula. Its multiplicity-weighted probabilities decay geometrically. The canonical perfect strategy of the game $G_{\mathrm{Cat}}^{(4)}$ below therefore has finite entropy: $S(k_{\mathrm{Cat}})$ is its infinite Catalan factor's contribution, and all its other canonical factors are finite dimensional. The channel of $G_{\mathrm{Cat}}^{(4)}$ instead forces the factor h_{Cat} and hence infinite entropy. The communication task itself creates the entropy demand through the square-root map.

19.2 A four-bridge Catalan assembly

The Catalan relations are not the four-creation syntax of Definition 4.4. We therefore specify their own assembly, extending [23, Section 6] from two bridges to four. Use the seventeen corners, twenty-five frame-bound parameters, cover $\mathfrak{c}_4 = 121$, axis equations, support-copy equations and block-unitary transport equations of Lemma 4.5. The transported unitary encodings are those of $B, C_{\mathrm{Cat}}, P, Q$: two unitaries b_+, b_- for B, one for the positive contraction C_{Cat}, and the reflections for P, Q. Thus $|\Lambda| = 5$. For each bridge use the six frame refinements and these five graph-label refinements, on both parties, retaining the original opposite FMS questions.

For $r = 0, \dots, 4$, denote the decoded corner copies by b_r, c_r, p_r, q_r. Impose the root reflection equations for p_0, q_0. Introduce one centralizer contraction $a = (a_+ + a_-)/2$. The remaining equations are

$$a = e_4 a e_0, \qquad a^* a = e_0, \qquad a a^* + b_4 b_4^* + p_4 = e_4, \tag{113}$$

$$b_0^* b_0 = e_0 - p_0 - q_0, \qquad p_0 q_0 = 0, \qquad c_0 p_0 = p_0, \tag{114}$$

$$c_0 q_0 = q_0/2, \qquad c_4 a = a c_0/2, \qquad c_0 b_0 = 2 b_0 c_0. \tag{115}$$

In particular the range equation is at the far corner. It is not $aa^* + b_0 b_0^* + p_0 = e_0$.

Let $v = w_4 w_3 w_2 w_1$. Extraction on the source gives

$$A = v^* a, \qquad B = b_0, \qquad C_{\mathrm{Cat}} = c_0, \qquad P = p_0, \qquad Q = q_0.$$

Conjugate the far range equation by v, using the actual transported copies. This gives $AA^* + BB^* + P = e_0$. The other local equations follow directly; for example

$$C_{\mathrm{Cat}} A = c_0 v^* a = v^* c_4 a = v^* a c_0/2 = A C_{\mathrm{Cat}}/2.$$

Since a is a centralizer contraction and $hv = 16vh$, its source density has $h_0 A = A h_0/16$. The other decoded labels commute with it. After removal of the finite scalar factors, use exactly the finite-amplification cancellation proof of Lemma 16.1, now for these five transported unitary labels. It gives a realization of (107)–(110) and the same spectrum identity with μ_4.

Conversely use the canonical Catalan operators and density k_{Cat} in $\mathrm{diag}(\mu_4) \otimes k_{\mathrm{Cat}}$, put $a = |e_4\rangle\langle e_0| \otimes A$, and use identity transports. The ratio makes a commute with the scaffold density. The blockwise polar encodings, actual graphs, label refinements and finite Boolean-row extension used in Lemma 4.5 realize all the displayed equations simultaneously. Thus they compute a finite rational game $G^{(4)}_{\mathrm{Cat}}$ with a perfect normal strategy.

19.3 Every normal code has infinite entropy

Set $N = N_{\Gamma(G^{(4)}_{\mathrm{Cat}})}$ and let m be the number of tagged local questions of $G^{(4)}_{\mathrm{Cat}}$. The canonical strategy just constructed has finite amplitude sum, by (111) and its finite scaffold and scalar factors. Theorem 15.3 therefore gives a normal m-message code.

Consider any normal code, including a mixed resource, and let ρ_B be its Bob marginal. The assemblage converse and Theorem 15.3 construct a pure perfect game strategy with

$$h_{\mathrm{game}} = \frac{\rho_B^2}{\operatorname{Tr} \rho_B^2}, \qquad \rho_B = \frac{\sqrt{h_{\mathrm{game}}}}{\operatorname{Tr} \sqrt{h_{\mathrm{game}}}}.$$

The assembly spectrum identity and Lemma 19.1 give

$$h_{\mathrm{game}} \simeq \mathrm{diag}(\mu_4) \otimes k_{\mathrm{Cat}} \otimes \rho_{\mathrm{spec}} \otimes I_3/3 \otimes I_{d_{\max}}/d_{\max}.$$

Here $\simeq$ means unitary equivalence of positive operators. Trace summability forces $\operatorname{Tr}\sqrt{\rho_{\mathrm{spec}}} < \infty$. Taking the normalized square root gives

$$\begin{aligned} \rho_B &\simeq \operatorname{diag}(\hat{\mu}_4) \otimes h_{\mathrm{Cat}} \otimes \rho_{\mathrm{spec,code}} \otimes I_3/3 \otimes I_{d_{\max}}/d_{\max}, \\ \hat{\mu}_{4,i} &= \frac{\sqrt{\mu_{4,i}}}{\sum_j \sqrt{\mu_{4,j}}}, \quad \rho_{\mathrm{spec,code}} = \frac{\sqrt{\rho_{\mathrm{spec}}}}{\operatorname{Tr}\sqrt{\rho_{\mathrm{spec}}}}. \end{aligned} \tag{116}$$

The Catalan factor is exactly h_{Cat}; the full marginal also includes the finite factors and the possible spectator $\rho_{\mathrm{spec,code}}$. Equation (112) gives $S(\rho_B) = \infty$ for every code.

For a pure resource its two marginal spectra coincide, so Alice's entropy is infinite as well. For a mixed resource $\omega = \sum_j t_j |\psi_j\rangle\langle\psi_j|$, each positive-weight component is a perfect code under the same encoding and decoding measurements. This follows by applying positivity to each forbidden error expectation. Each such pure component has infinite entropy on both marginals. Extended entropy concavity gives

$$S(\omega_A) \geq t_j S((\psi_j)_A) = \infty,$$

and similarly for Bob. This proves the second part of Theorem C for all normal codes, rather than only canonical ones.

20 Discussion and open problems

The hierarchy separates several ways to ask for certainty. Finite-dimensional perfection has a finite existential witness. Commuting perfection is controlled by compactness. Value one asks for arbitrarily accurate finite witnesses. Spatial and semifinite perfection combine an unbounded finite choice with an infinite verification. In the spatial reduction the chosen object is the order of a clock phase; in the semifinite reduction it is the depth of a critical tree. Without a trace, as in every type-III factor, the choice disappears and the commuting level returns.

20.1 Entropy budgets

The entropy restriction does not lower the complexity. Each YES instance G_e^{clk} of the spatial reduction has a finite-entropy witness. Corollary 3.6 shows that the least sufficient integer entropy budget, maximized over games of bit length at most n, is a function of Turing degree $0''$ that eventually dominates every total $0'$-computable function. Corollary 3.7 proves the same profile for the least sufficient Schmidt stratum. Finite-entropy compactness also controls entire prescribed tables by Corollary 3.4. The finite-entropy and arbitrary-spatial decision sets have the same completeness degree, although the sets themselves differ: the entropy-game construction in [23] has spatial perfection and no finite-entropy perfect strategy, and Corollary 14.2 classifies their difference.

20.2 Prescribed tables and escaping mass

The largest-Schmidt compactness argument of Theorem 3.1 does not directly control a prescribed correlation table without a uniform tail bound. For a concrete obstruction, put

$$D_{\mathrm{esc},n} = \operatorname{diag}(1, \underbrace{n^{-1/2}, \ldots, n^{-1/2}}_{n \text{ entries}}, 0, \ldots).$$

Then $\operatorname{Tr} D_{\mathrm{esc},n}^2 = 2$ and $D_{\mathrm{esc},n} \to |0\rangle\langle 0|$ in operator norm. Alice measures $|0\rangle\langle 0|$ and its complement; Bob has one trivial outcome. Every normalized vectorization of $D_{\mathrm{esc},n}$ gives Alice

probabilities $(1/2, 1/2)$. The normalized limit gives $(1, 0)$. The largest-amplitude normalization preserves zero equations while changing nonzero normalized probabilities.

For the finite-entropy realization class, Corollary 3.4 proves the Σ_2^0 upper bound for exact rational or real-algebraic tables. The remaining proposed bounds concern other restrictions.

Unproved remarks on prescribed-table bounds. The following upper-bound remarks are not proved in this paper and are not used in any theorem. For a prescribed table with rational or real-algebraic entries, direct coding of countable operators and a normal vector gives a Σ_1^1 upper bound. A supplied computable Schmidt-tail modulus makes the normalized state family effectively compact and gives a Π_1^0 upper bound. An existential choice from a uniformly total computable family of such moduli gives Σ_2^0. If an arbitrary program for a computable modulus is existentially supplied, totality and the tail conditions give a Σ_3^0 upper bound. These proposed bounds concern the respective restricted realization classes; establishing their effective coding details is outside the present proofs. We do not prove that every algebraic spatial table has a computable-tail realization, or give a completeness classification for unrestricted table membership.

20.3 The Fu–Miller–Slofstra remark

After Theorem 1.1, Fu, Miller and Slofstra state that their fixed-scenario membership problem is in RE for $t = q$ or $t = qs$ [26]. We found no proof of the general qs upper assertion in that discussion. The following consequence needs a uniform interpretation over finite scenarios.

Suppose membership of real-algebraic tables in C_{qs} has a uniform RE procedure. If every spatially perfect game in our reduction had an algebraic perfect table, enumerate all algebraic tables of its scenario, test the losing zeros exactly, and dovetail the membership procedures. This would enumerate precisely the YES inputs of a Σ_2^0-complete family, a contradiction. Under that interpretation, either the RE remark fails for qs, or some spatially perfect games in this family have no perfect correlation table with all entries algebraic. The conclusion is existential over YES games; it does not say that every YES game has only nonalgebraic perfect tables.

The literal remark is per fixed scenario. The scenario of G_e^{clk} grows with e, which is hardwired into the automaton alphabet, so separate nonuniform RE procedures for each scenario give no contradiction. Tension arises only if there is one procedure uniform across scenarios, together with algebraic perfect tables for every YES game of the reduction. This is an open discrepancy, not a refutation. Resolving the uniformity and algebraicity questions is a natural next problem.

20.4 Zero-error capacities

Section 18 gives exact finite-block cancellation, theta bounds and arithmetical upper bounds for rates. Capacity computability and strict normal versus finite-dimensional asymptotic separations remain open. The Catalan and parity examples expose respectively an entropy barrier and a positive game-value gap; neither turns one-shot hardness into a capacity-undecidability theorem.

Obtaining relative Hilbert–Schmidt control of an approximate game assemblage from a high-rate code, and quantitative game lower bounds, remain open.

20.5 Other questions

The type languages are existence languages: the host and its normal state may be chosen together. They do not decide whether a prescribed factor and state realize a correlation, or

determine the least resources of a strategy. Natural questions ask which syntactic classes force finite-entropy spatial perfection whenever spatial perfection is possible, and how quantitative entropy requirements interact with the difference classification. The shared-family theorem resolves the nested differences by an explicit finite-entropy model. A quantitative stability theorem for the new spectral factors is a separate problem.

Can the game compiler be made substantially smaller? Can the period reader be realized by a simpler predicate? What are the optimal effective bounds when one supplies an entropy budget or a Schmidt-tail modulus? A classification of unrestricted prescribed-table membership would require additional control of the normalization that escapes in the example above.

How this work was done

The authors selected the questions and proof standards, directed the research and take responsibility for the claims. AI systems assisted with derivation, refutation-first review, programming and writing. The derivations of the II-ONLY completion, Rényi ladder, nested assembly and capacity boundary were produced with one model family. A second model family performed refutation-first checks, reconstructing the operator arguments and complexity reductions and checking the proofs presented here. This describes computational assistance, not independent human refereeing.

Finite computation is separate evidence. Separate counters recompute both full-game dry counts from the producer's emitted polynomials. The polynomial expansion and the minimal compiler each have a single producer, and the minimal structural audit reuses the serialized checker in `spatial/compiler/toolkit/`. For the nested compiler, a producer emits words, rational rows and row plans; a separate checker re-expands their semantic expressions and verifies the exact projection and transport algebra. The five concrete programs are validation instances, not compiled representatives of every reduction program. Random models test finite algebraic identities, not normal-support extraction. The parity graph's theta witness, independence number, block code and complete Schrijver symmetry decomposition were recomputed in an independent exact implementation for this paper. The commuting loss estimate is an analytic telescoping proof, with its finite word identity and constants checked exactly.

The ancillary runner also checks finite selector models, equality readers, Boolean reductions, tree and comb spectra, explicit histories, finite-cutoff defects, channel correspondences, clique-saturation slicing and the rate identity. Exact rational or symbolic checks are distinguished from floating-point matrix calculations. Appendix C and Appendix D describe the arithmetic and limits. These computations do not prove infinite-dimensional existence, compactness, arbitrary-normal soundness, the group embedding theorems, type-III classification or hardness by sampling. Those conclusions rest on the written proofs and cited external theorems. This preprint has not been peer reviewed.

A The nested assembly

A.1 Input, output, and the scope of the calibration

An input is a finite real rational $*$-polynomial presentation in typed centralizer labels and contractions $X_1, \dots, X_s$. A label has type projection, unitary, positive contraction, or contraction. The modular symbols have the prescribed common ratio

$$h_0 X_i = \tfrac{1}{2} X_i h_0. \tag{117}$$

Each monomial contains at most two modular symbols, counting adjoints; centralizer coefficients can occur before, between, and after them. Centralizer polynomial equations and

degree-one endpoint equations are included. A normal realization has a nonzero Hilbert space and a faithful trace-class density h_0 of trace one, with all identities on its support. The ratio in (117) is part of the input convention, not an operator to be synchronized. Section B treats the common ratio 2^{-n}. Arbitrary unrelated real ratios are outside this stated lemma: the calibration used here supplies powers of two.

The proof uses two calibrated levels so that an entire inner activity, including off-diagonal lifted contractions, is transported by one outer map. A centralizer corner unitary then identifies equal-density corners. Products of the resulting modular operators can be translated into actual centralizer products. The whole-activity refinement is what prevents independent gauges from changing those products.

Theorem A.1 (Nested compiler)**.** *Every such presentation effectively determines a finite rational Boolean game. A perfect normal spatial strategy extracts a normal realization on a source corner of mass* $1/81$*. Every normal realization has one simultaneous perfect extension using a* 25*-corner scaffold, a qutrit frame, and a finite row ancilla. The extension preserves finiteness of every positive moment and of entropy. With types, centralizer conditions, and modular covariance exact, constructed approximate realizations obey*

$$1-\omega \le C_{\mathrm{nest}} \sum_j \|r_j\|_{2,h_0}^2, \qquad \|r\|_{2,h_0}^2 = \mathrm{Tr}(h_0 r^* r), \tag{118}$$

where C_{nest} *depends only on the finite compiled presentation.*

A.2 Corners, parameters, and activities

Begin with nine synchronized corner projections

$$f,\ m_{00}, m_{01}, m_{10}, m_{11},\ E_{00}, E_{01}, E_{10}, E_{11}.$$

Name a synchronized activity $E_{\mathrm{in}}^{\mathrm{act}}$ and impose

$$E_{\mathrm{in}}^{\mathrm{act}} = f + \sum_{ab} m_{ab}, \qquad E_{\mathrm{in}}^{\mathrm{act}} + \sum_{ab} E_{ab} = I. \tag{119}$$

The inner conditional FMS has flag f, activity $E_{\mathrm{in}}^{\mathrm{act}}$, and selected corner $e = m_{00}$. The outer conditional FMS has flag *the whole* $E_{\mathrm{in}}^{\mathrm{act}}$, activity I, and selected corner $E = E_{00}$. Retain every original question, both orientations of each event binding, and every inactive answer. The inner inactive event is $I - E_{\mathrm{in}}^{\mathrm{act}}$. The outer inactive event is zero. In particular its column partition has flag $E_{\mathrm{in}}^{\mathrm{act}}$, four E_{ab} answers, and a zero inactive answer. No new inner flag is substituted for $E_{\mathrm{in}}^{\mathrm{act}}$ in that partition.

The frame-bound parameter list is exactly the nine corners, $E_{\mathrm{in}}^{\mathrm{act}}$, and the four positive ZI, IX events, two for each bridge: fourteen parameters. Use the parameter-binding rows of Section 4.2. Set $M_{\mathrm{in}} = E_{\mathrm{in}}^{\mathrm{act}} - f$ and $M_{\mathrm{out}} = I - E_{\mathrm{in}}^{\mathrm{act}}$. The four unitary axis labels are equated to $I - 2M + 2E_{\mathrm{axis},+}$ for their respective bridge. Let $r_{ab} = x_{\mathrm{in}}^a z_{\mathrm{in}}^b e$ and $R_{ab} = x_{\mathrm{out}}^a z_{\mathrm{out}}^b E$ denote the actual magic corner maps. These are centralizer expressions in the bound parameters and axes. The two calibrated transports satisfy

$$u^*u = f,\ uu^* = e,\ hu = 2uh; \qquad \mathsf{U}^*\mathsf{U} = E_{\mathrm{in}}^{\mathrm{act}},\ \mathsf{U}\mathsf{U}^* = E,\ h\mathsf{U} = 2\mathsf{U}h. \tag{120}$$

The inner masses give $\mathrm{Tr}(hE_{\mathrm{in}}^{\mathrm{act}}) = 9\,\mathrm{Tr}(hf)$; the outer masses give $1 = 9\,\mathrm{Tr}(hE_{\mathrm{in}}^{\mathrm{act}})$. Thus $\mathrm{Tr}(hf) = 1/81$ without an assumed nonvanishing sector. The bridge theorem remains valid for a zero activity, and these equalities and the cover then prove nonvanishing.

The outer bridge carries the whole inner activity, including each lifted contraction, into a copy with one common gauge. A centralizer unitary identifies equal-density corners there. Consequently two modular factors can be represented by actual centralizer products; transporting the labels separately would not identify the intermediate gauges.

A.3 The complete transport list and the whole-activity block

Encode a projection P by $2P-I$, a positive contraction C by a unitary with $C = (2I+U+U^*)/4$, a general contraction by the average of two unitaries, and a unitary directly. Identities in a supported corner use that corner's identity. The outer transported list consists of:

1. the two inner projections f, e, regarded as projections in $E_{\mathrm{in}}^{\mathrm{act}}$;
2. all source label encodings that occur at any endpoint (transporting all source labels is an allowed deterministic choice);
3. both unitary encodings of every lifted contraction $a_i = ea_i f$;
4. the finitely many derived coefficients specified below when a coefficient occurs inside a quadratic word.

For every encoding name k, introduce global unitaries U_k, V_k with

$$(U_k - I)(I - E_{\mathrm{in}}^{\mathrm{act}}) = 0, \qquad (V_k - I)(I - E) = 0.$$

A root-supported encoding additionally fixes $I - f$. Its transported copy will then fix $E - f'$. Introduce a global unitary V_k^{blk} and impose

$$V_k^{\mathrm{blk}} = E_{\mathrm{in}}^{\mathrm{act}} U_k + \sum_{ab} R_{ab}(EV_k)R_{ab}^*. \tag{121}$$

There is no $I - E_{\mathrm{in}}^{\mathrm{act}} - M_{\mathrm{out}}$ term since the outer activity is I. In (121), $E_{\mathrm{in}}^{\mathrm{act}} U_k$ acts on the entire five-corner space $E_{\mathrm{in}}^{\mathrm{act}}\mathcal{H}$, including off-diagonal blocks between f and e. Replacing it by a sum of five unrelated diagonal encodings would lose the required common transport. The identity $(U_k - I)(I - E_{\mathrm{in}}^{\mathrm{act}}) = 0$ gives $U_k(I - E_{\mathrm{in}}^{\mathrm{act}}) = I - E_{\mathrm{in}}^{\mathrm{act}}$. Left multiplication by U_k^* gives $U_k^*(I - E_{\mathrm{in}}^{\mathrm{act}}) = I - E_{\mathrm{in}}^{\mathrm{act}}$; taking adjoints gives $(I - E_{\mathrm{in}}^{\mathrm{act}})U_k = I - E_{\mathrm{in}}^{\mathrm{act}}$. Thus $[U_k, E_{\mathrm{in}}^{\mathrm{act}}] = 0$. The same proof gives $[V_k, E] = 0$. Consequently $E_{\mathrm{in}}^{\mathrm{act}} U_k$ is a unitary on $E_{\mathrm{in}}^{\mathrm{act}}\mathcal{H}$, and each $R_{ab}(EV_k)R_{ab}^*$ is unitary on its own magic corner. Their supports are orthogonal and cover I, so (121) is block diagonal for $E_{\mathrm{in}}^{\mathrm{act}}\mathcal{H} \oplus \bigoplus_{ab} E_{ab}\mathcal{H}$ and is unitary.

For f use $U_f = I - 2E_{\mathrm{in}}^{\mathrm{act}} + 2f$, and for e use $U_e = I - 2E_{\mathrm{in}}^{\mathrm{act}} + 2e$; their outer unitary names determine

$$f' = \tfrac{1}{2}E(I + V_f), \qquad e' = \tfrac{1}{2}E(I + V_e).$$

For each bridge separately refine its retained signs for the six projections

$$F_1, \quad F_2, \quad F_3, \quad G_{12}(1), \quad G_{13}(1), \quad G_{23}(1).$$

For the outer bridge also refine them separately for every $G_{12}(V_k^{\mathrm{blk}})$. The unchanged opposite questions are retained. The six frame refinements first make both calibrated transports scalar on the qutrit frame. The outer graph commutator then has lower-left block

$$\mathsf{U}(E_{\mathrm{in}}^{\mathrm{act}} U_k) = (EV_k)\mathsf{U}. \tag{122}$$

This follows with the same U for every k, including the two encodings of a contraction. Hence $f' = \mathsf{U}f\mathsf{U}^*$, $e' = \mathsf{U}e\mathsf{U}^*$, and, with $\mathsf{U}_0 = \mathsf{U}f$, $\mathsf{U}_1 = \mathsf{U}e$,

$$a_i' = \mathsf{U}_1 a_i \mathsf{U}_0^*. \tag{123}$$

Both sides are actual operators, not independently gauged copies. There are six refinements at the inner bridge and $6 + |\Lambda|$ at the outer bridge, per retained question and per party.

Label refinement (Lemma 4.2) makes every retained sign commute with the six frame projections. In the frame coordinates these generate $\mathrm{Mat}_3 \otimes I$: the F_i give the diagonal matrix units and $F_i G_{ij}(1) F_j = e_{ij} \otimes I/2$ gives the others. Their commutant is $I_3 \otimes B(\mathcal{H}_{\mathrm{base}})$. Because the corners are frame-bound, the sign and its (c_0, f) block are frame scalar. Thus $\mathsf{U} = I_3 \otimes \widetilde{\mathsf{U}}$

(and likewise u). Writing the outer sign as $I_3 \otimes \widetilde{\mathcal{X}}_0$, the $(1,2)$ frame block of its commutator with

$$G_{12}(V_k^{\mathrm{blk}}) = \tfrac{1}{2}(F_1 + F_2 + e_{12} \otimes V_k^{\mathrm{blk}} + e_{21} \otimes V_k^{\mathrm{blk}*})$$

is $[\widetilde{\mathcal{X}}_0, V_k^{\mathrm{blk}}]/2$. On the outer sign support, with flag space $E_{\mathrm{in}}^{\mathrm{act}}\mathcal{H}$ and selected magic space $E\mathcal{H}$, the bridge lemma gives

$$\widetilde{\mathcal{X}}_0 = \begin{pmatrix} 0 & \widetilde{\mathsf{U}}^* \\ \widetilde{\mathsf{U}} & 0 \end{pmatrix}, \qquad V_k^{\mathrm{blk}} = \begin{pmatrix} E_{\mathrm{in}}^{\mathrm{act}} U_k|_{E_{\mathrm{in}}^{\mathrm{act}}\mathcal{H}} & 0 \\ 0 & EV_k|_{E\mathcal{H}} \end{pmatrix}.$$

Here c_0 means the selected outer magic direction E and f means its flag, namely the *whole* $E_{\mathrm{in}}^{\mathrm{act}}$, not the inner f. The (c_0, f) block is exactly $\widetilde{\mathsf{U}}(E_{\mathrm{in}}^{\mathrm{act}}U_k) - (EV_k)\widetilde{\mathsf{U}}$. $E_{\mathrm{in}}^{\mathrm{act}}U_k$ may mix all five inner corners. For example, if $E_{\mathrm{in}}^{\mathrm{act}}U_k = (U_{rs})$ in inner-corner coordinates, this equation simply says $EV_k = \mathsf{U}(U_{rs})\mathsf{U}^*$; Thus the commutator argument gives (122) with the same U for every label. Multiplying (122) by the unitary adjoints gives transport also for U_k^*, V_k^*. Decoding the same polynomials in these unitaries therefore transports projections, positive contractions and averages of two unitaries. In particular it gives (123), including both encodings of each a_i, rather than just its diagonal compressions.

Introduce a centralizer contraction t by two global unitaries and impose

$$t = f'te, \qquad t^*t = e, \qquad tt^* = f'. \tag{124}$$

The corner maps t and a_i are synchronized centralizer expressions; $u, \mathsf{U}, \mathsf{U}_0, \mathsf{U}_1$ are extracted transports and are never graph words. Set $\mathcal{H}_0 = f\mathcal{H}$ and $h_0 = 81fhf|_{\mathcal{H}_0}$. The extracted contractions are

$$X_i = \mathsf{U}_0^* t a_i. \tag{125}$$

Since $h\mathsf{U}_0^* = \mathsf{U}_0^* h/2$, all other factors centralize h, and $\mathrm{Tr}(hf) = 1/81$, (125) gives (117) for the faithful trace-one h_0.

A.4 All quadratic translations and arbitrary coefficients

Put $\ell = \mathsf{U}_0^* t \mathsf{U}_1^*$. It is a corner unitary from $e'\mathcal{H}$ to $f\mathcal{H}$. Direct multiplication, using only (123)–(124), gives

$$X_i^* X_j = a_i^* a_j, \tag{126}$$

$$\mathsf{U}_0 X_i X_j^* \mathsf{U}_0^* = t a_i a_j^* t^*, \tag{127}$$

$$X_i X_j = \ell a_i' t a_j. \tag{128}$$

For the last identity use $a_i \mathsf{U}_0^* = \mathsf{U}_1^* a_i'$; the intermediate projections are e, f', e' and cancel by their stated source/range rules. The following table is consequently an equivalence in both directions.

Root identity	Submitted centralizer identity
$X_i^* X_j = R_{\mathrm{tar}}$	$a_i^* a_j = R_{\mathrm{tar}}$ at f
$\sum_i X_i X_i^* = R_{\mathrm{tar}}$	$\sum_i t a_i a_i^* t^* = {R_{\mathrm{tar}}}'$ at f'
$X_i X_j = X_k X_l$	$a_i' t a_j = a_k' t a_l$
$X_i^* X_j = X_k X_l^*$	$a_i'^* a_j' = t a_k a_l^* t^*$
$L X_i = X_j R$	$L' t a_i = t a_j R$

Here ${R_{\mathrm{tar}}}' = \mathsf{U}_0 R_{\mathrm{tar}} \mathsf{U}_0^*$ and $L' = \mathsf{U}_0 L \mathsf{U}_0^*$ are obtained from the transported encodings. The mixed quadratic row uses $\mathsf{U}_0 a_i^* a_j \mathsf{U}_0^* = a_i'^* a_j'$. Conjugation by U_0, or multiplication by ℓ, is injective on the specified corner. Thus these are operator equivalences rather than implications inferred from traces. In particular they give the displayed quadratic identities.

For completeness of the admissible class, here is a finite procedure for interior coefficients. Give X degree 1 and X^* degree -1. In a density eigenbasis, a degree-d word has matrix entries only where the output eigenvalue is 2^{-d} times the input eigenvalue. Distinct degrees are disjoint and orthogonal in $\|\cdot\|_{2,h_0}$; split a polynomial into degrees $-2,-1,0,1,2$. Take adjoints of negative degrees. For centralizer coefficients K_L, K_M, K_R:

1. Degree one: $K_L X_i K_R$ lifts to $K_L{}' t a_i K_R$.
2. Degree zero: $K_L X_i K_M X_j^* K_R$ lifts to $K_L{}' t a_i K_M a_j^* t^* K_R{}'$ at f'. A term $K_L X_i^* K_M X_j K_R$ first becomes $K_L a_i^* t^* K_M{}' t a_j K_R$ at f; transport that whole supported centralizer expression to f' through its own derived encoding.
3. Degree two: put $H_{K_L} = t^* K_L{}' t$ at e and transport it to $H'_{K_L} = \mathsf{U}_1 H_{K_L} \mathsf{U}_1^*$ at e'. Then

$$K_L X_i K_M X_j K_R = \ell H'_{K_L} a'_i K_M{}' t a_j K_R.$$

Indeed $\ell H'_{K_L} = K_L \ell$, by multiplication and $tt^* = f'$.

A derived coefficient with a known rational norm bound K is encoded after division by $\max(1,K)$, and its defining equation is added as a centralizer polynomial. Such a bound follows by multiplying and summing the declared contraction bounds. Only the two displayed kinds of derived coefficients are needed for degree two. Define the base labels and t first, the original transports next, and then the derived coefficients and their transports. Definitions involve no newly transported derived coefficient. In the implemented application and regression routes the chosen bounds are $K = 1$, so division by $\max(1,K)$ is vacuous there; it is a provision of the general procedure. Adding their separate refinements constrains the original U, and never extracts a fresh gauge. This is a finite, non-circular assembly for the whole stated class.

A.5 Soundness and the Boolean game

Start with a normal strategy, purify without querying the environment, and reduce to the original marginal supports by the toolkit items (N1)–(N8) of Section 4. Perfect matching precedes every use of projectivity. Boolean row tests give the rational identities on these supports. Graphs and triangles then give actual common unitary words. Parameter bindings identify the actual FMS events; they do not replace them by free labels. Apply the two conditional bridge theorems to those bound events. The frame refinements apply first; the block refinements then give (122). Decode labels, impose (124), and apply (125) and the translation procedure. This order uses neither the intended application nor its spectral classification in extracting the scaffold.

All cancellations of a coefficient operator use its dense range on the original local support, as proved in (N1)–(N8). On a mixed-state cut $A \mid BE$ the coefficient map need not be injective; no assertion of injectivity on all of $B \otimes E$ is used. All graph calculations are bounded block identities. No density inverse, bounded or otherwise, occurs. The source density is faithful since f commutes with h and h is faithful on the reduced support.

Write $h = h_A$ on $P_A \mathcal{H}_A$. The synchronized corners are projections commuting with h; their sum identities make them orthogonal. Since $hu = 2uh$ and $h\mathsf{U} = 2\mathsf{U}h$, cyclicity of the trace of a trace-class operator times bounded operators gives

$$\begin{aligned} \operatorname{Tr}(he) &= \operatorname{Tr}(huu^*) = \operatorname{Tr}(u^* hu) = 2\operatorname{Tr}(hf), \\ \operatorname{Tr}(hE) &= \operatorname{Tr}(h\mathsf{U}\mathsf{U}^*) = \operatorname{Tr}(\mathsf{U}^* h \mathsf{U}) = 2\operatorname{Tr}(hE_{\mathrm{in}}^{\mathrm{act}}). \end{aligned}$$

The centralizer magic-corner maps give equal masses to the four inner selected copies and, separately, the four outer selected copies. Thus $\operatorname{Tr}(hE_{\mathrm{in}}^{\mathrm{act}}) = 9\operatorname{Tr}(hf)$ and $1 = \operatorname{Tr}(hE_{\mathrm{in}}^{\mathrm{act}}) + 4\operatorname{Tr}(hE) = 9\operatorname{Tr}(hE_{\mathrm{in}}^{\mathrm{act}})$, proving $\operatorname{Tr}(hf) = 1/81$. It follows that $\mathcal{H}_0 = fP_A\mathcal{H}_A \neq 0$ and $h_0 = 81fhf|_{\mathcal{H}_0}$ is faithful, trace class and trace one.

A supported unitary commutes with its support by the calculation above. On $\mathcal{H}_0$ the four decoding types are therefore respectively U, $(I+U)/2$ with $U^2=I$, $(2I+U+U^*)/4$, and $(U_+ + U_-)/2$. They are respectively unitary, projection, positive contraction, and contraction, and each commutes with h_0. The supported $a_i = ea_if$ is a contraction; (124) makes t a partial isometry from e onto f', and U_0 is one from f onto f'. Hence $X_i = \mathsf{U}_0^* t a_i$ maps $\mathcal{H}_0$ into itself with norm at most one. More explicitly,

$$h_0X_i = 81fh\mathsf{U}_0^*ta_i = \tfrac{81}{2}\mathsf{U}_0^*hta_i = \tfrac{81}{2}\mathsf{U}_0^*ta_ifhf = \tfrac{1}{2}X_ih_0.$$

Transport of decoded labels and their adjoints, followed by the bounded corner multiplications in Section A, now gives every submitted application identity on this actual $\mathcal{H}_0$ in both directions. The degree splitting uses the density eigenbasis on this support, never an inverse of h.

Bob's original marginal support is treated in the same way: matched coarse effects are its projective mirrors, the bridge and refinement lemmas apply on that support, and the same cyclicity gives source mass $1/81$. Mirroring products reverses their order, as in (N4a); no extra mirror equation is submitted. Alice's extracted realization already proves soundness; the symmetric support argument justifies all Bob-side bindings and refinements, including on a purified mixed-state cut with its original Bob marginal.

Expand the final finite centralizer polynomials in their real rational unitary words, cancel adjacent inverse tokens, and close under proper prefixes. The exported words cancel adjacent inverse tokens, but are not reduced using parameter involutions; this is stated redundancy in their counts. The exact row audit may use those involutions. Add all graphs, placement triangles, product triangles, inverse triangles, parameter bindings, and the off-diagonal polynomial rows. Apply the Boolean-row construction already proved above. Retain all FMS and refinement questions, pad the two question sets, and sample every pair uniformly with positive rational probability Q^{-2}. Accept every unspecified answer pair. Perfection gives zero probability for each individual rejecting rule, proving soundness.

A.6 One simultaneous completeness model

Let $\mu = \mathrm{diag}(1,2,2,2,2)/9$. On $\mathbb{C}^5_{\mathrm{out}} \otimes \mathbb{C}^5_{\mathrm{in}} \otimes \mathcal{H}_0$ take

$$h_{\mathrm{sc}} = \mu \otimes \mu \otimes h_0. \tag{129}$$

Outer index 0 is the inner activity; inner index 0 is f and inner index 1 is e. The selected outer index is 1. Thus

$$\begin{aligned} f &= |0,0\rangle\langle 0,0| \otimes I, & e &= |0,1\rangle\langle 0,1| \otimes I,\\ f' &= |1,0\rangle\langle 1,0| \otimes I, & e' &= |1,1\rangle\langle 1,1| \otimes I,\\ \mathsf{U} &= |1\rangle\langle 0| \otimes I_5 \otimes I, & t &= |1,0\rangle\langle 0,1| \otimes I,\\ a_i &= |0,1\rangle\langle 0,0| \otimes X_i, & a_i' &= |1,1\rangle\langle 1,0| \otimes X_i. \end{aligned}$$

The density weights at e and f' are both $2h_0/81$, so t centralizes h_{sc}. Commutation of a_i with h_{sc} is exactly $2h_0X_i = X_ih_0$. All the copies use the identity outer identifications. The inner and outer FMS effects are the real canonical ones on their respective five-dimensional factors, with the inner copy inactive off outer index 0. Their compressed sign vectors are $|f,f\rangle + (-1)^{xy}|c_x, r_y\rangle$, up to the common positive amplitude and spectator factors. The FMS sign completeness used here is the explicit calculation in Lemma 4.5, not a completeness assertion in Section 4.

Every positive density eigenspace is finite-dimensional. For a centralizer contraction T, its polar partial isometry extends on each such eigenspace to a unitary V. Then

$$U_\pm = V\big(|T| \pm i\sqrt{I-|T|^2}\big), \qquad T = (U_+ + U_-)/2. \tag{130}$$

Direct sums give bounded centralizer unitaries. On a prescribed support such as $E_{\mathrm{in}}^{\mathrm{act}}$, do this inside that support and extend by identity outside. For a root positive contraction use $U = 2C - I + 2i\sqrt{C(I-C)}$, and for a projection use $2P - I$. Encode source choices once and transport precisely those choices into E; (121) now holds simultaneously for every label, even a lifted a_i with off-diagonal inner blocks. Encode the global t by (130).

Append $I_3/3$ and all graph, triangle, and parameter-binding projections. For every refinement use actual joint products: its block unitary commutes with its own FMS signs. Distinct labels require no mutual commutation. Finally append $I_{d_{\max}}/d_{\max}$ and the Boolean-row effects. Alice uses these operators and Bob their transposes in a product density eigenbasis. The toolkit proves every matching, calibration, orthogonality, and row-zero test with this one state and assignment. Hence no separate realization is being chosen for separate rows.

Both nested bridges have magic density twice the flag density. For the inner one the flag spectator density is h_0 (up to its scalar outer weight); for the outer one it is $\mu \otimes h_0$. Denote either normalized spectator by ρ, and the total flag mass by λ. The canonical magic directions satisfy $\langle c_x, r_y\rangle = \epsilon_{xy}/\sqrt{2}$, where $\epsilon_{xy} = (-1)^{xy}$. Compressing Ω_h to the two sign supports gives, omitting only the unchanged frame and row factors,

$$\Omega_{xy} = \sqrt{\lambda}(|f, f\rangle + \epsilon_{xy}|c_x, r_y\rangle) \otimes \Omega_\rho.$$

Alice's sign observable exchanges f, c_x and Bob's exchanges f, r_y. Applying the swaps gives $(\mathcal{X}_x \otimes I)\Omega_{xy} = \epsilon_{xy}(I \otimes \mathcal{Y}_y)\Omega_{xy}$. Their spectral sign answers therefore never violate $ab = \epsilon_{xy}$. The other column/row projections give the retained answers and the calibration rules exactly. The inner measurements have inactive effect $I - E_{\mathrm{in}}^{\mathrm{act}}$ off outer index 0; that sector is matched on the two sides. The outer bridge is active everywhere.

Under $E\mathcal{H} \cong \mathbb{C}_{\mathrm{in}}^5 \otimes \mathcal{H}_0$, let $\widehat{U}_k$ be the entire chosen unitary on $E_{\mathrm{in}}^{\mathrm{act}}\mathcal{H}$. Identity outer identifications make

$$V_k^{\mathrm{blk}} = I_{\mathrm{out}} \otimes \widehat{U}_k.$$

Every outer FMS effect, including its noncentralizer sign effects, acts on the outer factor alone and hence commutes with V_k^{blk}, even when $\widehat{U}_k$ has off-diagonal inner blocks. The inner bridge has only the six frame refinements, which commute because the frame is a separate scalar factor. Thus every labelling refinement is the actual positive joint product of commuting projections, sums to the original coarse measurement, and has the required graph marginal. No commutation between distinct labels is needed. Transposition in the product density eigenbasis preserves these commutators and products on Bob's side. All centralizer matching and orthogonality tests follow from (125), and the appended row factor gives all Boolean row tests simultaneously with the same state. This proves the claimed nested completeness assignment.

The complete marginal is a tensor product of h_0 with a fixed finite-dimensional density η. Thus $\mathrm{Tr}[(\eta \otimes h_0)^\alpha] = \mathrm{Tr}(\eta^\alpha)\,\mathrm{Tr}(h_0^\alpha)$ for every $\alpha > 0$, and its entropy is $S(\eta) + S(h_0)$. Conversely compression to the commuting source corner of mass $1/81$ cannot create a finite positive moment from an infinite full moment. These observations give the stated resource preservation.

A.7 The loss constant, including the normalization factors

Keep exact structural hypotheses and choose all derived coefficient encodings from their actual expressions, so their defining rows stay exact. A homogeneous source residual r has a lifted centralizer residual p. For a degree-one or degree-two lowering residual the lift has input corner f, so

$$\|p\|_{2,h_{\mathrm{sc}}}^2 = \frac{1}{81}\|r\|_{2,h_0}^2.$$

A degree-zero residual transported to f' instead has factor $2/81$. For negative degrees take the adjoint; if $h_0 r = 2^{-d} r h_0$, cyclicity gives $\|r^*\|_{2,h_0}^2 = 2^{-d}\|r\|_{2,h_0}^2$. Therefore each chosen translation has factor $\beta_j/81$, with $\beta_j \le 4$. Splitting degrees preserves the sum of squared norms because their density matrix entries have disjoint ratios.

For the qutrit row S_j the two off-diagonal blocks are p_j, p_j^*. Since p_j is in the centralizer, trace cyclicity gives

$$\mathrm{Tr}(h_{\mathrm{old}} S_j^2) = \frac{2\beta_j}{243}\|r_j\|_{2,h_0}^2.$$

Let L_j clear the rational row denominators and let d_j be its padded power-of-four column dimension. Equation (19) then gives, for one oriented row-zero test,

$$Q^{-2} p(\Pi_0^{\mathrm{row}}, \Pi_{d_j}^{\mathrm{ref}}) = \frac{8L_j^2\beta_j}{243 d_j^3 Q^2}\|r_j\|_{2,h_0}^2. \tag{131}$$

Sum these probabilities; overlapping rules can only reduce total loss. If both orientations are retained, include both in the sum (equivalently multiply the maximum constant by two). One may therefore take the maximum displayed coefficient, with the explicit orientation multiplicity, as C_{nest}. It is independent of the Hilbert space, density spectrum, witness period, and computation time. If just one forbidden projection has root probability p, its residual squared norm is p, and (118) reads loss at most $C_{\mathrm{nest}} p$. This proves the completeness estimate, not a robust extraction assertion.

B The bridge-count and branching variants

Corollary B.1 (Common dyadic ratio and arbitrary finite branching)**.** *For every fixed $n \ge 1$, replace the inner bridge by an n-bridge chain and the outer bridge by an n-bridge chain with first flag the entire inner activity. The same degree-two compiler works with common ratio 2^{-n}. If $d_n = 1 + 4n$ and $Z_n = 1 + 4\sum_{g=1}^{n} 2^g$, the scaffold has d_n^2 scalar corners and source mass Z_n^{-2}. Any finite number b of creation labels may be used. For a full b-ary isometry presentation a normal realization requires $b2^{-n} < 1$ and contains the weighted Fock factor with weights $(1 - b2^{-n})2^{-n|w|}$.*

Proof. The explicit chain corner list is $e_0, m_{g,ab}$, $1 \le g \le n$, with $e_g = m_{g,00}$, activity $e_{g-1} + \sum_{ab} m_{g,ab}$, and cover $e_0 + \sum_{g,ab} m_{g,ab} = I$. The bridge transports U_g have ratio 2; their product has ratio 2^n. Use the block equation (121) at each link, including its inactive identity term when the link activity is not I. Transport the same encoding list through every link. This gives the actual common product transport, by induction on g. The inner chain density is

$$Z_n^{-1}\,\mathrm{diag}(1; 2^{\oplus 4}; 4^{\oplus 4}; \ldots; (2^n)^{\oplus 4}) \otimes h_0.$$

Take two such factors. The inner last selected corner and the outer last selected copy of the source both have density $2^n h_0/Z_n^2$, so the same t construction applies. Replace $\mathsf{U}_0, \mathsf{U}_1$ by restrictions of the common outer chain product. Every multiplication in the proof above remains valid, and the common quadratic factor still exists. The finite mass factors in (131) change accordingly and remain presentation-dependent constants.

The number b affects only the finite list of lifted a_i and its isometry, orthogonality, and range rows. No bridge has to possess b magic corners: its magic square still has four. If $S_i^* S_j = \delta_{ij} I$ and $hS_i = 2^{-n} S_i h$, cyclicity gives total range mass $b2^{-n}$. Pull a positive density eigenvector backwards through nonzero S_i^*; its eigenvalue increases by 2^n and so the process must terminate. The defect and all its word descendants exhaust the support. The word sectors are orthogonal, and their densities are $2^{-n|w|}$ times the root density. Normalization forces $b2^{-n} < 1$ and gives the stated factor. Its α-moment converges precisely when $b2^{-n\alpha} < 1$. This proves both the $n \ne 4$ and $b \ne 4$ variants. □

Exactly n bridges for the bare-tree presentations of Section 9. The bare-tree and degree-one endpoint presentations need only one chain, hence exactly n bridges, rather than the two chains needed for the quadratic extension. Here is its complete specialization. Use the preceding $1+4n$ corner list and cover, bind those corners and the $2n$ positive axis events to the frame, and set $v = \mathsf{U}_n \cdots \mathsf{U}_1$. For each transported encoding use the copies $u_{g,\ell}$ fixing $I - e_g$ and the block equation (25). Refine that link for the six frame projections and every $G_{12}(V_{g,\ell})$. This gives the same calibrated transport of every label through v, with no independent link gauges. Introduce $b_i = e_n b_i e_0$ and extract $S_i = v^* b_i$. The input rows are exactly

$$b_i^* b_j = \delta_{ij} e_0, \qquad \sum_i b_i b_i^* = e_n - P_n, \qquad L_n b_i = b_j R_0,$$

plus the root centralizer polynomial rows. These are respectively the isometry/orthogonality, range, and endpoint translations; multiply by v, v^* to obtain both implications. The cover fixes source mass Z_n^{-1}. The one-chain density displayed in the proof, the same finite-block encodings, and actual joint refinements give simultaneous completeness. The one-oriented loss constant is

$$\max_j \frac{8L_j^2 \beta_j}{3 Z_n d_j^3 Q^2}, \qquad \beta_j = 1 \text{ for source and endpoint rows}, \quad \beta_j = 2^n \text{ for range rows}.$$

For the two-chain quadratic version replace Z_n by Z_n^2 and use $\beta_j \le 2^{2n}$. Any finite b is handled just by the displayed finite i, j lists. Thus the one-chain n-bridge compiler used in Section 9 and the nested degree-two extension are both specified.

C Finite checks and reproduction

The accompanying `verification/` directory contains eight suites and one runner. Run `python -B run_all.py --output-dir DIR` there, with `DIR` new or empty. The runner writes logs and results to that directory, regenerates nested cores there, compares fresh counts and hashes with the shipped records, and fails if any check fails. Add `--full-counts` to repeat both large full-game dry counts. All suites can run from a relocated source copy. Its README states commands, dependencies and limits. `SHA256SUMS` identifies every bundled file apart from itself, with relative paths; the manifest uses LF line endings. The tree-and-clock checker needs NumPy; the communication suite needs NumPy, NetworkX, SymPy and mpmath; the nested checks additionally use SymPy and psutil; the capacity checks use CVXPY with an SDP solver. The exact parity certificates use SymPy and rational arithmetic.

Spatial suite (`spatial/`). The numerical tree checker constructs the truncated Fock model, Weyl pair, coordinate increments, clock, counter and exact-return candidate of Sections 5–6. Its pulse-driven cellular automaton is a small front-and-mark model. It is not an implementation of every Turing-machine decoder used by the reduction. The checker tests the creation and range identities, modular scaling, endpoint recurrences, clock and reader equations, tableau propagation, centralizer commutators and top-depth defects. The four diagonal isometry equations fail exactly at the top depth, as predicted by (51); all other reported residuals are at floating-point precision. The analytical mass is (50).

The primitive-phase checker tests the existence of a good primitive phase for every prime below 4000 and tests the chord lower bound only for primes below 400. Its trigonometric evaluations use floating point. Lemma 6.1 is the proof for all primes. The independent exact tree suite checks ten models, depths 0 through 4 and clock orders $2, 3$, with 2000 exact checks in total. A separate minimal compiler instance has four bridges and one modular isometry,

but no transported endpoint labels. It has 748 rational rows and 27 991 questions per party. Its independent structural audit checks 77 750 serialized bindings, 11 652 column-graph rules, all 748 row expansions, 333 scalar pair contexts and 2472 old forbidden events. It does not instantiate the full ten-label computation compiler. Table 4 is regenerated by the suite and compared with the paper.

Type suite (`types/`). The program uses exact rational identities for finite binary trees through depth 7, reader stand-ins through depth 4, an actual source-lifted graph residual with the scaffold and qutrit density, finite shift truncations with their top boundary defect, and chain scaffold masses through five bridges. It uses randomized finitely supported group elements with a fixed seed for (79), and numerical entropy and rotation checks. An exact-fraction enclosure certifies the golden-mean value at $y = 1$, the $y = 2$ comparison, and the rational base of the induction covering every $y \geq 3$ in (132). The Boolean profile core tests every reduction of Theorem 14.1 with unspecified coordinates adversarial, and enumerates all 7581 monotone five-input functions for an alternative input collection. Exact checks are distinguished from floating-point checks in its output. The finite stand-in is not asserted to be a representation of an arbitrary $\mathcal{G}_e$ or its HNN extension. It tests the level-reader and terminal algebra with an explicit balanced involution. Existence of the actual finite von Neumann algebra for any good n rests on the HNN normal-form theorem and regular representation in Lemma 11.4.

Communication suite (`zero-error/`). The magic-square graph has classical independence number five and a six-message entangled code. The flagged-magic-square graph has fifty vertices, classical independence number nine and a ten-message entangled code. The reverse map of Theorem 15.3 is also tested after a noncommutative row-and-column scrambling. These examples verify the implementation of the finite correspondence already proved in [37]; they are not novelty claims. The naive D^2 forward marginal is tested as a negative control. Floating-point matrix checks use explicit tolerances. The exact script checks $4n + 1$, $6n + 1$, $\mathfrak{c}_n$, normalized scaffold weights, source/far mass ratios, convergence and criticality of the amplitude series, the five- and eight-bridge constants (102) and (103), the five-bridge code entropy (105), the squared Catalan entropy, and all identities in (111). The infinite-entropy conclusion uses Catalan asymptotics in the proof, not a finite partial entropy sum.

Labels inside the programs. Some program outputs use internal relation labels, for example "(19)–(21)" in the type suite or "(20) JK = ZKJ" in the tree checker. The README maps these labels to the equations of this paper. The finite-check labels above do not identify proof-assistant results.

Lean thin core. Lean kernel-checks the 601 listed proof roots for the concrete row-game construction, selected actual-support and matrix-unit arguments, bridge/refinement and row-algebra steps, fixed-n=4 application syntax, word-coordinate and scalar Fock-entropy calculations, local clock/reader implications, elementary winning-functional identities, and the good-primitive-phase lemma for every prime. Only `propext`, `Classical.choice` and `Quot.sound` occur as solution axioms. Theorem A and the complete compiler, extraction and reduction are not Lean-checked. The general `Presentation.admissible` predicate omits transport of the root and of labels to the left of a creation symbol in a linear endpoint sum, but the actual fixed-$n = 4$ application table satisfies both requirements: its root is transported and it has no linear endpoint sums. The paper's synchronized-projection premise requires `NamedBinaryProjections R S` together with `RowsHold R S`; `RowsHold` alone permits nonprojective effects. Mathlib's 8624 mathematical-library Lean sources match

commit `56a005855ea17e0b0958f73ec801ae140b0a1679` byte for byte; the supplied vendor tree changes three cache-tool behaviors in `Cache/Requests.lean`, outside the proof import closure, and verification reuses dependency oleans rather than rebuilding Mathlib from zero. The shipped README gives the exact scope and replay details.

Table 6: Scope of selected Lean results; WHOLE refers only to the stated local assertion.

Local result	Paper reference (stable label)	Extent
Good primitive phase, every prime	Lemma 6.1 (`lem:goodphase`)	WHOLE
Actual application-table syntactic admissibility	Definition 4.4 (`def:admissible`); table (48)	WHOLE
Bridge component identity	Equation (12) (`eq:bridgevector`)	WHOLE
Magic-square rigidity and matrix units	Lemma 4.1 (`lem:bridge`)	STEP
Scalar Fock entropy series	Equation (33) (`eq:fockentropy`)	WHOLE
Conditional clock and reader implications	Equations (41), (44)–(47) (`eq:det`, `eq:coordinateclock`, `eq:reader`, `eq:return`, `eq:D`)	STEP

Build. The paper is built with `python -B build.py` from `paper/`. This runs `pdflatex`, BibTeX and the required repeat passes, and fails if the final log contains a warning, an undefined reference, or an overfull or underfull box. The plain-text abstract is `abstract.txt`.

The coupled selector and Rényi suites. The `ii-only/` programs check all sixteen selector edges at depths zero, one and two, their separate Gram system, coordinate and counter recursions, exact equality readers, the norm 1/8, Boolean coordinates and the three-star construction. They also exercise the injectivity hypothesis by a negative reader example. The `renyi/` programs check weighted-tree moments, quarter-plane moments, comb rounding and normalizers, tail and truncation formulas, Fibonacci reader bounds and nested transport algebra. They do not test an arbitrary total verifier or prove a computability classification from finite samples.

Capacity and the parity graph. The `capacity/` suite includes the independent exact `parity_certificates.py`: all 16384 vertex subsets, all thirty-nine edge orthogonalities, all 153 block-code pairs, all forty-eight symmetry automorphisms and the full fourteen-dimensional symbolic Fourier decomposition. It verifies the algebraic optimizer and every scalar or two-by-two positivity block. The free-word telescoping identity and the coefficient 1/18 are exact. No Schrijver claim for the strong square is included. Separate numerical scripts test nonproduct saturation slicing, theta vectors, code scramblings and the rate identity. Their numerical tolerances are reported in their logs; SDP values there are regression checks, not substitutes for the parity graph's exact certificate.

Full reduction dry counts. The `full-compile/` directory ships the two producers and independent SQLite counters, small exact audits and count records. The optional full-count run regenerates word streams and prefix databases in the fresh result directory and reproduces the literal 6 191 638 087 and factored 8 516 580 question counts. Neither is a serialized Boolean game. The small audit checks the tree boundary loss $t_m/(11616Q^2)$ for depths zero through four and the associated rational layout accounting. The nested generator and its separate row/semantic audits are described next.

C.1 Supplementary exact rotation margin

For $\vartheta = (\sqrt{5}-1)/2$ and $\zeta = e^{2\pi i\vartheta}$, use the lower bound of Section 14.2. Put $\gamma_y = 2^{y+1}|\zeta^y - 1|$. The global minimum of $\log_2 \gamma_y$ is attained at $y = 1$, and

$$2.89 < \log_2 \gamma_1 < 2.9, \qquad \log_2 \gamma_1 = 2.898452049757\ldots. \tag{132}$$

Here is a certificate, including the infinite tail. For $y \geq 3$, the preceding bound gives $\gamma_y \geq 2^{y+3}/(\sqrt{5}\,y + 1/2) > 8$: at $y = 3$ use $\sqrt{5} < 9/4$, and the ratio increases because $2(\sqrt{5}\,y + 1/2) > \sqrt{5}\,(y+1) + 1/2$. For $y = 2$ the exact certificate described below gives $\gamma_2 > 8$. For $y = 1$ it gives $1 < \gamma_1 < 8$ and $\gamma_{1,-}^{100} > 2^{289}$, $\gamma_{1,+}^{10} < 2^{29}$ for rational bounds $\gamma_{1,-} < \gamma_1 < \gamma_{1,+}$. To construct them, enclose $\sqrt{5}$ by integer square roots at scale 10^{80}, and π by $16\arctan(1/5) - 4\arctan(1/239)$ with 60 alternating terms and the next-term remainder. Evaluate $\sin(\pi\vartheta)$ at the rational endpoints using 40 alternating terms and the next term; sine is decreasing on this interval in $(\pi/2, \pi)$. Round outward to denominator 10^{40} and compare the displayed integer powers. For $y = 2$, use $|\sin(2\pi\vartheta)| = \sin(\pi(2\vartheta - 1))$; its argument is in $(0, \pi/2)$, where sine increases, and the same series gives $16\sin(\pi(2\vartheta - 1)) > 8$. The bundled exact-fraction routine implements these instructions. It certifies the $y = 1, 2$ enclosures and the rational base and coefficient inequality used in the tail estimate. Together with the induction above these prove the minimum over all positive integers; the 3999-point floating sweep is supplementary. The weaker positive margin is all that contractivity requires.

D Compiler validation instances

D.1 Parametric presentation input and exact size formulas

The general generator accepts a JSON presentation with typed `labels`, a list of `modular` names, and rational `relations`. Each relation is a finite sum of coefficient/word pairs; a token ending in `*` is an adjoint. It checks the modular degree bound, splits all five possible homogeneous degrees, adjoints negative degrees, and introduces the derived coefficients in Appendix A. It then emits the literal reduced words, proper prefixes, projection variables, rational rows, and row plans. The general interface is

```
python -I generate_nested.py --presentation INPUT.json
  --output-stem INSTANCE
```

The output stem is restricted to a filename inside the chosen output directory. The two application schemas can also take supplied finite cellular-automaton tables. The factored scan programs below avoid expanding a static track into a redundant product alphabet.

The word and rational-row schemas are emitted once in `compiler-schemas.json`. Let s count modular symbols; let c count unitary encodings of transported root labels and derived coefficients, with two encodings per contraction; let r count their projection labels; let d count derived defining equations; and let p_{app} count the final homogeneous application equations. Put $a = 14$ for the scaffold's bound parameters. Then

$$\begin{aligned} g &= 24 + 6s + 3c, \\ p_{\mathrm{str}} &= 13 + 7s + 3c + r + d, \qquad p = p_{\mathrm{str}} + p_{\mathrm{app}}, \\ V &= 15w + 1 - g + 19a, \\ R &= 12w - g + p + 17a - 1. \end{aligned} \tag{133}$$

Here g counts unitary generators, V projection variables, R rational rows, and w words including identity. The exact word count is $w = 1 + 2g + \sum_{k\geq 2} n_k$, where n_k is the number

of distinct length-k reduced prefixes of the expanded input words. This prefix profile is a presentation size parameter: a count using only the number and lengths of relations cannot account for shared prefixes.

To derive (133), there are $9w$ graph rows, one frame row, one reference triangle, $2(w-1)$ placement triangles, $w-1-2g$ proper-prefix product triangles, g inverse triangles, $17a$ binding rows, and p polynomial rows. Graphs use $12w$ projection names, triangles use $3w-2-g$, the frame uses three, and parameter binding uses $19a$. The outer transported list has $2+c+2s$ encodings, so the inner and outer bridge refinement counts per party are 6 and $8+c+2s$.

D.2 The concrete programs and both decision branches

Use a tiny predicate family, with indices $e = 0, 1, 2$:

$$R(0,n,s) = \text{false}, \qquad R(1,n,s) \iff n = 0, \qquad R(2,n,s) \iff (n,s) = (0,0). \tag{134}$$

The first and third are NO instances of $\exists n \forall s R$; the second has the single integer witness $n = 0$. A predicate true at only one *pair* does not have a universal-in-s witness. All three requested literal lists are supplied, so this distinction is tested rather than left implicit.

Each uniform verifier has eleven head states:

none, boot, scan1, scan2, scanbig, move1, move2, movebig, accept, reject, wait.

The return marker is a separate immutable track. At the origin, boot becomes move1, ignoring the marker at zero. A move state clears its cell and sends its corresponding scan state to the right. A scan on a nonmarker becomes the next saturated-count move state. At the first positive marker, scan2 certifies period $2 = F_3$; scan1 and scanbig reject. For $e = 1$ period two accepts; for $e = 0$ it rejects; for $e = 2$ it waits one step and rejects. Acceptance and rejection are absorbing. For these particular predicates every decoded $n > 0$ already fails at $s = 0$, so rejecting all other periods is exactly the required specialization of the Fibonacci verifier. It is unnecessary to compute a larger Fibonacci index only to reject it immediately.

For clarity, the actual update rows are finite factored centralizer polynomials. With P_h the current head projection and L_h its left-neighbor copy, a moving-head arrival is $(I - P_x)L_{\text{move}j}P_{\text{none}}$. For example $N_{\text{move2}} = (I - E_{\text{ret}})P_{\text{scan1}}$, and the positive instance has $N_{\text{accept}} = P_{\text{accept}} + E_{\text{ret}}P_{\text{scan2}}$. The none projection is identity minus all other successors. The code lists every successor polynomial. Their classical truth tables are deterministic even on head collisions, using center-state priority. Head projections have four synchronized neighbor/time families, and the same endpoint equations (62) and initial boot/none rows. This representation is equivalent to a finite automaton with a static marker track, and all its rows are in the admissible class.

For the comb choose the two explicit total programs

$$k_m = 1, \qquad k_m = m. \tag{135}$$

The constant program has critical index 0, and the linear program has critical index $1/2$. Both have nonzero B_0, B_1, branch vertices and leaves. Their six head states are none, boot, scan, move, accept, reject. The constant program checks at $x = 0$ whether $E_{\text{guard}} = \mathbf{1}_{\{\ell=0\}}$. The linear program scans until the first M or L marker (encoding C_{sp} or L): the guard should be one if the L marker occurs strictly first, and zero if the markers coincide or the M marker occurs first. Correct and flipped-guard decisions are both checked. Every head family commutes with $C_{\text{sp}}, L, E_{\text{guard}}, C_x, C_t$.

In the legal linear comb, $0 \le \ell \le m$, so the first marker is at $x = \ell$; the two possible stop cases are $\ell < m$ and $\ell = m$. Before that cell both marker bits are zero, giving a move followed by a scan one cell to the right. This proves termination and acceptance for every legal fiber

by induction, with no bound on m, ℓ. Flipping the guard reverses the stop decision. As a substantive negative control, the constant-depth comb can be fed to the linear checker: its fibers $m \geq 2, \ell = 1$ then reject. All other application identities still hold.

Instance	g	w	p	V	R
Uniform, never true	201	6995	426	104991	84402
Uniform, witness $n = 0$	201	6995	426	104991	84402
Uniform, one true pair	201	6995	426	104991	84402
Comb, $k_m = 1$	171	8414	412	126306	101446
Comb, $k_m = m$	171	8583	412	128841	103474

These literal lists are validation instances. The general theorem rests on the parametric lemma, as in the compiler construction of Section 4; the full reduction game for an arbitrary verifier program was not compiled.

D.3 Exact symbolic checks of the infinite models

The independent checker reads exported JSON and does not import the producer. It substitutes graph, complement, tilted-graph, triangle, and binding formulas in the exact rational free group ring, treating parameter reflections as involutions. Every noninput rational row becomes exactly zero. Every polynomial row becomes exactly its claimed pair of off-diagonal blocks, and every graph and triangle is separately checked to be a projection. Semantic expression trees are independently re-expanded into the literal input polynomials, closing the link between operator checks and exported rows.

Every structural polynomial is also checked directly on all 25 scaffold corners, at generic basis coordinates and each boundary. The a_i and t unitary encodings are explicit monomial completions over Gaussian rationals: on a source/range pair they are the swap with phases $1, \pm i$, and on unmatched coordinates the phase is $\pm i$. Root encodings are free diagonal phases keyed by basis coordinate; projection encodings are involutions. This proves every support, axis, whole-activity block, reflection, and t row, including actual unitarity of all encodings. Positive and general diagonal contraction encodings can then be specialized by (130). No density inverse enters. The same structural identities are checked again on rebuilt finite cutoff encodings.

Application checks use explicit decoded basis actions. S, T increment x, t, with adjoints vanishing at zero. On comb basis (m, w), V_{sp} increments m on the empty-word spine and vanishes off it, while B_i prepends i exactly when $|w| < k_m$; their adjoints remove the matching prefix or vanish at the boundary. All counters, markers, guards and readers are the explicit scalar formulas above. For the period-two uniform models, $Z = -I$, J swaps two Weyl basis vectors, $K = \mathrm{diag}(1, -1)$, $W = (-1)^x$, and the reader is $-2^{-x}/4$ on odd x and zero otherwise.

The checker evaluates the generic symbolic basis index and every lower boundary separately. Short tree words, both prefix bits, generic opaque tails, spine boundaries, branch vertices, and leaves are all included. Head fields are arbitrary symbolic functions for endpoint checks; their one-hot initialization and exhaustive local successor truth tables check the local update rule; the unbounded honest-history recurrence is justified by the prose induction above. The uniform programs have 5324 exact local update checks each and the comb programs 3456 each. The explicit period-two histories are also checked through the absorbing state. For the comb the two symbolic stop-order cases above give zero rejecting projection on every legal fiber. This is a prose induction, not a machine-checked induction certificate. The program `check_far_history.py`, implemented independently of the row producer, evaluates all head-dependent exported application rows with `cut=None`. For the linear comb it checks $m, \ell \leq 30$, $t \leq 65$ (successor reads through 66), over 8.2 million row actions. This finite

far-index engine check supports the explicit history formula without replacing the unbounded induction.

Nontrivial Fibonacci clock models. A sparse complex checker also evaluates the 187 exported application rows of `uniform-no` at $d_{\mathrm{clk}} = F_4, F_5, F_6 = 3, 5, 8$, with $a = 2, 3, 5$ respectively and $z = \exp(2\pi i a/d_{\mathrm{clk}})$. It uses $J|j\rangle = |j{+}1 \bmod d_{\mathrm{clk}}\rangle$, $K|j\rangle = z^{-j}|j\rangle$, $W(x) = z^x$, and the actual complex reader in (61), conjugating diagonal values for adjoints. A separate evolution of the local rule checks the explicit head history. The first positive marker is reached at time $2d_{\mathrm{clk}}$; only the reject row is nonzero, on $x = d_{\mathrm{clk}}, t \geq 2d_{\mathrm{clk}} + 1$. Its exact infinite root probability is $2^{-(3d_{\mathrm{clk}}+2)}$. All application rows are evaluated on both output corners, all Weyl coordinates and all $0 \leq x \leq 2d_{\mathrm{clk}} + 1$, $0 \leq t \leq 2d_{\mathrm{clk}} + 3$. The rebuilt finite shifts have only the two source-isometry boundary defects and this expected reject defect. These complex checks use tolerance $2 \cdot 10^{-12}$, not exact rational arithmetic. They allocate sparse basis actions, stay below 1.5 GB, and do not materialize a Boolean game. Value one as $d_{\mathrm{clk}} \to \infty$ is still a proof-only conclusion, supported by these three finite models and the uniform analytic reader bound. The script is `check_fibonacci_models.py`.

A negative uniform instance has no perfect completeness model. The certification checks that only the expected rejecting row is nonzero. In the generic symbolic pass, `reject_*` rows are skipped: for positive programs the formal reject field is assigned 0, so that pass does not independently prove absence of rejection. The concrete far-index and finite truncation runs do evaluate the reject rows, including the expected nonzero negative controls. With period two, the rejecting projection is $x = 2, t \geq 5$ for $e = 0$, and $x = 2, t \geq 6$ for $e = 2$. Their exact root probabilities are $1/256$ and $1/512$. The wrong-guard comb control has root reject probability $1/256$: the bad comb-fiber mass is $1/4$, checker spatial mass $1/4$, and time tail $1/16$. These exact values are the expected outcomes of the negative controls.

A binary-tree shift is locally finite in the word basis, but need not be banded under an arbitrary enumeration of vertices by one integer. Polar completions are likewise specified by density blocks rather than an assumed universal bandwidth. Explicit basis actions and exact block formulas specify the checks. Finite shifts are tested with their boundary defects.

D.4 Finite defects, the worst rows, and the loss bound

For a normalized geometric register on $0, \ldots, N$, put $t_N = (2^{N+1} - 1)^{-1}$. Covariance stays exact, while $S_N^* S_N - I = -|N\rangle\langle N|$ has weighted squared defect t_N. Its source lift has defect $t_N/81$ and its qutrit row $2t_N/243$. The isometry-row plans in these exports have $L = 8, d = 64$. Consequently the total one-oriented boundary loss before Q^{-2} is

$$\frac{t_{N_x} + t_{N_t}}{124416} \quad \text{(uniform)}, \qquad \frac{t_{N_x} + t_{N_t} + 2^{-N_m}/Z_{N_m}}{124416} \quad \text{(comb)}. \tag{136}$$

Here every complete bush with $m \leq N_m$ is retained, and

$$Z_N = 2(1 - 2^{-N}) \quad (k_m = 1), \qquad Z_N = 3 - (N + 3)2^{-N} \quad (k_m = m).$$

Only the two checker/register source-isometry rows and the top spine source-isometry row can contribute boundary loss. All bush rows and all scaffold rows remain exact. The worst boundary row is explicitly reported for each cutoff: in a balanced large linear comb it is a checker isometry, whereas for the constant comb it is the spine row. All these defects are $O(2^{-N})$ for balanced cutoffs.

The forbidden-reject row has $L = 2, d = 16$, so a root rejecting probability p adds $p/31104$ before Q^{-2}. Thus a fixed-period negative instance has a nonzero limiting reject defect. Its *boundary* defects still tend to zero. The value-one construction separately sends the chosen

Fibonacci period to infinity, making the reject loss at most $C_{\mathrm{nest}} 2^{-d_{\mathrm{clk}}}$ as already proved. A fixed negative model retains its nonzero rejecting defect in the limit.

The rectangular cutoffs $(0,0), (1,1), (2,3), (3,2), (7,3), (15,3), (31,7)$ and balanced cutoffs through 255 are evaluated by exact normalized boundary formulas. Small cutoffs are additionally evaluated exhaustively for every application polynomial, with complete finite bushes. For a general $k_m \le m^2$, keeping complete bushes still gives spine defect $2^{-N}/Z_N$, and the omitted comb mass is at most $2^{-N}(N^2 + 4N + 7)$. Larger cutoffs are certified by these exact formulas; exponentially large matrices are never allocated.

D.5 Random algebra checks and the general regression

The random-algebra script runs 200 instance-independent random models of the displayed nested transport and quadratic identities. Its grouping into five batches of forty does not use any instance rows or verifier program. Each model uses four full 32-dimensional corner spaces and unitary transports between them; (120) holds by construction. Five nilpotent modular contractions, complex corner gauges, and density-eigenspace gauges exercise the algebra. There are 145 checks per model, 29000 in total, with maximum Frobenius residual below $7.1 \cdot 10^{-15}$ against tolerance $5 \cdot 10^{-12}$. Tiny density weights occur, but full-space corners and unitary transports make any claimed test of dense-range support cancellation vacuous. This is an algebra check, not a game-to-operators soundness test. No Boolean question/predicate table is materialized for any instance; the exports contain the word lists, rational rows and row plans for the constructive game.

A further synthetic input exercises all five modular degrees and all interior-coefficient translation routes, with three derived coefficients. Its general compiler export has 43197 words and 518601 rational rows. The independent audit checks the row links and projections, and exact rational 4-dimensional modular root matrices verify that each compiled residual is exactly the claimed transported homogeneous residual, including the adjointed negative degrees. No claim is made that this arbitrary regression presentation is satisfiable. This check tests the translator rather than silently assuming the input equations vanish.

The merged runner copies the generator and checking scripts to a new result directory, regenerates the large literal lists there, checks their byte hashes and counts against the shipped records, and runs each audit from scratch. The large core JSON files are not shipped. The five reference byte hashes are retained in `nested/expected-core-records.json`; no Boolean predicate table is materialized for any validation instance. Generated lists are finite certificates of their stated algebra, not a game-to-operators soundness verification.

E Results used from other papers

Tables 7 and 8 list every load-bearing result imported from other papers, with the place where it is used. Results restated with proof here are marked as such.

Table 7: Imported results.

Result	Source	Used in
Perfect completeness of $\mathsf{MIP}^* = \mathsf{RE}$	[32, Theorem 12.6]	Sections 3, 14, 16.7
Commuting perfection is Π^0_1-complete	[53]; [42, Theorem 6.4]	Sections 12, 17
Finite-dimensional value one is Π^0_2-complete	[42, Theorem 6.9]	Table 1
Convergent moment hierarchy	[43]	Sections 3, 17
Flagged magic-square geometry	[37]; square [40, 49]	Section 4.1
Normal bridge, label refinement, rational rows, row loss	[23, Lemmas 4.1, 4.2, Theorem B, Eq. (32)]; restated with proof	Section 4
Catalan game, local relations and two-bridge assembly	[23, Theorem C, Sections 3 and 6, Lemma 3.1]	Sections 14, 19
Standard-form reduction of commuting strategies	[20, Lemma 1.6]; restated	Section 2.4
Commuting calibrated bridge	[20, Lemma 2.2]; proof recalled	Section 4.1.3
Expected-inclusion transfer	[54]; [20, Lemma 3.1]; proof given	Section 12
Tracial mechanism for linear-system games	[20, Proposition 7.4]	Section 14
Cuntz gauge KMS state and its factor	[45]; [29, Theorem 4.9, Example 4.13]; [20, Section 3.1]	Section 11.5
Bounded centralizer perturbation	[48, Lemma 4.5]	Section 4.1.3
Effective Higman embedding	[30]; [9, Theorem 2.2]; [16]	Section 11.3
HNN normal form	[31]	Section 11.3

Table 8: Imported results. (continued)

Result	Source	Used in
Asymptotic ratio set and S-invariant	[13, Section 3.6, Theorem 3.6.1]	Section 12
Absorption argument	[27, Lemma 3.8]	Section 12
Flow of weights classification	[15]	Section 12
Injective factor classification	[14]	Section 14.2
Linear-system games: nonclosure, solution groups, spatial implies finite	[52]; [10, Theorems 4–5]; [11]	Section 14
Assemblage characterization of zero-error codes	[18]	Section 15
Clique saturation, square-root map, Corollaries 7 and 8	[37, Lemma 5, Theorem 6, Corollaries 7–8]	Sections 15, 16.7
Douglas factorization	[19]	Section 15
Synchronous tracial representation	[47, Theorem 5.5]; [33, 42]	Section 17
Limit lemma	Shoenfield (1959) [51]	Section 3.4
Right-c.e.-in-$0'$ reals as computable limsups	[5, Lemma 4]	Section 10
Theta vector method	[6]	Section 18
$\alpha_q \le \vartheta^-$	[17]	Section 18.6
Magic-rectangle game	[2]	Section 18.6
Theta-capacity conjecture and incompatible conjectures	[34, Conjectures 19, 26, Corollary 27]	Section 18
Anchored repetition	[4, Theorem 1.3]	Section 18
Rényi spectral points dominating theta	[41]	Section 18

The dependence is one way. This paper uses the entropy-game paper [23] and the type-game paper [20]; neither uses any result proved here.

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