Worst-case transport and light cones for quasi-periodic operators*

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Abstract

We study the fastest transport exponent α_u^+ of supercritical analytic quasi-periodic Schrödinger operators at the worst phase. For finite Liouville exponent β, an upper law involving the Lyapunov exponent ℓ and acceleration ω keeps transport below one uniformly in phase. When $\beta \geq \ell$ its local rate is $1 - \ell/(2\omega\beta)$, and the factor 2ω is attained. For the almost Mathieu operator with $L = \log \lambda > 0$, we prove the sharper uniform upper bound $1 - L/(2\beta)$ also on $L/2 < \beta < L$. Thus whenever the even-denominator exponent satisfies $\beta_e = \beta > L/2$, with β finite, the uniform exponent, the supremum over phases and the exponent at phase $1/4$ all equal $1 - L/(2\beta)$. The same value holds on a dense G_δ set, while almost every phase has $\max\{0, 1 - L/\beta\}$. The new upper proof uses translated one-band and two-band frames, an exact inverse-Morse factorization, matrix shells and one fixed energy unit through the full depth range. For odd-harmonic analytic potentials, a half-period sign flip gives a lower bound through a closed central gap. In the Aubry–André XY chain, for even sources and the specified receivers, the fixed-site light cone follows the typical law at almost every phase; the light cone uniform over sites follows the exact worst-case law when $\beta_e = \beta > L/2$ is finite. For an explicit computable chain these exponents differ and both have Turing degree $\mathbf{0}''$. We also prove computability of the Lyapunov exponent for computable analytic potentials and a transport exponent of degree $\mathbf{0}''$ throughout an explicit computable even-potential neighborhood. No computation enters the new upper proof.

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1 Introduction

Consider a one-frequency quasi-periodic Schrödinger operator on $\ell^2(\mathbb{Z})$,

$$(H_{v,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + v(\theta + n\alpha)\, u(n), \tag{1}$$

with a real-analytic sampling function v on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, an irrational frequency α and a phase $\theta \in \mathbb{T}$. The almost Mathieu operator, $v(x) = 2\lambda \cos 2\pi x$, is the Aubry–André model of physics [1]. Let p_n/q_n be the continued-fraction convergents of α and

$$\beta(\alpha) = \limsup_{n\to\infty} \frac{\log q_{n+1}}{q_n} \in [0, \infty]. \tag{2}$$

When the Lyapunov exponent L is positive on the spectrum, transport is a competition between the localization strength L and the Liouville strength $\beta(\alpha)$. **KNOWN.** The companion papers [20, 21] determine the fastest transport exponent α_u^+ at almost every phase: it equals $\max\{0, 1 - \ell_*/\beta(\alpha)\}$, where ℓ_* is the minimum of L on the spectrum, for the almost Mathieu operator and for every supercritical analytic potential of type I.

Labels **NEW**, **KNOWN** and **KNOWN IN A SPECIAL CASE** distinguish this paper's contributions, including those of version 1, from the credited inputs. The extension to $L/2 < \beta < L$ is new in version 2. This paper asks what happens at the worst phase. The question has a physical reading. Moving the launch site of a packet by x sites moves the phase by $x\alpha$, so the worst phase governs transport uniformly over launch sites. Through the Jordan–Wigner transformation it bounds the uniform light cone of the quasi-periodic XY spin chain, and for the Aubry–André chain, at almost every phase, the typical law governs the light cone seen from a fixed site. Three points summarize the answers.

- **NEW.** At the worst phase the localization strength is divided by at most twice the acceleration: when $\max_\Sigma \ell \le \beta(\alpha) < \infty$, transport is never faster than $1 - \ell/(2\omega\beta)$, maximized over the spectrum, for every analytic potential with $L > 0$ on the spectrum and uniformly in the phase. The factor 2ω is attained.

- **NEW.** The worst value is generic in the sense of Baire: for the almost Mathieu operator, when $\beta_e(\alpha) = \beta(\alpha) \in (\frac{1}{2}\log\lambda, \infty)$, the exponent is $1 - \log\lambda/(2\beta)$ on a residual set of phases and $\max\{0, 1 - \log\lambda/\beta\}$ at almost every phase. In general the typical value is $\max\{0, 1 - \log\lambda/\beta\}$.
- **NEW.** In the Aubry–André XY chain, for the observables specified below, at almost every phase the fixed-site light cone follows the typical law, and when $\beta_e = \beta$ is finite and greater than $\frac{1}{2}\log\lambda$ the uniform light cone follows the worst-case law; for every supercritical analytic field the uniform light cone obeys the all-phase upper law. For an explicit computable chain both exponents are different reals of Turing degree $\mathbf{0''}$. The Lyapunov exponent itself is computable, so the uncomputability sits entirely in the frequency.

1.1 Transport exponents

Let $\psi_t = e^{-itH}\delta_0$, $a(j,T) = \frac{2}{T}\int_0^\infty e^{-2t/T}|\langle\delta_j, \psi_t\rangle|^2\,dt$, $P(N,T) = \sum_{|j|>N} a(j,T)$ and $M_p(T) = \sum_j |j|^p a(j,T)$. The upper transport exponents are $\beta^+(p) = \limsup_{T\to\infty} \log M_p(T)/(p\log T)$, and $\alpha_u^+ = \sup_{p>0}\beta^+(p)$ is the fastest one, as in [17, 20, 21]. A tail $P(T^a, T)$ decaying faster than every power forces $\alpha_u^+ \le a$, and a tail of size T^{-C} at distance T^a forces $\alpha_u^+ \ge a$. We write $\alpha_u^+(\theta)$ when the phase matters, and

$$\alpha_u^{\mathrm{unif}} = \inf\Big\{a \ge 0 :\ \sup_\theta P_\theta(T^a, T) = O(T^{-m}) \text{ for every } m > 0\Big\}$$

for the uniform exponent. It bounds $\sup_\theta \alpha_u^+(\theta)$, and it is also the exponent of all packets of one operator, uniformly in the launch site (Section 2). Let

$$G_\alpha = \{\theta \in \mathbb{T} :\ |\sin(2\pi q_n\theta)| \ge q_n^{-2} \text{ for all sufficiently large } n\}, \tag{3}$$

a set of full measure [20, Lemma 2.1]. Let $\delta_n = q_n\alpha - p_n$, $b_n = q_n^{-1}\log|\delta_n|^{-1}$, and let $\beta_e(\alpha)$ and $\beta_o(\alpha)$ be the limsups of b_n along even and along odd q_n, with value 0 along a finite set; then $\beta(\alpha) = \max\{\beta_e, \beta_o\}$.

1.2 The all-phase law

Write $\ell(E)$ for the Lyapunov exponent at energy E and Σ for the spectrum. The acceleration $\omega(E)$, from Avila's global theory [2], is the right derivative at $y = 0$ of $L(E,y)/(2\pi)$, where $L(E,y)$ is the Lyapunov exponent with phase complexified by iy; it is a positive integer at every spectral energy where $\ell > 0$. The operator is supercritical if $\ell > 0$ on Σ, and of type I if moreover $\omega = 1$ on Σ [22]. For $\ell > 0$, $w \ge 1/2$ and $b \ge 0$ let

$$f(b;\ell,w) = 1 - \frac{\ell}{2wb} \quad (b \ge \ell), \qquad f(b;\ell,w) = \frac{(2w-1)b}{(2w-1)b + \ell} \quad (b < \ell).$$

Theorem A (All-phase upper law)**.** ***NEW.*** *Let v be real analytic with $L > 0$ on the spectrum, and let $0 < \beta(\alpha) < \infty$. Put $F_{\mathrm{U}}(\beta) = \max_{E\in\Sigma} f(\beta;\ell(E),\omega(E))$. For every $a > F_{\mathrm{U}}(\beta)$ and every $m > 0$,*

$$\sup_{\theta\in\mathbb{T}} P_\theta(T^a, T) = O(T^{-m}) \qquad (T \to \infty).$$

Hence, at every phase, $\alpha_u^+(\theta) \le \alpha_u^{\mathrm{unif}} \le F_{\mathrm{U}}(\beta) < 1$. If $\beta \ge \max_\Sigma \ell$, then

$$F_{\mathrm{U}}(\beta) = 1 - \frac{1}{2\beta}\min_{E\in\Sigma}\frac{\ell(E)}{\omega(E)} \le 1 - \frac{\ell_*}{2\omega^+\beta}, \qquad \omega^+ = \max_\Sigma \omega.$$

For type I potentials the bound is $1-\ell_*/(2\beta)$ when $\beta \geq \ell_*$, and $\beta/(\beta+\ell_*)$ when $\beta < \ell_*$ (Corollary 5.6).

Remark 1.1. KNOWN endpoints; NEW finite-β consequence. At $\beta(\alpha) = 0$ the same argument gives 0; this endpoint is known [17], [33, Remark 1.12], [21, Corollary 5.10]. At $\beta(\alpha) = \infty$, Jitomirskaya and Zhang proved $\alpha_u^+ = 1$ at every phase [34, Theorem 3(2)]. With Theorem A: $\alpha_u^+ = 1$ at some, equivalently at every, phase if and only if $\beta(\alpha) = \infty$. The lower bound $\alpha_u^+(\theta) \geq \max\{0, 1-\ell_*/\beta\}$ holds at every phase [21, Theorem 4.1], so Theorem A brackets the exponent at every phase.

1.3 Sharpness

Theorem B (Sharpness). *1.* ***NEW.*** *Let $\lambda > 1$ and $L = \log\lambda$. For the almost Mathieu operator, $\alpha_u^+(H_{\lambda,\alpha,1/4}) \geq \max\{0, 1-L/(2\beta_e(\alpha))\}$ for every irrational α. For every finite $\beta(\alpha) > L/2$, $\alpha_u^{\mathrm{unif}} \leq 1-L/(2\beta(\alpha))$. If $\beta_e(\alpha) = \beta(\alpha) \in (L/2, \infty)$, then*

$$\alpha_u^{\mathrm{unif}} = \sup_\theta \alpha_u^+(\theta) = \alpha_u^+(1/4) = 1 - \frac{L}{2\beta(\alpha)}.$$

If $\beta_e(\alpha) \geq L$, then $\alpha_u^+(1/4) = \max\{0, 1-L/\beta_o(\alpha), 1-L/(2\beta_e(\alpha))\}$.

2. ***NEW.*** *For $v(x) = 2\lambda\cos 2\pi sx$ with $\lambda > 1$ and an integer $s \geq 2$, the acceleration is s on Σ, and for every $\beta_* > \log\lambda$ there is an α with $\beta(\alpha) = \beta_*$ and $\alpha_u^{\mathrm{unif}} = \sup_\theta \alpha_u^+(\theta) = 1-\ell_*/(2\omega^+\beta(\alpha))$. So the factor $2\omega^+$ in Theorem A cannot be improved in the class of analytic potentials.*

3. ***NEW; KNOWN IN A SPECIAL CASE for the almost Mathieu gap closure.*** *Let v be real analytic with $L > 0$ on Σ and $v(x+1/2) = -v(x)$. Then $0 \in \Sigma$, every even-denominator periodic approximant has a closed gap at $E = 0$ at some phase, and $\alpha_u^{\mathrm{unif}} \geq \max\{0, 1-\ell(0)/(2\beta_e(\alpha))\}$.*

Remark 1.2. KNOWN typical law and gap closure; NEW worst-phase equality. At almost every phase the exponent of the almost Mathieu operator is $\max\{0, 1-L/\beta\}$ [20, Theorem A]. The exact worst-case law has localization numerator $L/2$, with the typical value truncated at zero. When $\beta_e = \beta \in (L/2, \infty)$ the phase $1/4$ is strictly faster than almost every phase, including in the window $e^\beta < \lambda < e^{2\beta}$ where Avila, You and Zhou proved Anderson localization at almost every phase [5]. For the almost Mathieu operator the closed central gap of even-denominator approximants is classical [7], and the value of the Chambers polynomial at 0 that expresses it is due to Lamoureux and Mingo [36].

For odd-harmonic type I potentials, Theorem B(3) and Theorem A give $1-\ell(0)/(2\beta) \leq \alpha_u^{\mathrm{unif}} \leq 1-\ell_*/(2\beta)$ when $\beta_e = \beta$ is finite and at least ℓ_*. When moreover $\beta \geq \ell(0)$, the uniform exponent of packets filtered ever more narrowly near the closed gap at $E = 0$ converges to $1-\ell(0)/(2\beta)$ (Corollary 10.16); this is a statement about the uniform exponent in the limit of narrow filters, not about each phase or each fixed filter.

1.4 Measure versus category

Theorem C (Category). ***NEW.*** *Let $\lambda > 1$ and $\beta_e(\alpha) = \beta(\alpha) \in (\frac{1}{2}\log\lambda, \infty)$. Then $\alpha_u^+(H_{\lambda,\alpha,\theta}) = 1-\log\lambda/(2\beta(\alpha))$ for every θ in a dense G_δ set.*

KNOWN typical law; NEW residual equality. Since $\alpha_u^+ = \max\{0, 1 - \log\lambda/\beta\}$ at almost every phase [20, Theorem A], the exponent takes one value on a set of full measure and a strictly larger value on a residual set. The lower bound on a residual set holds whenever $\beta_e > \log\lambda/2$, and a residual lower bound $1 - \ell(0)/(2\beta_e)$ holds for every supercritical odd-harmonic analytic potential (Corollary 11.3). The Baire argument behind it is standard; the content lies in lower-bound mechanisms that depend on the phase only through an open set of local phases.

1.5 Light cones

Consider the XY chain $\sum_n (X_n X_{n+1} + Y_n Y_{n+1}) + \sum_n h_n Z_n$ with $h_n = v(\theta + n\alpha)$. For an even observable A near a site x and receivers B beyond distance T^a from x, either even observables on the two sides, such as parity strings, or arbitrary observables on a half-line, let $\Gamma_x(\theta)$ be the least a for which $\|[\tau_t(A), B]\|$ decays faster than every power of T uniformly for $|t| \le T$, and $\Gamma_{\mathrm{unif}}(\theta)$ the same uniformly over x (Definition 12.6).

Theorem D (Light cones). *1. **NEW consequence of the KNOWN typical transport law.** For the Aubry–André chain, $h_n = 2\lambda\cos 2\pi(\theta + n\alpha)$ with $\lambda > 1$, and for almost every θ, the light-cone exponent at every site is $\Gamma_x(\theta) = \max\{0, 1 - \log\lambda/\beta(\alpha)\}$.*

*2. **NEW.** For every real-analytic v with $L > 0$ on Σ and $\beta(\alpha) < \infty$, $\Gamma_{\mathrm{unif}}(\theta) \le F_{\mathrm{U}}(\beta(\alpha))$ at every phase. For the Aubry–André chain with $\beta_e(\alpha) = \beta(\alpha) \in (\frac{1}{2}\log\lambda, \infty)$, $\Gamma_{\mathrm{unif}}(\theta) = 1 - \log\lambda/(2\beta(\alpha))$ at every phase.*

*3. **NEW; KNOWN transport-degree input from Paper A.** There are a computable α and a computable phase θ_* such that the chain with $h_n = 4\cos 2\pi(\theta_* + n\alpha)$ has $\Gamma_{\mathrm{unif}} = 1 - 1/(2s_*)$ and $\Gamma_x = 1 - 1/s_*$ at every site, where $s_* \in [23/12, 2]$ has Turing degree $\mathbf{0}''$; both exponents have degree $\mathbf{0}''$. In the same chain at the phase $1/4$, every site has exponent $1 - 1/(2s_*)$.*

For the pair Z_0, P_R, an order-one commutator is equivalent to a one-body mass in R bounded away from both 0 and 1: $\|[\tau_t(Z_0), P_R]\| = 4\sqrt{p(1-p)}$, where p is the one-body mass in R at time $2t$ (Proposition 12.1). The bounds concern even sources and the stated receivers; they are not conventional exponential Lieb–Robinson bounds for arbitrary observables.

Remark 1.3. KNOWN. Reductions of XY dynamics to one-body dynamics go back to Lieb, Schultz and Mattis [38], and were used by Hamza, Sims and Stolz for disordered chains [24]. Damanik, Lemm, Lukic and Yessen identified anomalous Lieb–Robinson exponents of the Fibonacci XY chain with upper transport exponents [16]. The infinite-temperature OTOC for one receiver site is contained in the formula of Riddell and Sørensen [44].

1.6 Computability

Theorem E (Computability). *1. **KNOWN IN A SPECIAL CASE; no priority claim for the general algorithmic consequence.** For every computable real-analytic v, the Lyapunov exponent is a computable function of the frequency and the energy on $(\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R}$. At every computable irrational frequency, ℓ, ℓ_*, the density of states and the spectrum are computable, with no hypothesis of supercriticality.*

*2. **NEW extension of Paper A's coupling result.** Let α_* be the explicit computable frequency of [20, Theorem D(3)]. For every even analytic v with computable coefficients and $\sup_{|\operatorname{Im} z| \le 1/4} |v(z) - 4\cos 2\pi z| < 1/4$, the exponent $\alpha_u^+(H_{v,\alpha_*,1/4})$ and every moment exponent equal $1 - \ell_*(v)/\beta(\alpha_*) \in [0.3199, 0.6924]$, which has Turing degree exactly $\mathbf{0}''$; the value depends continuously on v.*

Part 1 is an algorithmic consequence of Avila's global theory; we make no priority claim for it, and computing spectra is not new (Remark 13.6). Paper A gave an exponent of degree $\mathbf{0}''$ that is robust along a coupling interval [20, Theorem D]; part 2 extends this to the computable members of an explicit neighbourhood of $4\cos 2\pi x$ in the even analytic potentials, open relative to that class. Computable coefficients are needed: a noncomputable coupling in the same set gives the computable value $49/100$ (Remark 13.10). Section 13 also shows that, for these transport laws at the prescribed good phases, the frequency is the only fragile datum, and that in the supercritical analytic class uncomputability is carried by lim sup-type quantities.

Version 2 contribution. **NEW.** Theorem 6.1 proves the uniform upper law on $L/2 < \beta < L$ without a parity hypothesis. Together with the even-scale lower bound it makes Theorems B, C and the uniform-site assertion of Theorem D exact for every finite $\beta_e = \beta > L/2$. **KNOWN.** Translated rational-flux effective Hamiltonians were developed by Wilkinson [46, p. 4337, abstract] and by Helffer–Sjöstrand [27]; the orbital correction is the classical Rammal–Wilkinson term, with an algebraic treatment by Rammal–Bellissard [43]. We use the quantitative global translated-well estimates stated in Paper D [19]. The new steps are the inner-edge curvature dichotomy, the exact inverse-Morse product, the replacement matrix shells, the fixed-unit all-depth estimates and their transport assembly. For $\beta_e < \beta$ the matching lower bound in terms of β remains open. No extension of the sharper upper bound to general type I potentials is asserted.

1.7 Relation to earlier work

KNOWN inputs and comparison. For phases outside the full-measure sets of the companion papers, little was known at positive finite $\beta(\alpha)$. Damanik and Tcheremchantsev proved vanishing upper exponents at every phase when $\beta(\alpha) = 0$, for trigonometric polynomial potentials [17, Theorems 4–5], with methods that also control individual times [18]. Jitomirskaya and Mavi proved arithmetic upper bounds that reach $\beta = 0$ for analytic potentials [33, Theorem 1.7 and Remark 1.12]. Han and Jitomirskaya proved upper bounds under a strongly bounded discrepancy condition [25], and Bradshaw, de Jong, Liu, Wang, Wang and Yang proved phase-uniform subpower bounds for weakly Liouville frequencies, which have $\beta = 0$ [10]. Jitomirskaya, Liu and Tcheremchantsev bounded fractal dimensions of spectral measures near the transition for α-Diophantine phases [32]. Jitomirskaya and Zhang settled $\beta = \infty$ [34]. In the literature we checked we did not find an all-phase upper law at positive finite β; this is a statement about our search, not a priority claim. The structural input is Avila's global theory [2]. The connection between the acceleration and zeros of finite-volume determinants is due to Han and Schlag [26], for Diophantine frequencies (phase restrictions enter only their application to localization), and the acceleration counts the vanishing dual Lyapunov exponents by Ge, Jitomirskaya, You and Zhou [23]; we prove the finite-scale count we need directly. The interpolation at every phase rests on the product lemma of Avila and Jitomirskaya [3, Lemma 9.7], used in the hierarchical analysis of Jitomirskaya and Liu [29]. Reflection resonances of the phase were developed by Jitomirskaya and Liu [30] and Liu [40], and Jitomirskaya, Liu and Mi gave a sharp criterion for semi-uniform dynamical localization, with almost Mathieu examples that are dynamically localized but not semi-uniformly [31]. The distinction between sitewise and uniform behaviour is therefore not new; the exact power laws are.

1.8 How the proofs work

NEW assembly from the credited inputs. *The upper bound* (Sections 3–5) combines two single-scale estimates through the covering argument of [20, 21]. At a resonant scale $q = q_n$,

the rational discriminant $t_q(x,E)$ is, as a function of the local phase x, close to a trigonometric polynomial of bounded degree. A Jensen identity bounds the number of its zeros in a strip of width proportional to q by about $2\omega(E)$ (Lemma 3.5). So the set where the discriminant is below an order-one threshold is covered by at most 2ω arcs, each of length about $e^{-\ell q/(2\omega)}$ (Lemma 3.7); the q-step resolvent identity of [21] then confines the packet within about $e^{(b-\min\{\ell,b\}/(2\omega))q}$ sites at every phase (Proposition 4.5). For the speed, determinants of intervals of length k are, up to an exponentially smaller error, trigonometric polynomials of degree about ωk in the starting phase (Lemma 4.2), and q consecutive orbit points interpolate polynomials of degree $q/2$ at every phase (Lemma 4.1). So every window of $2q$ sites contains a regular interval of length about $q/(2\omega)$, and a flux argument bounds the speed by $e^{-\ell q/(2\omega)}$ (Theorem 4.4). Optimizing the two radii over the time gives f.

NEW upper-bound assembly using KNOWN rational-flux machinery. *The sharper almost Mathieu upper law* (Sections 6–8) retains the nonlinear two-band symbol, keeps its exact diagonal blocks in the shell geometry and uses one energy unit at every depth. The cell weight changes only at bonds far from the frame anchors. The three-block boundary calculation is in Appendix B.

NEW lower-bound mechanism. *Sharpness* (Section 10). A half-period sign flip closes the central gap of every even-denominator approximant of an odd-harmonic potential (Proposition 10.1). Next to a closed gap the band is wide, its discriminant has derivative about $e^{\ell q/2}$, and an elliptic run of the Weyl solution from [20] carries a polynomial tail to distance $e^{(b-\ell/2)q}$.

NEW extension, with the priority qualification above. *Computability* (Section 13). Where the Lyapunov exponent is positive, quantization makes the complexified cocycle uniformly hyperbolic at small positive heights. There a finite certificate bounds the Lyapunov exponent from below, and convexity carries the bound to height zero exactly. Where it is zero, nonnegativity and finite-scale upper bounds suffice.

1.9 Organization

Section 2 fixes notation and lists the imported results. Sections 3–5 prove Theorem A. Sections 6–8 prove the sharper almost Mathieu upper law; Appendices A–C contain the longer calculations. Section 9 gives a second proof for the almost Mathieu operator. Section 10 proves Theorem B, Section 11 Theorem C, Section 12 Theorem D, and Section 13 Theorem E. Section 14 discusses open problems.

2 Setting and imported results

2.1 Operators, Lyapunov exponents and acceleration

Throughout, $v:\mathbb{T}\to\mathbb{R}$ is real analytic. It extends holomorphically to a strip $|\operatorname{Im} x|<\delta_v$, and we fix $0<\delta_1<\delta_v$. For an irrational α and a phase $\theta\in\mathbb{T}$ we write $H_\theta=H_{v,\alpha,\theta}$ for the operator (1), and $H_{\lambda,\alpha,\theta}$ for the almost Mathieu operator, $v(x)=2\lambda\cos 2\pi x$; for it we write $L=\log\lambda$. The spectrum Σ of H_θ does not depend on θ. We put

$$J=[-\|v\|_\infty-3,\ \|v\|_\infty+3],$$

an interval that contains Σ with room to spare.

For $E\in\mathbb{C}$ and $x\in\mathbb{C}$ with $|\operatorname{Im} x|<\delta_v$ let $T(E,x)=\left(\begin{smallmatrix}E-v(x) & -1\\ 1 & 0\end{smallmatrix}\right)$ and $A_r(E,x)=T(E,x+(r-1)\alpha)\cdots T(E,x)$. For a rational frequency p/q we write $A_r^{(p/q)}$ for the same products with α

replaced by p/q. We write $\ell(E)$ for the Lyapunov exponent at $E \in \mathbb{C}$, and, for real E and $|y| < \delta_v$, $L(E, y)$ for the Lyapunov exponent of the cocycle $x \mapsto T(E, x + iy)$ over the rotation by α; so $L(E, 0) = \ell(E)$. The function $L(E, \cdot)$ is even and convex [2, Section 2.1]. The *acceleration* is

$$\omega(E) = \lim_{y \downarrow 0} \frac{L(E, y) - \ell(E)}{2\pi y}.$$

We use three facts from Avila's global theory [2], in the form recorded in [21, Section 5.1].

(A1) *Quantization* [2, Theorem 5]. The acceleration of an analytic $SL(2, \mathbb{C})$ cocycle over an irrational rotation is an integer. Hence $L(E, \cdot)$ is convex and piecewise affine with slopes in $2\pi\mathbb{Z}$.

(A2) *Regularity* [2, Definition 3 and Theorem 6]. A cocycle is regular if its complexified Lyapunov exponent is affine for heights near zero. If its Lyapunov exponent is positive, it is regular if and only if it is uniformly hyperbolic.

(A3) *Uniform hyperbolicity* [2, Section 2.1]. A real energy lies outside Σ if and only if its cocycle is uniformly hyperbolic. Uniform hyperbolicity is an open condition, and the Lyapunov exponent is continuous on the uniformly hyperbolic cocycles.

The operator is *supercritical* if $\ell > 0$ on Σ. Then $\ell_* = \min_\Sigma \ell > 0$, and we put $\ell^+ = \max_\Sigma \ell$. At a supercritical spectral energy the acceleration is an integer $\omega(E) \geq 1$: if it were 0, evenness would make $L(E, \cdot)$ constant near 0, the cocycle would be regular with positive Lyapunov exponent, hence uniformly hyperbolic by (A2), and $E \notin \Sigma$ by (A3) [21, proof of Lemma 5.2]. The operator is *supercritical and type I* if moreover $\omega = 1$ on Σ [22, Definition 2.1], [21, Definition 5.1]. The almost Mathieu operator with $\lambda > 1$ is supercritical and type I, with $\ell = L$ and $\omega = 1$ on Σ [21, Section 5.1].

We also use the continuity of $E \mapsto L(E, y)$ for each fixed $|y| < \delta_v$, which holds for analytic $SL(2, \mathbb{C})$ cocycles at every irrational frequency [28, 4]; for $y = 0$ it is [9, Theorem 1]. Since $L(E, \cdot)$ is convex, pointwise convergence on an open interval is locally uniform, so $(E, y) \mapsto L(E, y)$ is jointly continuous on $\mathbb{R} \times (-\delta_v, \delta_v)$. Finally, by the Thouless formula [14],

$$\ell(E + i\eta) \geq \ell(E) \qquad (E \in \mathbb{R},\ \eta \geq 0), \tag{4}$$

as in [21, proof of Lemma 5.8]. We use (4) only at energies $E + i\eta$ with the same real part.

2.2 Continued fractions

Let p_n/q_n be the convergents of α and put

$$\delta_n = q_n\alpha - p_n, \qquad D_n = |\delta_n|^{-1}, \qquad b_n = \frac{\log D_n}{q_n}, \qquad \beta(\alpha) = \limsup_{n\to\infty} b_n = \limsup_{n\to\infty} \frac{\log q_{n+1}}{q_n}.$$

The two limsups agree because $q_{n+1} < D_n < 2q_{n+1}$ [21, eq. (5)]. At a fixed scale we drop the index and write q, p, δ, D, b. We also use the limsups along even and odd denominators,

$$\beta_e(\alpha) = \limsup_{n:\, q_n \text{ even}} b_n, \qquad \beta_o(\alpha) = \limsup_{n:\, q_n \text{ odd}} b_n,$$

with the convention that a limsup over a finite set is 0. Thus $\beta = \max\{\beta_e, \beta_o\}$.

2.3 Transport exponents

For a vector $\psi \in \ell^2(\mathbb{Z})$ let $\psi_t = e^{-itH}\psi$ and

$$\begin{aligned} a_\psi(j,T) &= \frac{2}{T}\int_0^\infty e^{-2t/T}|\langle \delta_j, \psi_t\rangle|^2\,dt, \\ P_\psi(N,T) = \sum_{|j|>N} a_\psi(j,T), &\qquad M_{p,\psi}(T) = \sum_j |j|^p a_\psi(j,T). \end{aligned} \tag{5}$$

For $\psi \neq 0$ the upper transport exponents are

$$\beta_\psi^+(p) = \limsup_{T\to\infty} \frac{\log M_{p,\psi}(T)}{p\log T}, \qquad \alpha_{u,\psi}^+ = \sup_{p>0}\beta_\psi^+(p). \tag{6}$$

They do not change when ψ is multiplied by a nonzero constant. When $\psi = \delta_0$ we drop the index, and we write $P_\theta(N,T)$ and $\alpha_u^+(\theta) = \alpha_u^+(H_{v,\alpha,\theta})$ when the phase matters. The fastest exponent α_u^+ is that of Damanik and Tcheremchantsev [17]. Always $0 \le \alpha_u^+ \le 1$. We use two facts from [21]: if $0 < a < 1$ and $P_\psi(T^a, T) = O(T^{-m})$ for every m, for a unit vector ψ with finite moments, then $\alpha_{u,\psi}^+ \le a$ [21, Lemma 2.3]; and the Parseval formula [21, Lemma 2.1]

$$P_\psi(N,T) = \frac{1}{\pi T}\int_{\mathbb{R}} \big\|\mathbf{1}_{\{|j|>N\}}(H - E - i/T)^{-1}\psi\big\|^2\,dE, \qquad \frac{1}{\pi T}\int_{\mathbb{R}}\big\|(H-E-i/T)^{-1}\psi\big\|^2\,dE = \|\psi\|^2. \tag{7}$$

Conversely, $M_p(T) \ge N^p P(N,T)$, so a tail $P(N_j,T_j) \ge T_j^{-C}$ along $T_j\to\infty$ with $\log N_j/\log T_j \to a$ gives $\alpha_u^+ \ge a$.

Definition 2.1 (Uniform exponent)**.** For v and α fixed, let

$$\alpha_u^{\mathrm{unif}} = \inf\Big\{a \ge 0 :\ \sup_{\theta\in\mathbb{T}} P_\theta(T^a,T) = O(T^{-m}) \text{ as } T\to\infty, \text{ for every } m>0\Big\}.$$

The set in the definition contains every $a > 1$, because hopping has range one, so $\alpha_u^{\mathrm{unif}} \le 1$. If $\sup_\theta P_\theta(T^a,T) = O(T^{-m})$ for every m, then $\alpha_u^+(\theta) \le a$ for every θ. Hence

$$\sup_\theta \alpha_u^+(\theta) \le \alpha_u^{\mathrm{unif}}. \tag{8}$$

Translation by x sites conjugates H_θ to $H_{\theta+x\alpha}$, and for fixed N,T the map $\theta \mapsto P_\theta(N,T)$ is continuous. Since the orbit $\theta + x\alpha$ is dense, $\sup_{x\in\mathbb{Z}} P_{\theta+x\alpha}(N,T) = \sup_\varphi P_\varphi(N,T)$ for every θ. So α_u^{unif} is also the exponent of the packets launched from all sites of one fixed operator H_θ, uniformly in the launch site, for any θ. We do not know whether equality holds in (8) in general.

For $0 < \beta < \infty$ we write $F_\beta(\ell) = \max\{0, 1 - \ell/\beta\}$, with $F_0 \equiv 0$ and $F_\infty \equiv 1$, as in [20, 21].

2.4 Rational discriminants

Let p/q be a convergent of α and $\Delta_\varphi(E) = \operatorname{tr} A_q^{(p/q)}(E,\varphi)$ the discriminant of the p/q-periodic operator at phase φ. It is invariant under $\varphi \mapsto \varphi + 1/q$, so

$$t_q(x,E) = \Delta_{x/q}(E)$$

is well defined, 1-periodic and analytic in x on $|\operatorname{Im} x| < q\delta_v$, and a monic real polynomial of degree q in E [21, Section 5.2]. For real x its roots are q simple real numbers, the *band centres* [21, Lemma 5.3]. For H_θ the *local phase* at site j is $x_j = q(\theta + j\alpha) \bmod 1 = x_0 + j\delta$. For the almost Mathieu operator, Chambers' formula [20, Lemma 3.1] gives

$$\Delta_\varphi(E) = P_q(E) - 2\lambda^q\cos(2\pi q\varphi) \tag{9}$$

with a real monic polynomial $P_q = P_{p/q}$ of degree q that does not depend on φ.

2.5 Imported results

We use the companion papers [20] (cited as A, version 2.0.1) and [21] (cited as B, version 1.0.0), with the numbering of those versions. From A: Lemma 2.1 (the set G_α of (3) has full measure), Lemma 2.4 (uniform growth on a neighbourhood U_h of Σ), Lemma 2.6 (rational spectra), Lemma 3.1 (Chambers' formula), Proposition 3.3 and equation (10) (the conserved polynomial and its normalization J_n, σ_n), Lemma 3.4, Lemmas 4.1–4.3 (barriers), Proposition 4.4, Proposition 7.1, Theorem 7.3, Lemma 8.2 (elliptic blocks), Lemmas 9.3, 9.4 and 9.6 (insertion, perturbed rotations, elliptic runs), Theorems A, B and D, and Lemma 11.1. From B: Lemmas 2.1, 2.3, 2.7 and 2.8, Theorem 4.1 (the lower bound), Lemmas 5.2–5.5, Corollary 5.6, Lemma 5.7 (the q-step identity), Theorem 5.9 (the per-denominator estimate), Corollary 5.10, the proof of Theorem 5.15 (its far-energy, forbidden-decay and cutting paragraphs, which do not use its hypothesis (W)), Lemma 5.20, Definition 5.21, Lemma 5.22, Theorem 5.26, Proposition 5.27, Lemma 5.28, Theorem 5.29, Remark 5.30, Lemmas 7.3, 7.5 and 7.6 (factorization, truncation and the mean lower bound), and Theorems A, B and D. Each use is stated where it occurs.

One further input is the trigonometric-polynomial product lemma of Avila and Jitomirskaya [3, Lemma 9.7]; see also [29, Lemma B.1]. For every irrational α, every $x \in \mathbb{R}$ and every n, if $0 \le l_0 \le q_n - 1$ minimizes $|\sin \pi(x + l\alpha)|$ over $0 \le l \le q_n - 1$, then

$$\Big| \sum_{0 \le l \le q_n - 1,\ l \ne l_0} \log |\sin \pi(x + l\alpha)| + (q_n - 1) \log 2 \Big| \le C \log q_n \tag{10}$$

with an absolute constant C. No condition on x or on the arithmetic of α is needed. We also use the elementary three-gap facts: for $0 < |d| < q_n$, $\|d\alpha\| \ge \|q_{n-1}\alpha\| \ge 1/(2q_n)$, where $\|\cdot\|$ is the distance to $\mathbb{Z}$; so the points $\theta + s\alpha$, s ranging over q_n consecutive integers, are at mutual distance at least $1/(2q_n)$.

For the sharper almost Mathieu upper law we also use Paper D [19], version 1.0.0: its complex-energy transfer bounds (Appendix A), contour Green estimates (Appendix B), translated local frames and self-adjoint compression (Appendix C), and supported global Bessel and boundary-weight estimates (Appendix D). The extensions to inner edges and to the raw low-b range are proved in Sections 7–8. Its spectral counting theorem is not used as a dynamical assertion.

3 Counting zeros near the real axis

In this section $H_{v,\alpha}$ is supercritical. The key fact is that, for all large convergent scales and uniformly for E in a fixed energy neighbourhood J of the spectrum, a truncation of the rational discriminant $t_q(\cdot, E)$ has at most $2\tilde{\omega}(E)$ zeros in a strip whose width is proportional to q (Lemma 3.5); at each fixed spectral energy this sharpens to $2\omega(E)$, with parameters and a threshold depending on the energy (Remark 3.6). We state it with finite-height quantities, which is what the dynamical argument needs.

3.1 The acceleration at a finite height

For real E and $0 < y < \delta_v$ put

$$\omega_y(E) = \frac{L(E, y) - \ell(E)}{2\pi y}.$$

For $\rho > 0$ write $\Sigma_\rho = \{E \in \mathbb{R} : \operatorname{dist}(E, \Sigma) \le \rho\}$.

Lemma 3.1. *1. For fixed y, $E \mapsto \omega_y(E)$ is continuous on $\mathbb{R}$.*

2. For fixed E, $y \mapsto \omega_y(E)$ is nondecreasing, and $\omega_y(E) \to \omega(E)$ as $y \downarrow 0$.

3. On Σ, ω is upper semicontinuous with values in $\{1, 2, \dots\}$. In particular $\omega^+ = \max_\Sigma \omega$ exists.

Proof. Part 1 is the continuity of $L(\cdot, y)$ and of ℓ. Part 2: the secant slope of a convex function from 0 is nondecreasing, and its limit is the right derivative. Part 3: $\omega = \inf_{y>0} \omega_y$ is an infimum of continuous functions, hence upper semicontinuous; its values on Σ are integers by (A1) and at least 1 by Section 2. An upper semicontinuous function on a compact set attains its maximum. □

Lemma 3.2 (Uniform complexified growth)**.** *Let $0 < y_0 \le \delta_1$ and $a > 0$. There are $\rho > 0$ and C such that*

$$\|A_r(z, x + iy)\| \le Ce^{(L(E,y)+a)r}$$

for every real $E \in J$, every $z \in \mathbb{C}$ with $|z - E| \le \rho$, every real x, every $|y| \le y_0$ and every $r \ge 0$. For all large q the rational products $A_r^{(p/q)}(z, \varphi + iy)$, $0 \le r \le q$, obey the same bound with C replaced by $C' = C \exp(C\|v'\|_{y_0})$, where $\|v'\|_{y_0} = \max_{|\operatorname{Im} x| \le y_0} |v'(x)|$.

Proof. This is the proof of [21, Lemma 5.5(2)], with the type I identity for $L(E, y)$ replaced by the joint continuity of $(E, y) \mapsto L(E, y)$. Fix $(E_0, y_0') \in J \times [-y_0, y_0]$ and pick r_0 with $r_0^{-1} \int_{\mathbb{T}} \log \|A_{r_0}(E_0, x + iy_0')\|\, dx < L(E_0, y_0') + a/4$. The integrand is jointly continuous in (z, y, x), and Birkhoff averages over the uniquely ergodic rotation by $r_0\alpha$ converge uniformly. This gives the bound $Ce^{(L(E_0,y_0')+3a/4)r}$ on a complex neighbourhood of (E_0, y_0'), and joint continuity of L replaces $L(E_0, y_0')$ by $L(E, y) + a/4$ nearby. Compactness of $J \times [-y_0, y_0]$ gives ρ and C. The rational products are handled as in [21, Lemma 5.5(3)]: the potentials of the rational and irrational products differ by at most $\|v'\|_{y_0}|\delta|$ per site, and expanding the product in these errors costs the factor $\exp(C\|v'\|_{y_0} q|\delta|) \le \exp(C\|v'\|_{y_0})$. □

Lemma 3.3 (A limit of maxima)**.** *Let $f : [0, \infty) \times (0, \infty) \times [1/2, \infty) \to \mathbb{R}$ be continuous and nondecreasing in its third argument, and fix $b \ge 0$. Then*

$$\lim_{y \downarrow 0,\ \rho \downarrow 0} \max_{E \in \Sigma_\rho} f\big(b, \ell(E), \max\{1, \omega_y(E)\}\big) = \max_{E \in \Sigma} f\big(b, \ell(E), \omega(E)\big),$$

where ρ ranges over values small enough that $\ell > 0$ on Σ_ρ.

Proof. The right side is a maximum: $E \mapsto f(b, \ell(E), \omega(E))$ is upper semicontinuous on Σ, because ω is integer valued and upper semicontinuous and f is continuous and monotone in its last argument. Call it F. Since $\omega_y \ge \omega \ge 1$ on Σ, the left side is at least F. Suppose it is not at most F in the limit. Then there are $\varepsilon > 0$, $y_k, \rho_k \to 0$ and $E_k \in \Sigma_{\rho_k}$ with $f(b, \ell(E_k), \max\{1, \omega_{y_k}(E_k)\}) \ge F + \varepsilon$. Pass to a subsequence with $E_k \to E_0 \in \Sigma$. For fixed $y > 0$ and large k, $\omega_{y_k} \le \omega_y$ by Lemma 3.1(2), so by monotonicity and continuity

$$F + \varepsilon \le \lim_k f\big(b, \ell(E_k), \max\{1, \omega_y(E_k)\}\big) = f\big(b, \ell(E_0), \max\{1, \omega_y(E_0)\}\big).$$

Letting $y \downarrow 0$ gives $F + \varepsilon \le f(b, \ell(E_0), \omega(E_0)) \le F$, a contradiction. □

3.2 A Jensen identity and the zero count

For a real trigonometric polynomial $\mathcal{T}(x) = \sum_{|k| \le K'} c_k e^{2\pi i k x}$ of exact degree $K' \ge 1$, with zeros $\zeta_1, \dots, \zeta_{2K'}$ in $\mathbb{C}/\mathbb{Z}$ counted with multiplicity, and for real y, put $m_y(\mathcal{T}) = \int_0^1 \log|\mathcal{T}(x + iy)|\, dx$.

Lemma 3.4 (Jensen identity). $m_y(\mathcal{T}) = \log|c_{K'}| + \pi\sum_k |y - \operatorname{Im}\zeta_k|$. *Hence, for* $Y > 0$,

$$m_Y(\mathcal{T}) + m_{-Y}(\mathcal{T}) - 2m_0(\mathcal{T}) = 2\pi \sum_{k:\,|\operatorname{Im}\zeta_k|<Y} \bigl(Y - |\operatorname{Im}\zeta_k|\bigr).$$

Proof. By [21, Lemma 7.3(1)], $|\mathcal{T}(x)| = |c_{K'}|4^{K'}\prod_k |\sin\pi(x-\zeta_k)|$ for complex x, and $\int_0^1 \log|\sin\pi(x+iy-\zeta)|\,dx = \pi|y - \operatorname{Im}\zeta| - \log 2$ [21, proof of Lemma 7.3(2)]. Sum over k. For the second identity note that $|Y-b| + |Y+b| - 2|b|$ equals $2(Y-|b|)$ if $|b| < Y$ and 0 otherwise. □

Let $\Gamma = \max\{\log\|T(E,x)\| : E \in J,\ |\operatorname{Im}x| \le \delta_1\}$ and $K_Z = \lceil(\Gamma+2)/(\pi\delta_1)\rceil$. For $E \in J$ write $t_q(x,E) = \sum_k c_{q,k}(E)e^{2\pi ikx}$ and let

$$T_{q,E}(x) = \sum_{|k|\le K_Z} c_{q,k}(E)e^{2\pi ikx}$$

be its truncation. For real E it is a real trigonometric polynomial.

Lemma 3.5 (Zero count). *Fix* $y_0 \in (0, \delta_1/2]$, $s \in (0,1)$, $a > 0$ *and* $\eta > 0$, *and put*

$$\tilde\omega(E) = \max\Bigl\{1,\ \frac{\omega_{y_0}(E)}{s} + \frac{2a+\eta}{2\pi s y_0}\Bigr\}.$$

For all large q *and every* $E \in J$:

1. $|t_q(x,E) - T_{q,E}(x)| \le e^{-q}$ *for* $|\operatorname{Im}x| \le qy_0$;
2. *the number* $N(E)$ *of zeros* ζ *of* $T_{q,E}$ *in* $\mathbb{C}/\mathbb{Z}$, *counted with multiplicity, with* $|\operatorname{Im}\zeta| \le (1-s)qy_0$, *satisfies* $N(E) \le 2\tilde\omega(E)$.

Proof. 1. For $|\operatorname{Im}x| \le q\delta_1$ every factor of the rational product defining $t_q(x,E)$ has a phase with imaginary part at most δ_1, so $|t_q(x,E)| \le 2e^{\Gamma q}$. Shifting the contour of the Fourier integral gives $|c_{q,k}(E)| \le 2e^{\Gamma q - 2\pi|k|q\delta_1}$ [21, proof of Lemma 7.5]. For $|\operatorname{Im}x| \le qy_0 \le q\delta_1/2$,

$$\sum_{|k|>K_Z} |c_{q,k}(E)|e^{2\pi|k|qy_0} \le \sum_{|k|>K_Z} 2e^{\Gamma q - \pi|k|q\delta_1} \le 8e^{(\Gamma - \pi(K_Z+1)\delta_1)q} \le 8e^{-2q} \le e^{-q}$$

for large q, since $\pi K_Z\delta_1 \ge \Gamma + 2$.

2. If $T_{q,E}$ is constant there is nothing to count. Otherwise apply Lemma 3.4 with $Y = qy_0$. On the line $\operatorname{Im}x = \pm Y$ we have $t_q(x,E) = \operatorname{tr}A_q^{(p/q)}(E, x/q)$ with $\operatorname{Im}(x/q) = \pm y_0$. So Lemma 3.2, part 1 and the evenness of $L(E,\cdot)$ give $m_{\pm Y}(T_{q,E}) \le \log\bigl(2C'e^{(L(E,y_0)+a)q} + e^{-q}\bigr) \le q\bigl(L(E,y_0)+2a\bigr)$ for large q. The mean lower bound [21, Lemma 7.6] gives $m_0(T_{q,E}) \ge q(\ell(E) - \eta)$; its proof applies verbatim to the truncation degree K_Z, since the truncation error on $\mathbb{R}$ is at most e^{-q} by part 1 and [21, Lemma 7.3(5)] loses only $K_Z\log 4$. Each zero counted in $N(E)$ contributes at least sY to the sum in Lemma 3.4. Therefore

$$2\pi s q y_0 N(E) \le m_Y + m_{-Y} - 2m_0 \le 2q\bigl(L(E,y_0) - \ell(E) + 2a + \eta\bigr) = 2q\bigl(2\pi y_0\omega_{y_0}(E) + 2a + \eta\bigr),$$

which is $N(E) \le 2\tilde\omega(E)$. □

All thresholds depend only on v, α and the fixed parameters, not on $E \in J$.

Remark 3.6 (Twice the acceleration)**.** Fix $E \in \Sigma$. Choosing y_0, a, η small and s close to 1, depending on E, makes $2\tilde{\omega}(E) < 2\omega(E)+1$ by Lemma 3.1(2), so $N(E) \le 2\omega(E)$ for all large q, with a threshold depending on E. In words: at a fixed spectral energy, in a strip of width proportional to q, the truncated discriminant has at most twice the acceleration many zeros. We do not claim a count that is uniform over energies with different accelerations; the arguments below keep the finite-height quantity $\tilde{\omega}$, which is uniform. Han and Schlag connected the acceleration with zero counts of finite-volume Dirichlet determinants [26, Theorem 1.2], under Diophantine assumptions, and Ge, Jitomirskaya, You and Zhou identified the acceleration with the number of vanishing dual Lyapunov exponents [23]. Lemma 3.5 is a finite-scale statement for rational discriminants at every irrational frequency; it is proved directly from the Jensen identity and the mean lower bound.

3.3 Sublevel sets

Write $\mathsf{g}(u) = |\sin \pi u| e^{-\pi |\operatorname{Im} u|}$ for $u \in \mathbb{C}/\mathbb{Z}$ and $c_0 = 2e^{-\pi}$, as in [21, Section 7.2].

Lemma 3.7 (Sublevel sets)**.** *In the setting of Lemma 3.5, let q be large, $E \in J$ and $\Lambda' > 0$, and put*

$$\varepsilon_X = 4^{K_Z} (\Lambda' + e^{-q})\, e^{-m_0(T_{q,E})}\, 0.49^{-2K_Z}.$$

If $\varepsilon_X < c_0^{2K_Z}$, then $X = \{x \in \mathbb{T} : |t_q(x,E)| < \Lambda'\}$ is empty when $N(E) = 0$, and otherwise lies in the union of $N(E)$ arcs of length $2r$, centred at the real parts of the zeros counted in $N(E)$, with $r = \varepsilon_X^{1/N(E)}/c_0$. In particular X is covered by at most $2\tilde{\omega}(E)$ arcs of length at most $2\varepsilon_X^{1/(2\tilde{\omega}(E))}/c_0$.

Proof. Write $\mathcal{T} = T_{q,E}$, of exact degree $K' \le K_Z$. For real x, [21, Lemma 7.3(3)] holds with equality: $|\mathcal{T}(x)| = 4^{K'} e^{m_0(\mathcal{T})} \prod_k \mathsf{g}(x - \zeta_k)$. A zero not counted in $N(E)$ has $|\operatorname{Im} \zeta_k| > (1-s)qy_0 \ge 1$ for large q, and then $\mathsf{g}(x-\zeta_k) \ge 0.49$ [21, proof of Lemma 7.3(2)]. If $x \in X$, then $|\mathcal{T}(x)| < \Lambda' + e^{-q}$ by Lemma 3.5(1), so the product of $\mathsf{g}(x-\zeta_k)$ over the counted zeros is less than ε_X. If $N(E) = 0$ the product is $1 > \varepsilon_X$, which is impossible. Otherwise some counted zero has $\mathsf{g}(x-\zeta_k) < \varepsilon_X^{1/N(E)} < c_0$. Since $\mathsf{g}(u) \ge c_0 \min\{\operatorname{dist}(u,\mathbb{Z}), 1\}$ [21, Lemma 7.3(2)], $\operatorname{dist}(x - \operatorname{Re}\zeta_k, \mathbb{Z}) \le \operatorname{dist}(x-\zeta_k, \mathbb{Z}) < \varepsilon_X^{1/N(E)}/c_0$. The last sentence follows from $N(E) \le 2\tilde{\omega}(E)$ and $\varepsilon_X < 1$. □

At an order-one threshold, Lemma 3.7 covers the sublevel set by at most $2\tilde{\omega}(E)$ arcs of length at most about $(e^{-m_0(T_{q,E})})^{1/(2\tilde{\omega}(E))}$, where $m_0(T_{q,E}) \approx q\ell(E)$; it gives upper bounds on the arcs, not their actual lengths. At a fixed spectral energy the limiting worst-cluster scale is $e^{-\ell(E)q/(2\omega(E))}$. This is the price of an exponentially tight cluster of zeros, which need not coincide, and it is the source of the factor 2ω below.

4 Velocity and confinement at every phase

In this section $H_{v,\alpha}$ is supercritical. We prove two single-scale estimates that hold at every phase: a bound $e^{-\ell(E)q/(2\omega(E))}$ on the speed (Theorem 4.4) and a bound on the distance the packet can reach before the drift stops it (Proposition 4.5). Both are stated with finite-height slack; the limiting rates are approached as the slack parameters tend to zero.

4.1 Interpolation at every phase

Lemma 4.1 (Full-period interpolation)**.** *There is an absolute constant C such that the following holds for every irrational α, every n with $q = q_n \ge 8$, every $\theta \in \mathbb{R}$ and every set S of $2K+1$*

consecutive integers with $2K+1 \in \{q-1, q\}$. *Let* $\varphi_s = \theta + s\alpha$ *and*

$$l_s(\varphi) = \prod_{s' \in S \setminus \{s\}} \frac{\sin \pi(\varphi - \varphi_{s'})}{\sin \pi(\varphi_s - \varphi_{s'})} \qquad (s \in S).$$

Then $\sup_{\varphi \in \mathbb{R}} |l_s(\varphi)| \le q^C$, *and every trigonometric polynomial* f *of degree at most* K *satisfies* $f = \sum_{s \in S} f(\varphi_s)\, l_s$.

Proof. Let $M \supset S$ be a set of q consecutive integers, so $|M \setminus S| \le 1$. By (10), applied with $x = \theta + m_0\alpha - \varphi$ where $m_0 = \min M$, the product $\prod_{m \in M,\, m \ne m_*} |\sin \pi(\varphi - \varphi_m)|$ is at most $2^{-(q-1)} q^C$, where m_* minimizes $|\sin \pi(\varphi - \varphi_m)|$ over M. The numerator of $|l_s(\varphi)|$ is the full product over M divided by the factors with m in $E_s = (M \setminus S) \cup \{s\}$, a set of at most two indices. If $m_* \in E_s$, its factor cancels. The other factors of E_s are at least $\sin(\pi/(4q)) \ge 1/(2q)$, since distinct nodes are at distance at least $1/(2q)$ and so at most one node is within $1/(4q)$ of φ. Hence the numerator is at most $2^{-(q-1)} q^C (2q)^2$. The denominator is at least $\prod_{m \in M \setminus \{s\}} |\sin \pi(\varphi_s - \varphi_m)|$, since the possible extra factor is at most one, and this is at least $2^{-(q-1)} q^{-C}$ by (10), whose minimizing index is now $m = s$. Dividing gives the bound.

A product of $2K$ factors $\sin \pi(\varphi - a_i)$ is a combination of $e^{2\pi i j \varphi}$ with $|j| \le K$, so each l_s is a trigonometric polynomial of degree K with $l_s(\varphi_{s'}) = \delta_{ss'}$. The nodes are distinct modulo one, and a nonzero trigonometric polynomial of degree at most K has at most $2K$ zeros on $\mathbb{T}$. This gives the interpolation formula. □

The lemma uses no condition on the phase, and this is the point: at every phase, $2K+1 \approx q$ nodes of one orbit segment control trigonometric polynomials of degree $K \approx q/2$, with only a polynomial loss.

4.2 Regular intervals

For an interval of length j whose first site has phase φ, write $F_j(z;\varphi) = \det(z - H_{[0,j-1]}(\varphi))$, the $(1,1)$ entry of $A_j(z,\varphi)$. Fix $y_0 \in (0, \delta_1]$ and $a \in (0, \min\{\ell_*, 1\}/100)$. For a convergent denominator q put $K = \lfloor (q-1)/2 \rfloor$ and, for real E,

$$\hat{\omega}(E) = \max\Big\{1,\ \omega_{y_0}(E) + \frac{3a}{2\pi y_0}\Big\}, \qquad k(E) = \Big\lfloor \frac{K}{\hat{\omega}(E)} \Big\rfloor.$$

Lemma 4.2 (Regular intervals)**.** *There is* $\rho > 0$ *such that for all large* q, *every phase* θ, *every real* E *with* $\operatorname{dist}(E, \Sigma) \le \rho$, *every* $z = E + i\eta$ *with* $0 \le \eta \le \rho$, *and every set* W *of* $2q$ *consecutive sites, there are* $k' \in \{k-2, k-1, k\}$, $k = k(E)$, *and an interval* $I \subset W$ *of length* k' *with*

$$|\det(z - H_I)| \ge e^{(\ell(E) - 4a)k'}.$$

Proof. Take ρ no larger than the ρ of Lemma 3.2 with these y_0, a, and so small that $\ell_*/2 \le \ell \le \ell^+ + 1$ on Σ_ρ. By (4), $\ell(z) \ge \ell(E)$. Since $\ell(z) = \inf_r r^{-1} \int_{\mathbb{T}} \log \|A_r(z,\varphi)\|\, d\varphi$, some real φ_* has $\|A_k(z,\varphi_*)\| \ge e^{k\ell(E)}$. The entries of $A_k(z,\varphi_*)$ are, up to sign, $F_k(z;\varphi_*)$, $F_{k-1}(z;\varphi_*)$, $F_{k-1}(z;\varphi_* + \alpha)$ and $F_{k-2}(z;\varphi_* + \alpha)$. So for some $k' \in \{k, k-1, k-2\}$ and some real φ', $|F_{k'}(z;\varphi')| \ge e^{k'\ell(E)}/2$.

Fix this k' and let $F = F_{k'}(z;\cdot)$, a 1-periodic function holomorphic on $|\operatorname{Im}\varphi| < \delta_v$. By Lemma 3.2, $|F(\varphi + iy)| \le \|A_{k'}(z, \varphi + iy)\| \le C e^{(L(E,y)+a)k'}$ for $|y| \le y_0$. Shifting the Fourier integral to $\operatorname{Im}\varphi = \mp y_0$ gives $|\hat{F}(m)| \le C e^{(L(E,y_0)+a)k' - 2\pi|m|y_0}$. So the truncation F_K of F to degree K satisfies on $\mathbb{R}$

$$|F - F_K| \le C_{y_0} e^{(L(E,y_0)+a)k' - 2\pi K y_0} = C_{y_0} e^{(\ell(E) + 2\pi y_0 \omega_{y_0}(E) + a)k' - 2\pi K y_0} \le C_{y_0} e^{(\ell(E) - 2a)k'},$$

because $2\pi K y_0 \ge 2\pi y_0 \hat\omega(E) k' \ge (2\pi y_0 \omega_{y_0}(E) + 3a)k'$.

Let $W = [w, w+2q-1]$ and $S = [w, w+2K]$. The intervals $[s, s+k'-1]$, $s \in S$, lie in W, since $2K + k' \le 3K < 2q$, and their first sites have the phases $\varphi_s = \theta + s\alpha$. Suppose $|F(\varphi_s)| \le e^{(\ell(E)-4a)k'}$ for every $s \in S$. Then $|F_K(\varphi_s)| \le 2C_{y_0} e^{(\ell(E)-2a)k'}$, and Lemma 4.1 gives $\sup_{\mathbb{R}} |F_K| \le (2K+1) q^C 2C_{y_0} e^{(\ell(E)-2a)k'}$. Hence $\sup_{\mathbb{R}} |F| \le q^{C+2} e^{(\ell(E)-2a)k'} < e^{k'\ell(E)}/2$ for large q, because $k' \ge k - 2 \ge q/(3\hat\omega_{\max}) - 3$ with $\hat\omega_{\max} = \max_{\Sigma_\rho} \hat\omega < \infty$. This contradicts the choice of φ'. So some $s \in S$ has $|F(\varphi_s)| > e^{(\ell(E)-4a)k'}$, and $I = [s, s+k'-1]$ works. □

Lemma 4.3 (Node inequality)**.** *Let $I = [x_1, x_2]$ be as in Lemma 4.2 and $x = x_1 + \lfloor (k'-1)/2 \rfloor$. If $(H - z)g = 0$ on I, then for $y \in \{x, x+1\}$,*

$$|g(y)|^2 \le 2\gamma_0^2 (|g(x_1 - 1)|^2 + |g(x_2 + 1)|^2), \qquad \gamma_0 = C e^{2\ell^+ + 2} e^{-(\ell(E)/2 - 5a)(k(E)-2)}.$$

Proof. The Poisson formula gives $g(y) = -G_I(x_1, y) g(x_1 - 1) - G_I(y, x_2) g(x_2 + 1)$ with $G_I = (H_I - z)^{-1}$, and Cramer's rule gives $|G_I(y, x_2)| = |\det(z - H_{[x_1, y-1]})| / |\det(z - H_I)|$ and the mirror formula for $G_I(x_1, y)$. The numerators are determinants of length at most $k'/2 + 1$, bounded by $Ce^{(\ell(E)+a)(k'/2+1)}$ by Lemma 3.2 at height zero. With the lower bound of Lemma 4.2, both Green's function entries are at most $Ce^{-(\ell(E)/2 - 5a)k' + \ell(E) + a} \le \gamma_0$, using $k' \ge k - 2$, $\ell(E)/2 - 5a > 0$ and $\ell(E) \le \ell^+ + 1$. Finally $(u+v)^2 \le 2u^2 + 2v^2$. □

The bound uses the common length $k(E) - 2$ rather than the length k' selected in each window, so that γ_0 does not depend on the window.

4.3 Velocity

Theorem 4.4 (Velocity at every phase)**.** *Let y_0, a be as above and ρ as in Lemma 4.2. There are n_0, T_0, C and $\mu > 0$ such that for all $n \ge n_0$, every phase θ, every $T \ge T_0$, every real E with $\operatorname{dist}(E, \Sigma) \le \rho$ and every $N \ge 1$, the resolvent column $\psi_E(j) = \langle \delta_j, (H_\theta - E - i/T)^{-1} \delta_0 \rangle$ satisfies*

$$W_E^+(N) := \sum_{j > N} |\psi_E(j)|^2 \le \exp\Big(-\frac{N/(4q_n) - 2}{1 + \bar\kappa_n(E) T} \Big) W_E^+(0),$$

and the mirror statement for $j < -N$, where

$$\bar\kappa_n(E) = 4\gamma_0^2 \le C \exp\Big(-\frac{(\ell(E) - 10a)\, q_n}{2\hat\omega(E)} \Big).$$

Consequently

$$P_\theta(N, T) \le 2 \sup_{\operatorname{dist}(E,\Sigma) \le \rho} \exp\Big(-\frac{N/(4q_n) - 2}{1 + \bar\kappa_n(E) T} \Big) + C e^{-\mu N}.$$

Proof. Choose $T_0 \ge 1/\rho$, so that $z = E + i/T$ satisfies the hypothesis of Lemma 4.2. Write $q = q_n$.

Right half-line. The vector $g(k) = \psi_E(k)$, $k \ge 0$, is square summable and solves $(H - z)g = 0$ at every $k \ge 1$. Let $W_i = [1 + 2(i-1)q, 2iq]$, $i \ge 1$. Lemma 4.2 gives a regular interval $I_i \subset W_i$; let x_i be its node, as in Lemma 4.3. Put $s_i = \sum_{k > x_i} |g(k)|^2$ and $M_i = s_i - s_{i+1}$. The flux identity

$$\frac{1}{T} \sum_{k > m} |g(k)|^2 = \operatorname{Im}\big(g(m) \overline{g(m+1)} \big) \qquad (m \ge 0),$$

which follows by summing the imaginary part of the recursion multiplied by $\overline{g(k)}$ over $k > m$, holds for every real potential [20, proof of Lemma 9.6]. For $i \geq 2$ the two sites adjacent to I_i lie in (x_{i-1}, x_i) and (x_i, x_{i+1}), so Lemma 4.3 gives $|g(x_i)||g(x_i+1)| \leq 2\gamma_0^2(M_{i-1}+M_i)$, and therefore

$$s_i \leq T|g(x_i)||g(x_i+1)| \leq \frac{\bar\kappa T}{2}\,(s_{i-1}-s_{i+1}), \qquad \bar\kappa = 4\gamma_0^2.$$

Since $s_{i+1} \leq s_i$, this gives $s_{i+1} \leq \frac{\bar\kappa T}{\bar\kappa T+2} s_{i-1} \leq \exp(-\frac{1}{1+\bar\kappa T}) s_{i-1}$. Iterating from $i=2$, $s_{2l+1} \leq \exp(-l/(1+\bar\kappa T))s_1 \leq \exp(-l/(1+\bar\kappa T))W_E^+(0)$. Since $x_{2l+1} \leq 2(2l+1)q$, the choice $l = \lfloor N/(4q)\rfloor - 1$ gives $x_{2l+1} \leq N$ and $l \geq N/(4q) - 2$, which is the claim when $l \geq 0$; when $l < 0$ the claim is trivial.

Left half-line. The reflection $(Ru)(n) = u(-n)$ conjugates $H_{v,\alpha,\theta}$ to $H_{\tilde v,\alpha,-\theta}$ with $\tilde v(x) = v(-x)$. Reversing transfer products and using the evenness of $L(E,\cdot)$, $\tilde v$ has the same complexified Lyapunov exponent $L(E,y)$ for every y, hence the same ℓ, ω_{y_0} and $\hat\omega$. Apply the right half-line argument to it.

The rate. $\bar\kappa = 4\gamma_0^2 \leq Ce^{-(\ell(E)-10a)(k(E)-2)}$, and $k(E) - 2 \geq K/\hat\omega(E) - 3 \geq (q-2)/(2\hat\omega(E)) - 3$.

The outside probability. By (7),

$$P_\theta(N,T) = \frac{1}{\pi T}\int_{\mathbb{R}} \|\mathbf{1}_{\{|j|>N\}}\psi_E\|^2\,dE, \qquad \frac{1}{\pi T}\int_{\mathbb{R}} \|\psi_E\|^2\,dE = 1.$$

For $\operatorname{dist}(E,\Sigma) \leq \rho$ use the two half-line bounds and $W_E^+(0) + W_E^-(0) \leq \|\psi_E\|^2$. For $\operatorname{dist}(E,\Sigma) > \rho$ the Combes–Thomas bound [21, Theorem 5.15(3)] gives $\|\mathbf{1}_{\{|j|>N\}}\psi_E\|^2 \leq 64 g^{-2} g'^{-1} e^{-g'N/4}$ with $g = \operatorname{dist}(E,\Sigma)$ and $g' = \min\{g,1\} \geq \min\{\rho,1\}$, which integrates to at most $Ce^{-\mu N}$. All constants are independent of θ. □

The theorem bounds Abel averages at the time scale T. A bound at each individual time needs a logarithmic allowance in the radius (Lemma 12.4 below); the exponents are not affected.

4.4 Confinement

Proposition 4.5 (Confinement at every phase)**.** *Let $y_0 \in (0, \delta_1/2]$, s, a, η be as in Lemma 3.5, and let $\tilde\omega_{\max} = \max_J \tilde\omega$. Fix $c > 0$ and $0 < d < B$, and let $\zeta > 0$ and $\xi > 0$ satisfy*

$$2a + \eta + \zeta \leq \xi < \frac{\min\{c, \ell_*, 1\}}{1000\,\tilde\omega_{\max}}.$$

For all large n with $b_n \geq c$, every phase θ, every $T \in [e^{dq_n}, e^{Bq_n}]$ and every real E with $\operatorname{dist}(E,\Sigma) \leq e^{-\xi q_n}$,

$$\|\mathbf{1}_{\{|j|>R(E)\}}\psi_E\| \leq 2T^2\exp(-3\xi q_n e^{\xi q_n}), \qquad R(E) = \exp\Big[\Big(b_n - \frac{\min\{\ell(E), b_n\}}{2\tilde\omega(E)} + 7\xi\Big)q_n\Big],$$

where $\psi_E = (H_\theta - E - i/T)^{-1}\delta_0$.

Proof. We follow the confinement steps of [21, proof of Theorem 5.15], which do not use its hypothesis (W), and replace its sublevel geometry by Lemma 3.7. Write $q = q_n$, $b = b_n$, $\delta = \delta_n$, $D = e^{bq}$, $x_j = q\theta + j\delta$, $z = E + i/T$ and $d_j = t_q(x_j, z)$. Let $N_\Omega = \lfloor D/16\rfloor$ and $\Omega = [-N_\Omega, N_\Omega] \cap \mathbb{Z}$; the map $j \mapsto x_j \bmod 1$ is injective on Ω, and its image lies in an arc of length at most $1/8$. Apply the q-step identity [21, Lemma 5.7] with its parameter a equal to $\zeta/2$; it applies because $1/T \leq e^{-dq} \leq 1/q$. Its error coefficient satisfies $\varepsilon_{\mathrm{res}} \leq e^{(\ell(E)+\zeta-b)q}$ for large q [21, (E2) in the proof of Theorem 5.15]. Put $\Lambda = e^{3\xi q}(2 + 2\varepsilon_{\mathrm{res}})$, so that $\vartheta = (2+\varepsilon_{\mathrm{res}})/(\Lambda - \varepsilon_{\mathrm{res}}) \leq e^{-3\xi q}$.

Comparison. For real x, $t_q(x,\cdot)$ is monic with real roots E_k [21, Lemma 5.3], so $|t_q(x,E+i/T)| = \prod_k |E+i/T-E_k| \geq |t_q(x,E)|$. Hence $\mathcal{A} = \{j \in \Omega : |d_j| < \Lambda\}$ is contained in $\mathcal{I} = \{j \in \Omega : |t_q(x_j,E)| < \Lambda\}$.

Sublevel geometry. We have $\Lambda \leq 4e^{3\xi q}\max\{1, e^{(\ell(E)+\zeta-b)q}\}$ and $m_0(T_{q,E}) \geq q(\ell(E)-\eta)$ by [21, Lemma 7.6], as in the proof of Lemma 3.5. So the number ε_X of Lemma 3.7, with $\Lambda' = \Lambda$, satisfies

$$\varepsilon_X \leq C^{Kz} e^{(3\xi+\zeta+\eta)q} e^{-\min\{\ell(E),b\}q},$$

which is less than c_0^{2Kz} for large q, because $\ell(E) \geq 2\ell_*/3$ for large q and so $\min\{\ell(E),b\} \geq \min\{2\ell_*/3, c\} > 4\xi \geq 3\xi+\zeta+\eta$. By Lemma 3.7, $\{x : |t_q(x,E)| < \Lambda\}$ lies in at most $2\tilde{\omega}(E)$ arcs of length $2r$ with

$$r \leq c_0^{-1} C^{Kz} e^{(3\xi+\zeta+\eta)q} e^{-\min\{\ell(E),b\}q/(2\tilde{\omega}(E))};$$

here we used $N(E) \leq 2\tilde{\omega}(E)$ on the factor $e^{-\min\{\ell(E),b\}q} < 1$ and $N(E) \geq 1$ on the remaining factor. Each such arc meets the arc $\{x_j : j \in \Omega\}$ in one arc, which pulls back to an integer interval of at most $2rD+1$ sites. So $\mathcal{I} \subset \mathcal{I}'$, a union of at most $2\tilde{\omega}(E)$ integer intervals with

$$\#\mathcal{I}' \leq 2\tilde{\omega}(E)(2rD+1) \leq e^{(b-\min\{\ell(E),b\}/(2\tilde{\omega}(E))+4\xi)q}$$

for large q, since $3\xi+\zeta+\eta < 4\xi$ and $b - \min\{\ell(E),b\}/(2\tilde{\omega}(E)) \geq 0$.

Forbidden decay. Let $q' = q+1$ and $\mathcal{B} = \mathcal{A} \cup [-q', q'] \cup (\mathbb{Z} \setminus [-N_\Omega + q, N_\Omega - q])$. By the forbidden-decay paragraph of [21, proof of Theorem 5.15], which uses only the q-step identity and $|d_j| \geq \Lambda$ off $\mathcal{A}$, $|\psi_E(j)| \leq \vartheta^k T$ whenever $\operatorname{dist}(j,\mathcal{B}) \geq kq$.

Cutting. The set $\mathcal{J} = \mathcal{I}' \cup [-q', q']$ has at most $2\tilde{\omega}(E)+1$ components and $\#\mathcal{J} \leq \#\mathcal{I}' + 2q' + 1$. Let $M_0 = q\lceil e^{\xi q}\rceil$. The M_0-neighbourhood of $\mathcal{J}$ has at most $\#\mathcal{J} + (2\tilde{\omega}(E)+1)(2M_0+1)$ sites. So the first site $c_+ > q'$ at distance at least M_0 from $\mathcal{J}$ satisfies

$$c_+ \leq q' + 1 + \#\mathcal{J} + (2\tilde{\omega}(E)+1)(2M_0+1) \leq e^{(b-\min\{\ell(E),b\}/(2\tilde{\omega}(E))+6\xi)q}.$$

This is far below $N_\Omega - q - M_0$, because $\min\{\ell(E),b\}/(2\tilde{\omega}(E)) \geq \min\{\ell_*, c\}/(3\tilde{\omega}_{\max}) > 100\xi$ and $N_\Omega \geq e^{bq}/16 - 1$. Hence $\operatorname{dist}(c_+,\mathcal{B}) \geq M_0$ and $|\psi_E(c_+)| \leq e^{-3\xi q e^{\xi q}} T$. The cutting estimate of [21, proof of Theorem 5.15] gives $\|\mathbf{1}_{(c_+,\infty)}\psi_E\| \leq T|\psi_E(c_+)|$. The left side is the same. Since $c_\pm \leq R(E)$, the claim follows. □

Only the total size of the allowed set enters the position of the first cut, not the location of its components.

5 The all-phase upper law

5.1 The rate function

For $\ell > 0$, $w \geq 1/2$ and $b \geq 0$ put

$$\varrho(b;\ell,w) = b - \frac{\min\{\ell,b\}}{2w}, \qquad \nu(\ell,w) = \frac{\ell}{2w}, \qquad f(b;\ell,w) = \frac{\varrho(b;\ell,w)}{\varrho(b;\ell,w)+\nu(\ell,w)}.$$

Here ϱ is the exponent of the confinement radius of Proposition 4.5 and ν the exponent of the velocity in Theorem 4.4, both per unit of q. Explicitly,

$$f(b;\ell,w) = \begin{cases} 1 - \dfrac{\ell}{2wb}, & b \geq \ell, \\ \dfrac{(2w-1)b}{(2w-1)b+\ell}, & 0 \leq b < \ell. \end{cases} \tag{11}$$

Lemma 5.1. *The function f is continuous, nondecreasing in b and in w, nonincreasing in ℓ, and $0 \le f < 1$, with $f(0;\ell,w) = 0$. The function ϱ is nondecreasing in w and nonincreasing in ℓ. For $\varrho_0 \ge 0$ and $\nu_0 > 0$,*

$$\sup_{\tau>0} \frac{\min\{\varrho_0, (\tau-\nu_0)_+\}}{\tau} = \frac{\varrho_0}{\varrho_0+\nu_0}, \tag{12}$$

attained at $\tau = \varrho_0 + \nu_0$.

Proof. Since $w \ge 1/2$, $\varrho \ge 0$, and $\varrho + \nu = b$ when $b \ge \ell$. The formulas (11) agree at $b = \ell$, and the monotonicity is read off from them. For (12): the function vanishes for $\tau \le \nu_0$, equals $(\tau-\nu_0)/\tau$, which increases, for $\nu_0 \le \tau \le \varrho_0+\nu_0$, and equals ϱ_0/τ, which decreases, afterwards. □

5.2 The theorem

Theorem 5.2 (All-phase upper law)**.** *Let v be real analytic, α irrational with $\beta = \beta(\alpha) < \infty$, and $H_{v,\alpha}$ supercritical. Put*

$$F_{\mathrm{U}}(\beta) = \max_{E\in\Sigma} f\big(\beta; \ell(E), \omega(E)\big).$$

For every $s' > F_{\mathrm{U}}(\beta)$ and $m > 0$,

$$\sup_{\theta\in\mathbb{T}} P_\theta(T^{s'}, T) = O(T^{-m}) \qquad (T\to\infty).$$

Hence $\sup_\theta \alpha_u^+(\theta) \le \alpha_u^{\mathrm{unif}} \le F_{\mathrm{U}}(\beta) < 1$. If $\beta \ge \ell^+$, then

$$F_{\mathrm{U}}(\beta) = 1 - \frac{1}{2\beta}\min_{E\in\Sigma}\frac{\ell(E)}{\omega(E)} \le 1 - \frac{\ell_*}{2\omega^+\beta}.$$

The maximum defining F_{U} exists because $E \mapsto f(\beta;\ell(E),\omega(E))$ is upper semicontinuous on Σ (proof of Lemma 3.3). The last formula holds because $f(\beta;\ell(E),\omega(E)) = 1 - \ell(E)/(2\omega(E)\beta)$ when $\beta \ge \ell(E)$.

Proof. It suffices to treat $s' < 1$, since $P_\theta(N,T)$ is nonincreasing in N. Fix $s' \in (F_{\mathrm{U}}(\beta), 1)$, $m > 0$ and $\varepsilon' > 0$ with $F_{\mathrm{U}}(\beta) < s' - 5\varepsilon'$. The parameters are chosen in the following order; none depends on θ.

Height and energy neighbourhood. By Lemma 3.3, applied with $b = \beta$, there are $y_0 \in (0, \delta_1/2]$ and $\rho_0 > 0$ with $2\ell_*/3 \le \ell \le \ell^+ + 1$ on Σ_{ρ_0} and

$$\max_{E\in\Sigma_{\rho_0}} f\big(\beta; \ell(E), \max\{1, \omega_{y_0}(E)\}\big) < s' - 4\varepsilon'.$$

Let $W = 2\max_J \max\{1, \omega_{y_0}\} + 1$. We shall take $s \in [1/2, 1)$ and $a, \eta \le \pi y_0/3$, so that $\tilde\omega \le W$ and $\hat\omega \le W$ on J.

The integer K and the slack ξ. Fix an integer $K \ge 1$ with $K\ell_*/(4W) \ge 4$, and then $\xi > 0$ with

$$8K\xi < \varepsilon', \qquad \xi < \frac{s'}{10K}, \qquad \xi < \frac{\min\{s'/(2K), \ell_*, 1\}}{1000W}.$$

Slack in the zero count. Choose $s \in [1/2, 1)$, $a \in (0, \min\{\ell_*, 1, \pi y_0\}/100)$, $\eta \in (0, \pi y_0/3]$, $\zeta > 0$ and $B_0 > \beta$ with $2a + \eta + \zeta \le \xi$ and

$$\max_{E\in\Sigma_{\rho_0}} f\big(B_0; \ell(E) - 10a, \Omega(E)\big) < s' - 3\varepsilon', \qquad \Omega(E) = \max\{\tilde\omega(E), \hat\omega(E)\}.$$

This is possible: as $s \to 1$ and $a, \eta \to 0$, Ω converges to $\max\{1, \omega_{y_0}\}$ uniformly on J, and f is uniformly continuous on $[\beta, \beta+1] \times [\ell_*/2, \ell^+ + 2] \times [1, W]$. With these choices $\nu(\ell(E) - 10a, \Omega(E)) \geq \ell_*/(4W)$ on Σ_{ρ_0}. Let $\rho \leq \rho_0$ be as in Lemma 4.2 for y_0 and a, and fix the thresholds of Theorem 4.4 and of Proposition 4.5, the latter with $c = s'/(2K)$, $d = 1/K$ and $B = (B_0+1)/s'$.

Covering. For large T let j be the least index with $q_j \geq K \log T$. As $T \to \infty$, $j \to \infty$.

Case A: $q_j \leq T^{s'}/(\log T)^2$. By Theorem 4.4 at scale j, for $\operatorname{dist}(E, \Sigma) \leq \rho$,

$$\bar{\kappa}_j(E) T \leq CT \exp\Big(-\frac{\ell_* q_j}{4W}\Big) \leq CT^{1 - K\ell_*/(4W)} \leq 1$$

for large T. So, with $N = T^{s'} \geq 8q_j(m \log T + 2)$, which holds for large T,

$$P_\theta(T^{s'}, T) \leq 2 \exp\Big(-\frac{N/(4q_j) - 2}{2}\Big) + Ce^{-\mu N} \leq 2e^{-1} T^{-m} + Ce^{-\mu T^{s'}} = O(T^{-m}).$$

Case B: $q_j > T^{s'}/(\log T)^2$. Put $n = j - 1$, $q = q_n < K \log T$ and $b = b_n$. Since $q_{n+1} < D_n = e^{bq}$, we have $bq > s' \log T - 2 \log\log T$. Since $b \leq B_0$ for large n, $q \geq (s' \log T - 2 \log\log T)/B_0$, so $\log\log T = o(q)$, and $b > s'/(2K) = c$ for large T. With $\tau = \log T/q$ we get $1/K < \tau \leq (B_0 + o(1))/s'$, so $T \in [e^{dq}, e^{Bq}]$. Split the energy axis.

Far energies, $\operatorname{dist}(E, \Sigma) \geq e^{-\xi q}$. The Combes–Thomas bound [21, Theorem 5.15(3)] with $N = T^{s'} \geq e^{s'q/K} \geq e^{2\xi q}$ shows that these energies contribute at most $Ce^{-e^{\xi q}/8}$ to (7), exactly as in [21, proof of Theorem 5.15].

Near energies, $\operatorname{dist}(E, \Sigma) \leq e^{-\xi q}$. For such E consider two radii:

$$N_1(E) = 8q(m \log T + 2)\big(1 + \bar{\kappa}_n(E) T\big), \qquad R(E) \text{ of Proposition 4.5.}$$

Theorem 4.4 gives the relative bound $\|\mathbf{1}_{\{|j| > N_1(E)\}} \psi_E\|^2 \leq T^{-2m} \|\psi_E\|^2$, and Proposition 4.5 the absolute bound $\|\mathbf{1}_{\{|j| > R(E)\}} \psi_E\|^2 \leq 4T^4 e^{-6\xi q e^{\xi q}}$. Let $\mathcal{E}_1$ be the set of near energies with $N_1(E) \leq R(E)$ and $\mathcal{E}_2$ the rest; both are Borel sets. We first show that $\min\{N_1(E), R(E)\} \leq N$ for every near energy, where $N = T^{s'}$ is the radius used for the far energies, and then bound the whole tail at this one radius.

The radius. Fix a near energy E and write $\nu = \nu(\ell(E) - 10a, \Omega(E))$ and $\varrho' = \varrho(b; \ell(E), \Omega(E))$. Since $\nu \leq (\ell(E) - 10a)/(2\hat{\omega}(E))$, Theorem 4.4 gives $\bar{\kappa}_n(E) T \leq Ce^{(\tau - \nu)q}$, so $\log N_1(E) \leq (\tau - \nu)_+ q + O(\log\log T)$. Since ϱ is nondecreasing in w and $\Omega \geq \tilde{\omega}$, $\log R(E) \leq (\varrho' + 7\xi)q$. Hence, by (12) and $\tau > 1/K$,

$$\frac{\log \min\{N_1(E), R(E)\}}{\log T} \leq \frac{\min\{\varrho', (\tau - \nu)_+\} + 7\xi}{\tau} + o(1) \leq \frac{\varrho'}{\varrho' + \nu} + 7K\xi + o(1).$$

Now $\varrho(b; \ell, w) \leq \varrho(b; \ell - 10a, w)$, so $\varrho'/(\varrho' + \nu) \leq f(b; \ell(E) - 10a, \Omega(E)) \leq f(B_0; \ell(E) - 10a, \Omega(E)) < s' - 3\varepsilon'$. The $o(1)$ is uniform in E. Therefore $\min\{N_1(E), R(E)\} \leq T^{s' - 3\varepsilon' + 7K\xi + o(1)} \leq T^{s'} = N$ for every near energy E, once T is large.

The tail at radius N. On $\mathcal{E}_1$ this gives $N_1(E) \leq N$, and on $\mathcal{E}_2$ it gives $R(E) \leq N$. Since the tails are nonincreasing in the radius, the relative bound holds at radius N on $\mathcal{E}_1$ and the absolute bound holds at radius N on $\mathcal{E}_2$; the far energies were estimated at radius N above. So (7) gives

$$\begin{aligned} P_\theta(T^{s'}, T) &\leq \frac{1}{\pi T}\Big[\int_{\mathcal{E}_1} T^{-2m} \|\psi_E\|^2 \, dE + \int_{\mathcal{E}_2} 4T^4 e^{-6\xi q e^{\xi q}} \, dE\Big] + Ce^{-e^{\xi q}/8} \\ &\leq T^{-2m} + CT^3 e^{-6\xi q e^{\xi q}} + Ce^{-e^{\xi q}/8}. \end{aligned}$$

Since $q \geq c' \log T$ with $c' = s'/(2B_0)$, the last two terms are $O(T^{-m})$, so $P_\theta(T^{s'}, T) = O(T^{-m})$.

All constants and thresholds are independent of θ. This proves the tail bound. The bound $\alpha_u^{\mathrm{unif}} \leq F_{\mathrm{U}}(\beta)$ follows from Definition 2.1, and $\sup_\theta \alpha_u^+ \leq \alpha_u^{\mathrm{unif}}$ is (8). Finally $F_{\mathrm{U}}(\beta) < 1$ because $f < 1$ and the maximum is attained. □

5.3 Consequences

Corollary 5.3 (Finite Liouville strength caps the speed)**.** *Let v be real analytic with $H_{v,\alpha}$ supercritical. If $\beta(\alpha) < \infty$, then $\sup_\theta \alpha_u^+(H_{v,\alpha,\theta}) \le F_{\mathrm{U}}(\beta(\alpha)) < 1$.*

Remark 5.4 (The endpoint)**.** The converse is due to Jitomirskaya and Zhang. If $\beta(\alpha) = \infty$, the Lyapunov exponent has no zeros, since it is positive on Σ and, by (A3), off Σ, and their theorem gives quasiballistic transport, $\beta^+(p) = 1$ for every $p > 0$, at every phase [34, Theorem 3(2)]; the numbering is the same in the published version and in arXiv:1510.07086v2. Together with Corollary 5.3: for supercritical analytic potentials, at every phase, $\alpha_u^+ = 1$ if and only if $\beta(\alpha) = \infty$.

Remark 5.5 (An all-phase bracket)**.** For $0 < \beta(\alpha) < \infty$ the lower bound $\alpha_u^+(\theta) \ge F_\beta(\ell_*)$ holds at every phase [21, Theorem 4.1]; it needs only a Lipschitz potential whose Lyapunov exponent is continuous near a spectral minimizer, which holds for analytic v [9]. So at every phase

$$F_\beta(\ell_*) \le \alpha_u^+(\theta) \le F_{\mathrm{U}}(\beta), \qquad \text{and for } \beta \ge \ell^+ : \quad 1 - \frac{\ell_*}{\beta} \le \alpha_u^+(\theta) \le 1 - \frac{1}{2\beta} \min_\Sigma \frac{\ell}{\omega}.$$

For type I potentials $\omega = 1$ on Σ, and since f is nonincreasing in ℓ, $F_{\mathrm{U}}(\beta) = f(\beta; \ell_*, 1)$.

Corollary 5.6 (Type I)**.** *If $H_{v,\alpha}$ is supercritical and type I and $\beta(\alpha) < \infty$, then $\alpha_u^{\mathrm{unif}} \le f(\beta; \ell_*, 1)$. That is, $\alpha_u^{\mathrm{unif}} \le 1 - \ell_*/(2\beta)$ if $\beta \ge \ell_*$, and $\alpha_u^{\mathrm{unif}} \le \beta/(\beta + \ell_*)$ if $\beta < \ell_*$. In particular, for the almost Mathieu operator with $\lambda > 1$ and $L = \log \lambda$,*

$$\alpha_u^{\mathrm{unif}} \le 1 - \frac{L}{2\beta} \quad (\beta \ge L), \qquad \alpha_u^{\mathrm{unif}} \le \frac{\beta}{\beta + L} \quad (0 \le \beta < L).$$

For the almost Mathieu operator, Theorem 6.1 sharpens the second bound to $1 - L/(2\beta)$ when $L/2 < \beta < L$. The general type I conclusion above is unchanged.

For comparison, at almost every phase the exponent is $F_\beta(\ell_*)$ [21, Theorem A]. So, for type I potentials with $\ell_* \le \beta < \infty$, the worst phase costs at most a factor 2 in the localization strength; Section 10 shows that this factor is attained. For general type I potentials with $\beta < \ell_*$ the typical exponent is 0 and this argument gives $\beta/(\beta + \ell_*)$. The sharper almost Mathieu law below L uses the additional frame analysis.

5.4 An example with acceleration two

Proposition 5.7. *Let $v(x) = 2\lambda \cos 2\pi x + 2\mu \cos 4\pi x$ with $\lambda \in \mathbb{R}$ and $|\mu| > 1$, and α irrational. Then $\ell(E) \ge \log|\mu|$ for every real E, so $H_{v,\alpha}$ is supercritical. On Σ, $\omega \in \{1, 2\}$, and $\omega(E) = 2$ if and only if $\ell(E) = \log|\mu|$.*

Proof. Let $w = e^{2\pi i x}$ and $w_k = w e^{2\pi i k\alpha}$. Then $w_k^2 T(E, x + k\alpha)$ is a matrix polynomial in w_k whose value at $w_k = 0$ is $\mathrm{diag}(-\mu, 0)$, because the coefficient of $e^{-4\pi i x}$ in $E - v$ is $-\mu$. So $w^{2n} A_n(E, x)$ is, up to a unimodular constant, a matrix polynomial $M(w)$ with $\|M(0)\| = |\mu|^n$. For $y \ge 0$, $|w| = e^{-2\pi y} \le 1$ on the line $\mathrm{Im}\, x = y$, and subharmonicity of $\log\|M\|$ gives $\int_0^1 \log\|A_n(E, x + iy)\|\, dx \ge n \log|\mu| + 4\pi n y$. Hence $L(E, y) \ge \log|\mu| + 4\pi|y|$ for all real E and y, using evenness, and in particular $\ell(E) \ge \log|\mu|$. On the other hand $\|T(E, x + iy)\| \le |\mu| e^{4\pi y} + O(e^{2\pi y})$, so $L(E, y) \le \log|\mu| + 4\pi y + o(1)$ as $y \to \infty$; v is entire. By (A1), $L(E, \cdot)$ is convex and piecewise affine with slopes in $2\pi\mathbb{Z}$. The two bounds force every slope on $[0, \infty)$ to be at most 4π, and $L(E, y) - 4\pi y \to \log|\mu|$. At $E \in \Sigma$, $\omega(E) \ge 1$, so $\omega(E) \in \{1, 2\}$. If $\omega(E) = 2$, the slope is 4π on all of $(0, \infty)$, so $L(E, y) = \ell(E) + 4\pi y$ and the limit gives $\ell(E) = \log|\mu|$. Conversely, if $\ell(E) = \log|\mu|$, then $L(E, y) \ge \ell(E) + 4\pi y$ and slopes at most 4π force equality, so $\omega(E) = 2$. □

Suppose $\omega = 2$ somewhere on Σ. Then $\ell_* = \log|\mu|$, and $\ell > \log|\mu|$ where $\omega = 1$, so $\min_\Sigma \ell/\omega = \log|\mu|/2$. Theorem 5.2 gives, for $\beta(\alpha) \geq \ell^+$,

$$\sup_\theta \alpha_u^+(\theta) \leq 1 - \frac{\log|\mu|}{4\beta(\alpha)}.$$

If $\lambda \neq 0$ the potential is primitive. Conditionally on the weak separation hypothesis WXS(v, α) of [21, Definition 7.1], the exponent at almost every phase is $1 - \log|\mu|/\beta(\alpha)$ [21, Theorem D]. Whether the worst phase reaches $1 - \log|\mu|/(4\beta)$, or only $1 - \log|\mu|/(2\beta)$, is open (Section 14).

6 The almost Mathieu upper law down to $L/2$

The almost Mathieu operator admits a sharper upper bound than Corollary 5.6 below L. Throughout this section, $L = \log\lambda > 0$. No parity assumption is needed for the upper bound.

Theorem 6.1 (The sharper almost Mathieu upper law)**.** *Let α be irrational and $L/2 < \beta(\alpha) < L$. For every $a > 1 - L/(2\beta(\alpha))$ and $m > 0$,*

$$\sup_{\theta\in\mathbb{T}} P_\theta(T^a, T) = O(T^{-m}).$$

The constants can depend on λ, α, a, m. Consequently $\alpha_u^{\mathrm{unif}} \leq 1 - L/(2\beta(\alpha))$.

Together with Corollary 5.6, this proves the upper bound $1 - L/(2\beta)$ for every finite $\beta > L/2$. The matching lower bound in Section 10 uses β_e, so equality in terms of β still requires $\beta_e = \beta$.

6.1 Reduction to a quantitative frame package

At one convergent write

$$q = q_n, \quad p = p_n, \quad \delta = q\alpha - p, \quad b = -q^{-1}\log|\delta|, \quad \Lambda = \lambda^q, \quad R_* = 2\Lambda + 2, \quad \kappa = (2\pi q)^2\Lambda\delta^2. \tag{13}$$

The letter P below denotes the Chambers polynomial P_q. A cell has q physical sites. Fix $\xi > 0$, put $\tau = \xi/100$, and choose $h = L + d_h > L$ after the finite list of derivative estimates used below. A factor denoted $\mathcal{R}_q$ has the form $q^C e^{Cd_h q}$. Different occurrences can have different fixed C. We first bound all these constants, then choose d_h so that every such loss fits its assigned fraction of τq, and finally take q large. Neither $d_h q$ nor τq is called $o(q)$ at fixed positive d_h, τ.

The frame phase interval has half-width $t_{\mathrm{fr}} = e^{-\xi q/4}/(2\pi q)$. Its translated copies have physical centers and local phases as in [19, Appendix C, Covariance and well density]. The upper-law argument uses their raw frame estimates at $2\xi \leq b \leq L - \xi$. This range includes $b \leq L/2$. It does not use a small-κ eigenvalue-counting theorem there.

Definition 6.2 (The package FP*)**.** For each inner edge class and scale, the following data are required, uniformly in the phase, the well and the launch site.

1. A finite positive energy unit ϵ, a covariant orthonormal frame V, its projection Π, and summable physical kernel bounds for the frame and its orthogonalization. At a closed pair the unit must be defined without dividing by $P'(E_a)$.

2. A gap m for $QHQ - E$ on $Q = 1 - \Pi$ throughout the class window. After declaring $A = D\epsilon$, the bulk compression error γ and bulk coupling c_0 satisfy

   $$\gamma \leq A/100, \qquad c_0^2/m \leq A/100, \qquad c_0 = o(m), \qquad A = o(m).$$

   A coupling at the ends of a frame interval may instead be kept as a separate localized block. It must satisfy the boundary calculation of Appendix B.

3. In each fine energy window, at most a fixed number of shell intervals per well and a bounded self-adjoint frame penalty give a frame gap at least $3A$. Their widths and forbidden paddings in cells are at most
$$C\exp(C\xi q)(1+\kappa^{-1/2}). \tag{14}$$
The constant C in the exponent is fixed before ξ is chosen.

4. For the physical operator with this penalty, a capped cell-constant weight φ gives a perturbation Δ with
$$\|\Pi\Delta\Pi\| = o(A), \quad \|Q\Delta\Pi\|^2 + \|\Pi\Delta Q\|^2 = o(mA), \quad \|Q\Delta Q\| = o(m).$$
Its jumps are at the specified bonds. With p_{pad} cells of forbidden padding, $\mu p_{\mathrm{pad}} \geq e^{c\xi q}$ for a fixed $c > 0$. The same bounds hold in the opposite weight direction.

5. Smooth fine windows cover the full class and overlap the interior thresholds below. There are $e^{O(q)}$ fine windows and $O(q)$ coarse classes. All fixed-order filter derivatives, inverse window widths and inverse constants are $e^{O(q)}$.

The little-o comparisons refer to $q \to \infty$ after the slack parameters have been fixed.

This is a static package. In particular, the shell gap is a statement about the linear energy-space operator $H - E$. A scalar identity for $P(H)$ does not replace it. The factors D are allowed to grow at low b. Their growth must still imply (14).

Proposition 6.3 (From the package to transport)**.** *Suppose* FP* *holds at every inner edge class for all sufficiently large convergents with* $2\xi \leq b \leq L - \xi$*, for every sufficiently small* $\xi > 0$*. Then Theorem 6.1 holds.*

Proof. We give the scale and energy bookkeeping. The resolvent, functional calculus and launch-shell argument are proved in Appendix C. They turn the package into absolute filtered-packet tails smaller than e^{-Mq} for every fixed M, on every fixed time window $|t| \leq e^{Bq}$. Their physical radius is
$$R_{\mathrm{edge}} \leq Cqe^{C\xi q}(1+\kappa^{-1/2}), \qquad q\kappa^{-1/2} = (2\pi)^{-1}e^{(b-L/2)q}. \tag{15}$$
The top edge has the one-band package from the translated-well construction in [19, Appendices C–E], with the cell-weight and penalty estimates in Appendix C. The bottom edge follows from $u(j) \mapsto (-1)^j u(j)$ and $\theta \mapsto \theta + 1/2$.

For interior energies set $e_P(E) = \min\{|P(E) - 2\Lambda|, |P(E) + 2\Lambda|\}$. If $0 < \eta \leq e_P/4$, then
$$\big|\{x \in \mathbb{T} : |P(E) - 2\Lambda\cos 2\pi x| < \eta\}\big| \leq \frac{2\eta}{\pi\sqrt{\Lambda e_P}}. \tag{16}$$
The set has at most two arcs, up to the circle seam; it is empty if $|P(E)| > 2\Lambda$. On each monotone cosine branch its derivative is at least $2\pi\sqrt{\Lambda e_P}$, which proves the assertion. Replace only the sublevel step in Proposition 4.5 by this estimate. Retaining the rational-product residual gives
$$R_{\mathrm{int}}(E) \leq Ce^{(L/2+4\xi_C)q}e_P(E)^{-1/2} + Cqe^{\xi_C q}, \qquad e_P \geq 16e^{(L-b+4\xi_C)q}. \tag{17}$$
Here $\xi_C > 0$ includes the Lyapunov-growth slack in the shrinking spectral neighborhood. The resolvent tail is the absolute bound of Proposition 4.5. It holds for any fixed exponential time window. Appendix C passes it to smooth interior filters on the same type of window. No doubly exponential time claim is needed for the interior packet.

Fix a requested exponent $a > 1 - L/(2\beta)$. Choose

$$1 - L/(2\beta) < a_0 < \min\{a, 1/2\}, \qquad \nu = L/2, \quad t_v = \nu/(1 - a_0), \quad r_v = a_0 t_v = t_v - \nu.$$

Then $\beta < t_v < L$ and $r_v > 0$. Choose $\beta < B_0 < t_v$. In the covering of Theorem 5.2, choose K first, and then ξ, ξ_C, d_h small enough that all radius losses fit inside the strict gaps $t_v - B_0$ and $a - a_0$. We may require $2\xi < r_v$ and $B_0 < L - \xi$. Case A of that proof is unchanged.

In Case B put $t = \log T/q$. The three regimes are

$$\begin{array}{r|l} t \le t_v & \text{velocity,} \\ t \ge t_v,\ b \le 2r_v & \text{old confinement,} \\ t \ge t_v,\ b > 2r_v & \text{frames and interior.} \end{array}$$

The velocity radius is $\exp(((t-\nu)_+ + \text{chosen loss})q)$, up to polynomial factors. For $t \le t_v$ it is bounded by $T^{a_0 + \text{chosen loss}}$. For $t \ge t_v$, a radius exponent at most r_v suffices. If $b \le 2r_v$ the old radius exponent $b/2$ in Proposition 9.4 suffices, with the final slack assigned to $a - a_0$. Otherwise $b > 2r_v > 2\xi$ and $b \le B_0 < L - \xi$. Apply the frame package. The exponent in (15) is at most $(B_0 - \nu)_+ + C\xi < r_v$. For the interior use the threshold

$$e_P \ge \exp((2L - 2t_v + 8\xi_C)q).$$

Since $b > 2r_v = 2t_v - L$, this dominates the threshold in (17) after arbitrarily small slack. That radius is at most $e^{(r_v + \text{chosen loss})q}$.

The edge windows extend to P-depth $4\Lambda e^{-\xi q}$. Choose their plateaux strictly between the last interior threshold and this cap. This is possible because $2L - 2t_v < L$. On $b \ge L/2 + \xi$ one may take plateau $\Lambda e^{-5\xi q/4}$, with $\xi_C \le \xi/16$. If elementary cutoffs are $\chi_1, \dots, \chi_k$, use

$$g_j = \chi_j \prod_{i<j} (1 - \chi_i), \qquad g_{\text{rest}} = \prod_{i=1}^{k} (1 - \chi_i).$$

These sum to one. The remainder is supported in the interior class, after inserting a smooth spectral-neighborhood cutoff. Fixed-order derivatives and the number of fine windows cost $e^{O(q)}$. Overlapping windows require no false separation from the resolvent energy. Their functional-calculus factors have $e^{O(q)}$ bounds on $T \le e^{Bq}$.

Summing absolute packet tails costs only the number of filters. For the Abel average, truncate time at kqT; the omitted part is e^{-2kq} and kqT is still in a fixed exponential window. Choose M, k for the prescribed inverse power of T. Far energies are treated by Combes–Thomas exactly as in Theorem 5.2. Since q is comparable with $\log T$ in Case B, the resulting bound is $O(T^{-m})$ at T^a. All choices are uniform in phase and launch site. □

Remark 6.4 (The lower-ratio scales)**.** At the limiting target $a_0 = 1 - L/(2\beta)$, the range requiring new confinement starts at $b > 2\beta - L$. For $\beta < 3L/4$ it includes $b < L/2$. This is why the raw frame construction and its coarse energy unit are needed down to $b \ge 2\xi$.

7 Inner edges and the two-band normal form

We now construct the data used in Definition 6.2. The imported global reduction is the translated-well reduction of [19, Appendices A–D]. We use its complex-energy transfer bounds, physical Green kernels, Gram orthogonalization and localized boundary weights. We do not import its Mathieu counting theorem as a transport theorem. The changes needed at inner edges are proved below. The lattice product estimates are recorded in Appendix A.

7.1 Two branches and the exact inverse Morse map

Chambers' formula, in a common periodic Bloch gauge, is

$$\det(E - B_t(k)) = P(E) - 2\cos k - c(t), \qquad c(t) = 2\Lambda \cos 2\pi q t. \tag{18}$$

This is the classical identity already used in Section 10; see [11] and [19, Methods and conventions]. The phase-union bands are the intervals on which $|P| \le R_*$. An inner gap γ, possibly closed, has edges $E_a \le E_a'$ and one critical point c_γ. Write

$$P(E_a) = P(E_a') = sR_*, \quad P(c_\gamma) = s(R_* + \mathrm{ex}), \quad s \in \{-1, 1\}, \quad \mathrm{ex} \ge 0.$$

The turning phase satisfies $c(t_a) = 2s\Lambda$ and $\cos k_a = s$. Roots of $P + sR_*$ are counted with multiplicity. Their real-rootedness follows by realizing the polynomial as a Hermitian fibre characteristic polynomial.

Lemma 7.1 (Curvature and the Morse chart)**.** *Let $d_S = 4\lambda + 5$ and $M_j = \sup |P^{(j)}|$ in the fixed complex neighborhood used for the rational derivative bounds. For large q,*

$$|P''(c_\gamma)| = (2R_* + \mathrm{ex}) \sum_{f : P(f) = -sR_*} (c_\gamma - f)^{-2} \ge 2R_* q / d_S^2, \tag{19}$$

$$|P'(E_a)| \le M_2 |E_a - c_\gamma|, \qquad \mathrm{ex} \ge |P'(E_a)|^2/(2M_2). \tag{20}$$

There is $\rho_c = e^{-d_h q}/(C_0 q^2)$, with C_0 sufficiently large, on a contracted disc about c_γ on which

$$P(E) = s(R_* + \mathrm{ex}) - s\zeta(E)^2, \quad \zeta(c_\gamma) = 0, \quad z_\gamma := \zeta'(c_\gamma) = \sqrt{|P''(c_\gamma)|/2} > 0, \tag{21}$$

whenever the two edges lie in its inner quarter. The map ζ is analytic and univalent there, is real on the real axis, and $\zeta'(E)/z_\gamma = 1 + O(1/q)$.

Proof. Put $Q = P + sR_*$. On the root-free interval containing the gap, $(Q'/Q)' = -\sum_f (E - f)^{-2}$. At c_γ the first logarithmic derivative vanishes and $Q(c_\gamma) = s(2R_* + \mathrm{ex})$. This proves (19). Integrating the same identity also gives $|P'(E_a)| \ge 2R_* q |c_\gamma - E_a|/d_S^2$. The mean-value theorem gives the first inequality in (20). Integrate $|P'(E)| \ge |P'(E_a)| - M_2|E - E_a|$ up to its zero, which lies before c_γ, to get the second.

The derivative bounds below are used only in the fixed complex neighborhood of the rational phase-union bands. To justify their use at a narrow-gap center, integrate the logarithmic derivative once more:

$$\log\left(1 + \frac{\mathrm{ex}}{2R_*}\right) \ge \frac{q|E_a - c_\gamma|^2}{2d_S^2}.$$

Thus the center of an N gap tends to its edge and belongs to that neighborhood. The triangular integral proving (20) needs only an $O(1/q)$ segment at the edge. If the critical point is farther away, take the same segment and enlarge the fixed derivative constants. No growth bound at a far gap center is assumed.

The rational derivative bounds are $M_j \le C_{j,h} q^j e^{hq}$, by [20, Proposition 3.3(2)] and [19, Appendix A, Polynomial derivatives and the top contour]. Taylor-expand at c_γ and take the analytic square root of the nonzero quadratic factor. Bound that factor and its derivative on a larger disc before contracting to ρ_c. Increasing C_0 makes its derivative $O(1/q)$ relative to its value at the center. This proves univalence and all assertions about ζ. □

On an isolated two-band chart let $h(k,t)$ be the frozen restriction of $B_t(k)$ to these two bands, in an orthonormal analytic frame. Put

$$T_s(k) = 2 - 2s\cos k, \quad U(t) = 2\Lambda - sc(t), \quad x = \mathrm{ex} + T_s(k) + U(t).$$

Cayley–Hamilton and (18) give the exact identity

$$Z = \zeta(h), \qquad Z^2 = xI_2, \qquad \operatorname{tr} Z = 0. \tag{22}$$

Indeed the two distinct energies in the chart have opposite ζ-values. Continuation proves the trace identity at a touching point as well. For real k, t, Z is Hermitian.

Write $c = c_\gamma$ and keep the full inverse

$$\psi(w) = z_\gamma(\zeta^{-1}(w) - c), \qquad Y = \psi(Z).$$

Define analytic scalar functions

$$\vartheta(x) = \frac{\psi(\sqrt{x}) + \psi(-\sqrt{x})}{2}, \qquad \Omega(x) = \left(\frac{\psi(\sqrt{x}) - \psi(-\sqrt{x})}{2}\right)^2. \tag{23}$$

They are analytic also at $x = 0$. With $\widetilde{Y} = Y - \vartheta(x)I_2$, one has

$$\widetilde{Y}^2 = \Omega(x)I_2, \qquad (Y - Y_0)(Y + Y_0 - 2\vartheta(x)) = \Phi(x)I_2, \quad \Phi(x) = \Omega(x) - (Y_0 - \vartheta(x))^2. \tag{24}$$

For admissible Y_0 in the contracted chart, $\Phi' = 1 + O(1/q) > 0$. The inverse estimates imply this after contracting the chart and choosing ξ larger than its fixed losses. These are exact frozen identities. In general $Y \neq Z$. Replacing the inverse by its tangent would already introduce a nonzero error at $\delta = 0$.

7.2 The wide and narrow classes

Use the final threshold

$$\mathrm{ex}_* = \Lambda e^{-4d_h q}/q. \qquad \mathrm{W}: \ \mathrm{ex} > \mathrm{ex}_*, \qquad \mathrm{N}: \ \mathrm{ex} \le \mathrm{ex}_*. \tag{25}$$

Class W uses separate one-band frames at its two edges. Class N uses one frame containing both bands. Every exactly closed pair belongs to N. The threshold is an eventual asymptotic threshold, not a classification inferred from a finite numerical sample.

Lemma 7.2 (Isolation and localized frames)**.** *For every sufficiently small $\xi > 0$, choose $d_h > 0$ sufficiently small after all fixed derivative orders. For large q the following hold uniformly on $|t - t_a| \le t_{\mathrm{fr}}$. In W each edge has a one-root contour of radius at least $e^{-6d_h q}/q^C$, with discriminant on its boundary at least $\Lambda e^{-11d_h q}/q^C$. In N the Morse contour contains exactly two roots, with boundary discriminant at least $\Lambda e^{-2d_h q}/q^C$. Both give Wannier frames with, for the finite derivative orders used here,*

$$|\partial_t^j w_{t,\ell}(x)| \le \mathcal{R}_q e^{-(L-4\tau)\operatorname{dist}(x,\mathcal{A})}, \qquad \#\mathcal{A} \le 2. \tag{26}$$

Here $\mathcal{A}$ is a finite set of lifted anchors in one cell. The translated global synthesis and its orthogonalization obey physical weighted row and column Schur bounds, retaining an interwell error at most $\mathcal{R}_q e^{-c/|\delta|}$.

Proof. In W, if c_γ is outside a fixed edge neighborhood, the logarithmic-derivative lower bound gives $|P'(E_a)| \ge c_\lambda q\Lambda$. Otherwise the segment from E_a to c_γ is inside the derivative neighborhood, and $\mathrm{ex} \le M_1|c_\gamma - E_a|$ combines with that lower bound. In either case the threshold (25) gives $|P'(E_a)| \ge \Lambda e^{-5d_h q}/q^C$. Take a fixed contraction of $\min\{|P'(E_a)|/(2M_2), R_*/M_1\}$ as the one-root radius. Taylor's theorem and Rouché give the stated contour estimates. In N, (19) and (25) put both roots in the inner quarter of the Morse disc for large q. Rouché counts exactly two roots there. The phase drift on the frame chart is $O(\Lambda e^{-\xi q/2})$ and is smaller than the boundary discriminant. Contract the complex Bloch strip accordingly.

The periodic Green-function argument in [19, Appendix B, Periodic Green function and Analytic fibre data] uses only this large discriminant and the complex transfer bounds. It therefore applies on these contours. It gives cyclic Green kernels and analytic fibre data on a strip of width $(L - O(d_h) - \text{chosen loss})q$. Choose a nondegenerate anchor Gram matrix at the reference real fibre. In rank two, pivoted Cholesky gives determinant at least $2/q^2$; the overlap anchors in Lemma 7.4 will give another valid choice at an open N gap. For its complex continuation use the periodic trace and determinant of the Gram matrix. Endpoint phases cancel in principal minors. Their Fourier coefficients are bounded on the large strip, so their variation on a contracted strip is small. The identity

$$G^{1/2} = \frac{G + \sqrt{\det G}\, I_2}{\sqrt{\operatorname{tr} G + 2\sqrt{\det G}}}$$

continues the inverse square root holomorphically from the positive real reference matrix. This does not assert entrywise Fourier flatness in an incompatible gauge.

Include the anchor endpoint phases before inverse Bloch transform, and choose a common periodic gauge for both columns. Place their finite lifts along the shorter arc, of length $r \le q/2$. Contour shifting gives (26). The tail of one Wannier function is measured from this finite set, rather than from its periodic repetition. Translating each well to the reference chart and applying the physical Schur orthogonalization of [19, Appendix C, Overlap cancellation and global orthogonalization] gives the global assertions. Across wells sum over distinct physical centers; the number of anchors per center is bounded. This proves both Schur sums without summing a nonzero error once per well. □

The raw global construction remains valid when $b \ge 2\xi$. Only its freezing comparison uses a lower bound on b. On patches of $\ell_c = \lceil (q|\delta|)^{-1/2} \rceil$ cells the phase variation is $O(\sqrt{|\delta|/q})$. It is smaller than the frame chart and than the complementary energy margin below. The estimates normalized by a level spacing need not be small on these scales. We keep the raw estimates and enlarge D instead.

Proposition 7.3 (Compression and direct complement exclusion)**.** *For $2\xi \le b \le L - \xi$, the global frames above have a self-adjoint effective operator h_s and, throughout their contracted class windows,*

$$\|(QHQ - E)v\| \ge m\|v\| \quad (v \in Q\ell^2), \qquad m \ge q^{-C} e^{-(\xi/2 + Cd_h)q}. \tag{27}$$

The bulk coupling is $c_{\mathrm{bulk}} \le |\delta|\mathcal{R}_q$. In N the safe compression error and separate end coupling are

$$\gamma_{\mathrm{cmp}} \le |\delta|\mathcal{R}_q e^{-(L-4\tau)(q-r)} + \delta^2\mathcal{R}_q + \mathcal{R}_q e^{-c/|\delta|}, \qquad c_{\mathrm{end}} \le e^{-(L/2 - \textit{chosen loss})q}. \tag{28}$$

The end leg is localized in the end strips, with decaying Fourier and interwell tails. In W the corresponding one-anchor exponent is retained, together with the positive boundary weight of [19, Appendix D, Interior coupling and omitted translates].

Proof. The translated matrix expansion, its Hermitian compression, and the separation of bulk and end couplings are the kernel calculations in [19, Appendices C–D]. They use nearest-neighbor hopping, the physical localization kernels and finitely many phase derivatives. With two anchors the convolution distance for a nonzero cell displacement is $|m|q - r$, rather than $|m|q$. This gives (28), with both factors at an end interface retained. It also proves the differentiated seminorms in Appendix A. A column-frozen matrix is replaced by its Hermitian part before any comparison.

For complement exclusion the omitted frozen branches are at least

$$g_{\mathrm{W}} \geq \min\{\rho_a/2, M''/(C|P'(E_a)|)\}, \quad g_{\mathrm{N}} \geq \min\{\rho_c/2, M''/[C\sqrt{M_2(\mathrm{ex} + M'')}]\}, \quad M'' = c_\lambda \Lambda e^{-\xi q/2}$$

from the class window. Inside a frame patch this follows from the one-root or two-root contour. Outside it, the phase value is separated from the turning value by M''; the inverse derivative or Morse divided difference gives these bounds. Use a common lower bound $g \geq q^{-C} e^{-(\xi/2 + C d_h)q}$ in all patches.

Put $\eta_{\mathrm{fr}} = C\sqrt{|\delta|/q}$. On each patch let $A_a = B_a - E$ be the frozen operator and Π_a its retained one- or two-band projection. The common margin g gives $\|A_a f\| \geq g\|(1 - \Pi_a)f\|$. Freezing and $(x - y)_+^2 \geq x^2/2 - y^2$ imply

$$\|(H - E)f\|^2 \geq \frac{g^2}{2}(\|f\|^2 - \|\Pi_a f\|^2) - \eta_{\mathrm{fr}}^2 \|f\|^2.$$

For $v \in Q\ell^2$, apply the genuine IMS formula to $(H - E)^2$ and sum with $f = \chi_a v$. Its error is $C|\delta|/q$. The supported joint Bessel estimate of [19, Appendix D, Excluding the complementary subspace], summed over the at most two anchors, is

$$\sum_a \|\Pi_a \chi_a v\|^2 \leq \mathcal{R}_q |\delta| q^{-1} \|v\|^2.$$

The patch-support factors are included in both Schur sums. Hence

$$\|(H - E)v\|^2 \geq \left[\frac{g^2}{2} - \frac{g^2}{2}\mathcal{R}_q \frac{|\delta|}{q} - C\frac{|\delta|}{q}\right] \|v\|^2.$$

Since $b \geq 2\xi$ and d_h is chosen after the fixed constants, $\mathcal{R}_q|\delta|/q = o(1)$ and $|\delta|/q = o(g^2)$. Therefore $\|(H - E)v\| \geq (g/\sqrt{2} - o(g))\|v\|$. The complete coupling, including the conservative rank-two end term, is $o(g)$. Consequently

$$\|(QHQ - E)v\| \geq \|(H - E)v\| - \|\Pi H Q\|\|v\| \geq (g/2)\|v\|.$$

This proves the gap for the compression directly. A spectral gap of H alone would not imply it. Both nearby branches have been included in N, also when their gap is positive but very small. □

7.3 The curvature dichotomy and overlap anchors

Lemma 7.4 (Overlap anchors at every open N gap)**.** *At an open class-N gap, let u, u' be the normalized real edge vectors at (t_a, k_a) and let $\mathcal{R}$ be the signed reflection commuting with that fibre. There is a non-fixed site x_* such that*

$$|u(x_*)u'(x_*)| \geq q^{-C} e^{-C d_h q}, \qquad \det G_{\{x_*, \mathcal{R}x_*\}} = 4u(x_*)^2 u'(x_*)^2. \tag{29}$$

If $r = \mathrm{dist}_{\mathbb{Z}/q\mathbb{Z}}(x_*, \mathcal{R}x_*) \leq q/2$, then

$$|P'(E_a)| \leq e^{hq} \mathcal{R}_q e^{-(L - 2\tau)r}. \tag{30}$$

Proof. Let $V' = \partial_t B_{t_a}$ and $M = \langle u', V'u\rangle$. Twice differentiating Chambers' identity, then using simple-eigenvalue perturbation theory, gives

$$\frac{c''(t_a)}{P'(E_a)} = \langle u, V''u\rangle + \frac{2|M|^2}{E_a - E'_a} + 2\sum_{j\neq a,a'} \frac{|\langle u_j, V'u\rangle|^2}{E_a - E_j}, \qquad |c''(t_a)| = 8\pi^2 q^2 \Lambda. \tag{31}$$

The derivative in this identity is signed. Other degenerate subspaces are summed by their projectors. A whole phase-union band separates the pair from every other same-sign root, so the other energy denominators have modulus at least $2R_*/M_1$. Also $\|V'\| \le 4\pi\lambda$ and $\|V''\| \le 8\pi^2\lambda$.

If the partner term has modulus less than $|E''_a|/3$, the other terms imply $|P'(E_a)| \ge c_\lambda q \Lambda e^{-d_h q}$. Equation (20) then gives $\mathrm{ex} \ge c'_\lambda \Lambda e^{-3d_h q}$. For large q this is W under (25). Otherwise

$$|M|^2 \ge \frac{4\pi^2 q^2 \Lambda}{3M_2} \ge c_\lambda e^{-d_h q}.$$

Every open N gap is therefore in this second alternative. Reflection anticommutes with V', so its two edge vectors have opposite reflection parities. Since $|M| \le 4\pi\lambda q \max_x |u(x)u'(x)|$, a site x_* satisfies (29). At a reflection-fixed site one of these opposite-parity vectors vanishes. Hence x_* is not fixed. The Gram determinant follows by writing the two reflected columns in the edge basis.

Let Π_γ be the frozen pair projector and $ed \in \{-1, 1\}$ the reflection sign at x_*. The exact two-dimensional pencil identity is

$$(E_a - E'_a)(\Pi_\gamma)_{x_* x_*} = 2ed(B\Pi_\gamma)_{x_*, \mathcal{R}x_*} - ed(E_a + E'_a)(\Pi_\gamma)_{x_*, \mathcal{R}x_*}.$$

The two off-diagonal kernels decay in the cyclic distance r by the contour Green bounds. The diagonal is at least $2|u(x_*)u'(x_*)|$. Thus $|E_a - E'_a| \le \mathcal{R}_q e^{-(L-2\tau)r}$. Finally $|P'(E_a)| \le M_2|E_a - E'_a|$, which proves (30). No maximizer of a projector diagonal has been assumed non-fixed. □

At an exactly closed pair we use the rank-two Riesz projector; (31) is not asserted for an individual vector at a double eigenvalue. For closed and near-closed pairs the general pivot anchors in Lemma 7.2 give the longer-arc tail with $r \le q/2$. In the shallow comparison it suffices to use the cutoff $\sqrt{\mathrm{ex}} \le e^{(\xi/2+6\tau)q}$. Open N gaps outside that cutoff use Lemma 7.4. At the longer-arc jump bond both endpoints are included. If

$$F = \mathcal{R}_q e^{-(L-4\tau)(q-r)}, \tag{32}$$

then $\|\mathbf{1}_{\mathrm{bonds}}\Pi\|^2 \le F$. This is a physical operator bound from row and column Schur sums, rather than a single frozen projector diagonal.

7.4 The one-band W package

For a W edge put $P'_a = |P'(E_a)|$ and $\sigma_{\mathrm{out}} = s\,\mathrm{sgn}\,P'(E_a)$. The positive inward depth operator is $P'_a\sigma_{\mathrm{out}}(E_a + s_a(t_a) - h_s)$. Here s_a is the orbital term, including the small parameter. The inverse branch $E(y)$ satisfies $P(E(y)) = sR_* - sy$, and

$$F_a(y) = P'_a|E(y) - E_a|, \qquad |F'_a(y) - 1| \le 4M_2 y/(P'_a)^2 \le q^C e^{-(\xi - Cd_h)q}$$

on $0 \le y \le C\Lambda e^{-\xi q}$. Taylor's theorem proves this after contracting the inverse domain. The orbital correction is the classical Rammal–Wilkinson term [46, 43, 27]; its uniform local compression is [19, Appendix C]. A possible odd part has size $\mathcal{R}_q\sqrt{\kappa}\sqrt{U}$, without an extra $|\delta|$. Its first two phase derivatives move the minimum by $\mathcal{R}_q$ cells and preserve quadratic growth there. Thus a margin D

gives one shell of width $C\sqrt{D/\kappa}+\mathcal{R}_q$ near the minimum, or two shells of width $CD/\sqrt{\kappa e}+\mathcal{R}_q$ away from it. The coarse unit is

$$A_a = D_a/P'_a, \qquad D_a = C\{1+P'_a(\gamma_{\mathrm{cmp}}+c_{\mathrm{bulk}}^2/m+\|W_{\mathrm{bd}}\|)\}.$$

Diagonal dominance after the shell penalty gives a gap $3A_a$. The one-anchor midpoint tail, the weight estimates of Appendix C, and Proposition 7.3 prove the W package. At low b the raw estimate gives

$$\sqrt{D_a/\kappa} \le \mathcal{R}_q(\kappa^{-1/2}+e^{\xi q/4}).$$

The padding $e^{(\xi/2+12\tau)q}$ absorbs the additional term. Top and bottom use the same one-band argument and the same coarse unit. These estimates are uniform at all depths of their class. The remaining N shell and weighted estimates are proved next.

8 Matrix shells, fixed units and assembly

8.1 One energy unit for the whole N class

Fix a class-N gap and the scale of (13). Use ϑ, Ω, Φ from (23)–(24). The self-adjoint effective symbol is $h+\varepsilon M$, with $\varepsilon = \delta/q$. If m_0 is the chosen scalar part of the orbital matrix at the reference point, use the consistent convention

$$c' = c+\varepsilon m_0, \qquad W = z_\gamma(M-m_0 I_2), \qquad y = Y+\varepsilon W. \tag{33}$$

Then $y = z_\gamma(h+\varepsilon M-c')$. For an energy E, put $Y_0 = z_\gamma(E-c')$. The operator $\mathrm{Op}(y)-Y_0$ is linear in E. The endpoint-average to midpoint error is kept separately.

A single physical energy margin will be used for the entire gap class. The reference size N_{ref} records the pair at the reference edge and does not include the moving parameter Y_0. The factor D pays raw compression and bulk Schur errors. Depth enlarges the shell threshold and penalty while ϵ_{ref} and A remain fixed. Let $\overline{H} \ge 1$ dominate the finite off-cell seminorms after the fixed losses have been reserved. We may take $\overline{H} \le e^{2\tau q}\mathcal{R}_q$. Put

$$\begin{aligned} N_{\mathrm{ref}} &= \sqrt{\mathrm{ex}}+1+\overline{H}, \\ \epsilon_{\mathrm{ref}} &= (z_\gamma N_{\mathrm{ref}})^{-1}, \\ D &= C_D\{1+z_\gamma N_{\mathrm{ref}}(\gamma_{\mathrm{cmp}}+c_{\mathrm{bulk}}^2/m+\|W_{\mathrm{bd}}\|)\}, \\ A &= D\epsilon_{\mathrm{ref}}, \qquad c_A = C_W z_\gamma A = C_W D/N_{\mathrm{ref}}. \end{aligned} \tag{34}$$

Here C_D is a sufficiently large fixed constant for all assigned bulk budgets, and C_W is fixed below. The bulk coupling in this definition is unweighted. The separate rank-two end coupling is excluded from D. A boundary weight is retained when present. This is one unit per gap and scale, independent of E. It is finite at $\mathrm{ex} = 0$.

The full class window is the contracted real chart with

$$|\zeta(E)|^2 \le \mathrm{ex}+4\Lambda e^{-\xi q}. \tag{35}$$

All assertions below hold for both branches and both signs of Y_0. There is no restriction to shallow values of $|Y_0|$.

8.2 The ordered product and the replacement shells

On the full cell lattice let

$$\overline{X} = \mathrm{Op}(y) - Y_0, \qquad \overline{V} = \mathrm{Op}(y) + Y_0 - 2\,\mathrm{Op}(\vartheta(x)).$$

Apply the partner after $\overline{X}$. The calculus of Appendix A gives

$$\begin{aligned} \overline{V}\overline{X} &= \mathrm{Op}(p_1) + E_2, \\ p_1 &= \Phi(x) I_2 + \varepsilon\{\widetilde{Y}, W\}_+ + \varepsilon^2 W^2 + \frac{i\delta}{2}[y_k, y_t] + i\delta\{y, \vartheta\}, \\ \|E_2\| &\le \kappa \overline{H}\mathcal{R}_q =: e_2. \end{aligned} \tag{36}$$

The braces with subscript + denote the anticommutator; the last braces are the ordered Poisson bracket. Both derivatives of $\vartheta(x)$ are retained, including $\partial_t \vartheta = \vartheta'(x)U'(t)$. The first four terms of p_1 are Hermitian on real k, t. The last term is anti-Hermitian. We use singular values and never assert that the ordered product is self-adjoint.

Let $\mathcal{D}_j = (p_1)_0(t_j)$ be its exact zero Fourier block and write $\mathcal{D}_j = \mathcal{D}_j^h + \mathcal{D}_j^a$. Set

$$\mathcal{D}^h(t) = \overline{\Phi}(t) I_2 + B(t), \qquad \overline{\Phi}(t) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(\mathrm{ex} + T_s(k) + U(t))\, dk.$$

The two ordered eigenvalues of $\mathcal{D}^h$ are denoted $\lambda_-(t) \le \lambda_+(t)$.

Lemma 8.1 (Block variation and global hopping)**.** *For the fixed derivative losses above,*

$$\|B(t)\| \le b_1\sqrt{\mathrm{ex} + 4 + U(t)} + b_2, \qquad b_1 = C\sqrt{\kappa}\mathcal{R}_q, \quad b_2 = C(\sqrt{\kappa} + \kappa)\overline{H}\mathcal{R}_q, \tag{37}$$

$$\|B'(t)\| \le C\sqrt{\kappa}\mathcal{R}_q(\sqrt{\mathrm{ex} + 4 + U(t)} + z_\gamma + \overline{H}) + C\kappa\mathcal{R}_q. \tag{38}$$

There is a core of $k_{\mathrm{core}} \le C\overline{H}\mathcal{R}_q$ cells about the cell nearest t_a. On either side of the core each ordered eigenvalue increases away from t_a, and

$$\lambda_\ell(t) - \lambda_\ell(u) \ge \tfrac{3}{4}(1 - 1/q)(U(t) - U(u)) \tag{39}$$

when t is farther from t_a than u on that side. For the full hopping kernel $\mathcal{P}_{ji}$ of $\mathrm{Op}(p_1)$,

$$\sup_j \sum_{i \ne j} \|\mathcal{P}_{ji}\|, \quad \sup_i \sum_{j \ne i} \|\mathcal{P}_{ji}\| \le H_p := C\{1 + (\sqrt{\kappa} + \kappa)\overline{H}\mathcal{R}_q\}. \tag{40}$$

The bound is independent of ex *and Y_0.*

Proof. The differentiated convolution estimates in Appendix A bound Y and W, with one extra factor q for W. Differentiating the dipole product uses $\varepsilon(Y_t W + Y W_t)$; differentiating the spin term uses y_{kt} and y_{tt}. The stated fixed seminorm orders control all of them. They give (37)–(38). The anti-Hermitian block has a uniform bound a_∞ with

$$a_\infty \le C\sqrt{\kappa\overline{H}}\mathcal{R}_q + |\delta|\mathcal{R}_q/\rho_c.$$

The last term is exponentially small for $b \ge 2\xi$.

Weyl's min-max inequality, integrated along a side of the core, gives

$$\lambda_\ell(t) - \lambda_\ell(u) \geq \overline{\Phi}(t) - \overline{\Phi}(u) - \int_u^t \|B'(v)\|\,|dv|.$$

Since $|U'| \geq 2\pi q\sqrt{\Lambda}\sqrt{U}$ on the chart, each of the four ratios from (38) to $(1-1/q)|U'|$ is at most $1/16$ outside an enlarged $C\overline{H}\mathcal{R}_q$ core. Here $\sqrt{U}$ changes by $\sqrt{\kappa}(1+o(1))$ per cell, $\mathrm{ex} \leq \mathrm{ex}_* \leq \Lambda$, and $z_\gamma \leq q\sqrt{\Lambda}\mathcal{R}_q$. Their sum is at most $1/4$, proving (39). No convexity or unique matrix-eigenvalue minimum is asserted inside the core.

For (40), retain the zero Fourier matrix exactly. Every nonzero Fourier coefficient of $\widetilde{Y}W$ contains at least one off-cell factor. The bounds

$$Y_{\mathrm{off}} \leq \overline{H}\mathcal{R}_q, \quad W_{\mathrm{off}} \leq q\overline{H}\mathcal{R}_q, \quad \|Y_0^{\mathrm{Fourier}}\| \leq Cz_\gamma\mathcal{R}_q, \quad \|W_0^{\mathrm{Fourier}}\| \leq qz_\gamma\mathcal{R}_q$$

are bounds for absolute weighted Fourier sums. Thus its product with ε has row and column sums at most $C|\delta|qz_\gamma\overline{H}\mathcal{R}_q \leq C\sqrt{\kappa}\overline{H}\mathcal{R}_q$. The ε^2W^2, spin and scalar-Poisson terms cost at most $C\kappa\overline{H}\mathcal{R}_q$ and $C\sqrt{\kappa}\overline{H}\mathcal{R}_q$. Expand the scalar Φ in bounded $T_s(k)$; $\Phi' = 1 + O(1/q)$ and the contracted-chart Cauchy bounds give off-cell sum $2 + o(1)$. Midpoint shifts and Fourier tails are included in these sums. This proves the global estimate in both directions. A large diagonal matrix has not been used as an off-cell bound. □

Keep the full diagonal matrix exactly. A cell belongs to the shell only when one of its exact Hermitian block eigenvalues is close to zero. The constant threshold pays hopping, non-Hermitian and interface errors. Adding the diagonal bound again would overestimate the width.

Fix a well J. Products of compressed factors retain an interface norm $I_{\mathrm{int}} \leq C\overline{H}(\overline{H} + h_\vartheta)$, where h_ϑ is the off-cell sum of $\mathrm{Op}(\vartheta)$. Choose a constant floor

$$\begin{aligned} W_0 &\geq C_0(1 + H_p + a_\infty + e_2 + I_{\mathrm{int}} + \kappa\mathcal{R}_q), \\ W_0 &\leq C\mathcal{R}_q^C(1 + \overline{H}^2 + \kappa\overline{H}). \end{aligned} \tag{41}$$

The second line follows by enlarging the fixed losses. Write $n = |Y_0| + \sqrt{\mathrm{ex}} + 1$ and define the replacement shell by

$$\begin{aligned} W' &= \max\{W_0,\, 2c_A(n + \overline{H} + 1),\, 4c_A^2\}, \\ \mathcal{S} &= \mathcal{K}_E \cup \{j \in J : \min_{\ell\in\{-,+\}} |\lambda_\ell(t_j)| < W'\}. \end{aligned} \tag{42}$$

Here $\mathcal{K}_E$ is the whole core if the displayed inequality holds somewhere in the core, and is empty otherwise. In particular, the full diagonal majorant in (37) is absent from the threshold. It is already in the exact eigenvalues.

Proposition 8.2 (Replacement shell geometry and penalized gap)**.** *For $2\xi \leq b \leq L - \xi$, large q, and every E in* (35)*, the shell has at most five components per well, at most four if it misses the core. Each has width at most*

$$C\left\{1 + \sqrt{W_0/\kappa} + \frac{c_A(1+\overline{H})}{\sqrt{\kappa}} + \sqrt{\frac{c_A(\sqrt{\mathrm{ex}}+1)}{\kappa}}\right\} + 2k_{\mathrm{core}}. \tag{43}$$

These widths obey (14)*. There is a bounded self-adjoint shell penalty, in energy units, $M_E\mathbf{1}_{\mathcal{S}}$, such that the effective frame operator has gap at least $4A$ at the window center and $3A$ for $|E - E_c| \leq A$ with the shell fixed at E_c. Its size satisfies*

$$M = z_\gamma M_E \leq C\{n + \sqrt{W_0} + \overline{H}^2 N_{\mathrm{ref}}/D + c_A\}. \tag{44}$$

Proof. Put $\mathcal{C} = J \setminus \mathcal{S}$. Off the core each ordered eigenvalue is monotone on each side and the threshold is constant in the cell. Each branch gives at most one interval on either side. Adding the conditional core gives the component count. For an interval with endpoint potential values $U_1 \le U_2$, (39) gives $U_2 - U_1 \le CW'$. Its cell count is at most

$$C \min\{\sqrt{W'}, W'/\sqrt{U_2}\}/\sqrt{\kappa} + 1.$$

If $W' = W_0$ or $4c_A^2$, use the square-root bound. For $W' = 2c_A(n + \overline{H} + 1)$ and $U_2 \ge n^2/16$, use the second bound. If $U_2 < n^2/16$, the shell inequality and the full diagonal bound (37) imply

$$n \le C(\sqrt{\mathrm{ex}} + 1 + \sqrt{W'} + b_1 + \sqrt{b_2}).$$

The quantities b_1^2, b_2 are absorbed in W_0. AM–GM applied to $c_A\sqrt{W'}$ proves (43). No cell-dependent diagonal-size threshold has been added.

Here are both Schur sums for the off-shell gap. The floor gives $d_j := \sigma_{\min}(\mathcal{D}_j) \ge W' - a_\infty \ge 0.99W'$ on $\mathcal{C}$. For $\mathcal{T}_{ji} = \mathcal{D}_j^{-1}\mathcal{P}_{ji}$,

$$\sup_j \sum_i \|\mathcal{T}_{ji}\| \le H_p/(W' - a_\infty) \le 1/32,$$
$$\sup_i \sum_j \|\mathcal{T}_{ji}\| \le (W' - a_\infty)^{-1} \sup_i \sum_j \|\mathcal{P}_{ji}\| \le 1/32.$$

Use operator norms, separately, for E_2 and the interface product: $\|\mathcal{D}_{\mathcal{C}}^{-1}(E_{2,\mathcal{C}} - \mathrm{Int})\| \le 1/32$. Thus the total normalized remainder has norm at most $1/16$. A norm-controlled remainder has not been assigned a row-sum bound.

Let N be cellwise scalar, with

$$N_j = \sqrt{\mathrm{ex} + 4 + U_j} + |Y_0| + \overline{H} + \sqrt{W'} + 1.$$

The diagonal part of $N^{-1}\overline{V}_{\mathcal{C}\mathcal{C}}$ is bounded. Since $N_j \ge \overline{H}$, its off-cell row and column sums satisfy, respectively,

$$\sup_j \sum_{i \ne j} \frac{\|\overline{V}_{ji}\|}{N_j} \le H_V/\overline{H} \le C,$$
$$\sup_i \sum_{j \ne i} \frac{\|\overline{V}_{ji}\|}{N_j} \le \overline{H}^{-1} \sup_i \sum_{j \ne i} \|\overline{V}_{ji}\| \le C.$$

Here the fixed seminorm losses have been absorbed into $\overline{H}$. Hence its norm is at most a fixed C_V. The product, with partner after X, now yields

$$C_V\|X_{\mathcal{C}\mathcal{C}}v\| \ge \|N^{-1}\mathcal{D}_{\mathcal{C}}(I + \mathcal{T}_{\mathrm{total}})v\| \ge \tfrac{15}{16} \min_j(d_j/N_j)\|v\|.$$

This uses the order shown. It commutes neither N nor $\mathrm{Op}(\vartheta)$ past X. For $U \le C(Y_0^2 + \mathrm{ex} + W')$ the last minimum is at least $cW'/(n + \sqrt{W'} + \overline{H} + 1) \ge cc_A$. In the outer region the scalar Φ dominates the matrix majorant, the anti-Hermitian error and the floor, giving ratio at least $c\sqrt{U}$, which is larger. Thus $\|X_{\mathcal{C}\mathcal{C}}v\| \ge g\|v\|$ for the declared floor $g = cc_A$.

Take $M = \|X_{\mathcal{S}\mathcal{S}}\| + g + 3\overline{H}^2/g$. Its shell block has gap $G \ge g + 3\overline{H}^2/g$. Diagonal scaling by $\mathrm{diag}(\sqrt{G}, \sqrt{g})$ bounds the normalized off-diagonal norm by $1/\sqrt{3}$. The whole penalized operator therefore has singular gap at least $(1 - 1/\sqrt{3})g$. Increase C_W once and divide by z_γ to get $4A$, with

the endpoint error already paid in D. Energy variation of size A leaves $3A$. If a block is empty, use the remaining gap or penalize the whole finite well directly. On shell cells the full diagonal bound implies $\sqrt{U} \le C(n + \sqrt{W'} + \sqrt{W_0})$. Substituting $g = cc_A$ gives (44). In particular its $\overline{H}^2 N_{\mathrm{ref}}/D$ term is retained.

Finally expand the unweighted definition of D in (34), using (28) and $c_{\mathrm{bulk}}^2/m \le \delta^2 e^{\xi q/2}\mathcal{R}_q$. Since $N_{\mathrm{ref}} \le \sqrt{\Lambda}\mathcal{R}_q$ and $r \le q/2$,

$$\begin{aligned} \frac{c_A(1+\overline{H})}{\sqrt{\kappa}} &\le \mathcal{R}_q^C(1+\kappa^{-1/2}), \\ \frac{c_A(\sqrt{\mathrm{ex}}+1)}{\kappa} &\le CD/\kappa, \\ \sqrt{D/\kappa} &\le e^{C\xi q}\mathcal{R}_q^C(1+\kappa^{-1/2}). \end{aligned}$$

For example the δ^2 term in D/κ is at most $\mathcal{R}_q^C(N_{\mathrm{ref}}/\sqrt{\Lambda})e^{\xi q/2}$. The distance-dependent term is at most $\mathcal{R}_q^C N_{\mathrm{ref}} e^{-(L-4\tau)(q-r)}/(\sqrt{\Lambda}|\delta|)$. Its square root is bounded by the same type of slack times $1+\kappa^{-1/2}$. The retained boundary weight and interwell tails are smaller. Equations (41) and (43) now give (14). This calculation keeps the low-b numerator D throughout. □

8.3 Weighted estimates at every depth

Proposition 8.3 (All-depth weighted budgets)**.** *Use the physical frame penalty synthesized from Proposition 8.2. Let the weight vanish on shell cells, jump at both-endpoint longer-arc bonds by at most*

$$\mu = e^{-(\xi/2+10\tau)q}, \qquad p_{\mathrm{pad}} = \lceil e^{(\xi/2+12\tau)q} \rceil,$$

and cap it before the frame ends, keeping it constant in exterior gaps. Throughout (35)*, the three weighted budgets of Definition 6.2 hold. All shells with their padding lie inside the frame, and their fine-window count and inverse widths are* $e^{O(q)}$*.*

Proof. Let F be (32). Physical weighted Schur estimates, with both weight orientations, give for shell synthesis $U_{\mathcal{S}}$

$$\|(e^{\pm\varphi}-1)U_{\mathcal{S}}\| \le C\mu\sqrt{F}, \qquad \|V^*(e^{\pm\varphi}-1)U_{\mathcal{S}}\| \le C\mu F. \tag{45}$$

The second bound uses two localized tails. Within a well the weights grow more slowly than the frame tails. Across wells retain $e^{\mu D_{\mathrm{cap}}}\mathcal{R}_q e^{-c/|\delta|}$, which is still negligible. The proof is given in Appendix C.

For open N, (30) and the Morse derivative comparison imply

$$\begin{aligned} z_\gamma\sqrt{\mathrm{ex}}F &\le e^{(d_h+6\tau)q}\mathcal{R}_q, \\ \mathrm{ex}F &\le e^{(2d_h+4\tau)q-Lr}\mathcal{R}_q. \end{aligned} \tag{46}$$

For closed or near-closed pairs use $\sqrt{\mathrm{ex}} \le e^{(\xi/2+6\tau)q}$ and $F \le \Lambda^{-1/2}e^{2\tau q}\mathcal{R}_q$ instead. In either case

$$K_{\mathrm{ref}} = z_\gamma N_{\mathrm{ref}} F \le e^{(\xi/2+8\tau)q}\mathcal{R}_q^C, \qquad \mu K_{\mathrm{ref}} = o(1). \tag{47}$$

The physical weight perturbation has norm $O(\mu)$ and is supported on the jump bonds. Its frame block is $O(\mu F) = o(A)$ by (47). Its cross blocks are $O(\mu\sqrt{F})$. After squaring and dividing by mA, the extra factor is $C\mu/m$, which tends to zero.

The penalty is nonlocal in physical space. Expanding its conjugation with (45) gives frame error $CM_E\mu F$, cross errors $CM_E\mu\sqrt{F}$ and complement error $CM_E\mu^2 F$. The penalty is the only factor that grows with depth. Normalize its frame error by $A = D/(z_\gamma N_{\mathrm{ref}})$; this leaves $\mu M N_{\mathrm{ref}} F/D$. The deep part of M is paid by $z_\gamma\sqrt{\mathrm{ex}}F$, and its reference-mass part by $\mathrm{ex}F$. Thus the same edge unit works at every depth. More precisely, use the corrected penalty bound:

$$MN_{\mathrm{ref}}F \le e^{(8\tau+2d_h)q}\mathcal{R}_q^C. \tag{48}$$

Here the depth-dependent term is paid by

$$\sqrt{\Lambda}e^{-\xi q/2}N_{\mathrm{ref}}F,$$

which is bounded using the first line of (46) or the near-closed cutoff. The squared reference term is paid by $\mathrm{ex}F$ in its second line. From (41), $\sqrt{W_0} \le C\mathcal{R}_q^C(1+\overline{H}+\sqrt{\kappa\overline{H}})$. This controls its term, and $\overline{H}^2 N_{\mathrm{ref}}^2 F$ is controlled by the same $\mathrm{ex}F$ bound. The last term is $c_A N_{\mathrm{ref}}F = C_W DF$, with D expanded in unweighted quantities as in Proposition 8.2. This proves (48). Consequently $\mu M N_{\mathrm{ref}}F = o(1)$. The normalized penalty cross budget has an additional $M_E\mu/m = o(1)$, since $M_E \le e^{4\tau q}\mathcal{R}_q^C$. The complement physical error is $O(\mu) = o(m)$; its penalty error has two one-tail factors and is $O(M_E\mu^2 F) = o(m)$. The adjoint cross block obeys the same bounds.

For location, a sufficient bound about the well bottom is

$$|j - j_a| \le \mathcal{R}_q^C\left\{\sqrt{e/\kappa} + (\mathrm{ex}/\kappa)^{1/4} + \sqrt{W_0/\kappa} + c_A/\sqrt{\kappa} + k_{\mathrm{core}}\right\}, \tag{49}$$

where $e \le 4\Lambda e^{-\xi q}$ bounds the spectral depth. The extra orbital position term is at most $e^{(b/2+\text{chosen loss})q}$. It is well inside the frame half-width $e^{(b-\xi/4)q}/(2\pi q)$ for $b \ge 2\xi$. The other terms and p_{pad} are inside it as well. This position bound does not count distance from the bottom as shell width. Also $\mu p_{\mathrm{pad}} \ge e^{2\tau q}$.

Finally

$$e^{-(L+\text{chosen loss})q} \le \epsilon_{\mathrm{ref}} \le Cq^{-1/2}\Lambda^{-1/2}.$$

The lower exponent is L, because ex can be large. The class width is bounded, so there are $e^{O(q)}$ fine windows. The weight and shell may be rebuilt in each smooth fine window. This changes neither the fixed unit nor the absolute depth. □

Remark 8.4 (Choice of energy unit)**.** The available upper estimates do not prove the weighted budgets with the varying unit $(z_\gamma\sqrt{\mathrm{ex}+e})^{-1}$ at all depths. This is a limitation of that sufficient estimate. It is not a theorem excluding every construction with varying units. The fixed reference unit above is sufficient.

8.4 Assembly

Proof of Theorem 6.1. We verify the five parts of Definition 6.2. The finite units, covariant frames and physical kernels are given by Lemma 7.2, Lemma 7.4 and (34). Direct complement exclusion is Proposition 7.3. The unweighted definition of D pays the compression and bulk squared-coupling budgets. It also gives $A = o(m)$, because

$$A \le C(\epsilon_{\mathrm{ref}} + \gamma_{\mathrm{cmp}} + c_{\mathrm{bulk}}^2/m + \|W_{\mathrm{bd}}\|) = o(m).$$

The shell gap and widths are Proposition 8.2; the all-depth weighted budgets are Proposition 8.3. The localized three-block calculation in Appendix B supplies the retained end coupling. W, top and bottom use the one-band package at the end of Section 7.

The full window (35) covers both sides of an N pair to P-depth $4\Lambda e^{-\xi q}$. The W and extreme windows have the corresponding contracted depths. Their smooth cover has $O(q)$ coarse classes and $e^{O(q)}$ fine windows. It overlaps the high-b plateau $\Lambda e^{-5\xi q/4}$ and the target-dependent low-b interior threshold in Proposition 6.3. No window is reset to shallow absolute depth. Thus all five parts hold, including closed pairs.

For any strict target $a > 1 - L/(2\beta)$, first fix the covering slack as in Proposition 6.3. There are finitely many constants in $e^{C\xi q}\mathcal{R}_q^C$. Choose ξ inside that slack, then d_h after their largest derivative constant, and then q large. The factor $e^{2\tau q}$ and all other fixed exponential losses are retained in this order. Proposition 6.3 now proves the stated tail estimate. □

Remark 8.5 (Finite certificates; no computation in the proof)**.** Independent interval calculations give the following enclosures for sampled maxima of a frozen-fibre diagnostic. With $\xi = L/4$, let S_B be the maximum, over the two real Bloch corners, of the sum of pair-projector diagonals at both jump-bond endpoints, and put $Q_{\mathrm{old}} = z_\gamma\sqrt{\mathrm{ex} + 4\Lambda e^{-\xi q}}S_B$. For the finite proxy class $\mathrm{ex} \le \Lambda/q^3$, both turning signs, the certified intervals for the maximum are

λ	p	q	$\max Q_{\mathrm{old}}$
2	8	13	$[65.567943, 65.567944]$
2	13	21	$[917.687046, 917.687047]$
2	21	34	$[56263.586146, 56263.586147]$
3	21	34	$[6992639.730639, 6992639.730640]$
7	8	17	$[280360.523174, 280360.523175]$

These are finite certificates, rounded outward from rational interval endpoints. The central touching pair at even q is certified as a rank-two projector using the exact touching identity, residual and Gram bounds. The proxy class is different from (25); two Bloch corners do not certify a uniform physical operator estimate. The intervals do not prove an asymptotic obstruction for varying units. None is used above.

9 A second proof for the almost Mathieu operator

For the almost Mathieu operator the upper bound of Corollary 5.6 has a second proof, with different ingredients for both halves. The velocity comes from reflection pairs of interpolation nodes instead of full-period interpolation, and the confinement from the conserved polynomial of [20] instead of the zero count. We include it because it shows where the factor $1/2$ comes from in this model: at a turning phase every node has a mirror partner with almost the same cosine, so only about $q/2$ distinct nodes are available in a period.

In this section $\lambda > 1$, $L = \log\lambda$, $H = H_{\lambda,\alpha,\theta}$, and $\|x\|$ is the distance from x to $\mathbb{Z}$.

9.1 Reflection half-sets

Lemma 9.1 (Uniform nodes at every phase)**.** *There is an absolute constant C_L such that the following holds for every irrational α, every n with $q = q_n \ge 8$, every $\psi \in \mathbb{R}$ and every integer m_0. Put $\varphi_m = \psi + m\alpha$ and $M = \{m_0, \dots, m_0 + q - 1\}$. There is a set $\mathcal{J} \subset M$ of $\lfloor (q-1)/2 \rfloor$ consecutive integers such that for every $j \in \mathcal{J}$ and every $c \in [-1, 1]$,*

$$\prod_{i \in \mathcal{J}\setminus\{j\}} \frac{|c - \cos 2\pi\varphi_i|}{|\cos 2\pi\varphi_j - \cos 2\pi\varphi_i|} \le q^{C_L}.$$

The same holds, with a larger absolute constant, for every $\mathcal{J}' \subset \mathcal{J}$ with $|\mathcal{J} \setminus \mathcal{J}'| \le 3$.

Proof. Write $\mathsf{s}(x) = |\sin \pi x|$.

Reflection centre. Let c_* minimize $\|2\psi + c\alpha\|$ over the integers $c \in R_c = [2m_0, 2m_0 + 2q - 2]$, and let $\varepsilon_* \in (-1/2, 1/2]$ represent $2\psi + c_*\alpha$ modulo one. Since R_c contains q consecutive integers, the three-gap facts give $|\varepsilon_*| \le 2/q$. Put $\mu = c_*/2 \in [m_0, m_0 + q - 1]$. If $\mu \le m_0 + q/2 - 1$, let $\mathcal{J}$ be the set of integers in $(\mu, \mu + q/2)$; otherwise the set of integers in $(\mu - q/2, \mu)$. In both cases $\mathcal{J} \subset M$ is an interval of integers, and after removing one endpoint if necessary, $|\mathcal{J}| = \lfloor (q-1)/2 \rfloor$.

The involution. For $m \in M$ let $\iota(m)$ be the element of M with $\iota(m) \equiv c_* - m \pmod q$. Both $m + \iota(m)$ and c_* lie in R_c, so $m + \iota(m) = c_* + t_m q$ with $t_m \in \{-1, 0, 1\}$, and

$$\varphi_m + \varphi_{\iota(m)} \equiv \varepsilon_* + t_m \delta_n \pmod 1, \qquad |\varepsilon_* + t_m \delta_n| \le 3/q. \tag{50}$$

For distinct $j, m \in \mathcal{J}$ we have $0 < |j + m - c_*| < q$, because both lie on the same side of μ at distance less than $q/2$. So $\iota(\mathcal{J}) \cap \mathcal{J} = \emptyset$. Let $E_j = M \setminus ((\mathcal{J} \setminus \{j\}) \cup (\iota(\mathcal{J}) \setminus \{\iota(j)\}))$; then $|E_j| \le 6$.

Separation. Distinct $m, m' \in M$ have $\|\varphi_m - \varphi_{m'}\| \ge 1/(2q)$. For distinct $j, m \in \mathcal{J}$ put $d = j + m - c_*$, so $0 < |d| < q$ and $\|d\alpha\| \ge 1/(2q)$. Since $j + m \in R_c$, minimality gives $\|\varepsilon_* + d\alpha\| \ge \|\varepsilon_*\|$. Hence

$$\|\varphi_j + \varphi_m\| = \|\varepsilon_* + d\alpha\| \ge \max\{\|\varepsilon_*\|, \|d\alpha\| - \|\varepsilon_*\|\} \ge \frac{\|d\alpha\|}{2} \ge \frac{1}{4q}.$$

This lower bound holds even when $\varepsilon_* = 0$; it is what fails if both members of a mirror pair are kept.

Splitting. Use $|\cos 2\pi a - \cos 2\pi b| = 2\mathsf{s}(a+b)\mathsf{s}(a-b)$. For real a,

$$S(a) := \sum_{i \in \mathcal{J} \setminus \{j\}} \log |\cos 2\pi a - \cos 2\pi \varphi_i| = (|\mathcal{J}| - 1)\log 2 + S_-(a) + S_+(a),$$

with $S_-(a) = \sum_{i \in \mathcal{J} \setminus \{j\}} \log \mathsf{s}(a - \varphi_i)$ and, by (50), $S_+(a) = \sum_{i' \in \iota(\mathcal{J}) \setminus \{\iota(j)\}} \log \mathsf{s}(a + \eta_{i'} - \varphi_{i'})$ with $|\eta_{i'}| \le 3/q$.

Upper bound at every a. Let m_* minimize $\mathsf{s}(a - \varphi_m)$ over M. For $m \ne m_*$, $\|a - \varphi_m\| \ge 1/(4q)$, so $\mathsf{s}(a - \varphi_m) \ge 1/(2q)$, and the ordered distances give $\sum_{m \ne m_*} 1/\mathsf{s}(a - \varphi_m) \le Cq \log q$. Drop the m_*-term wherever it occurs; it is at most zero. For the shifted terms with $i' \ne m_*$, $\log \mathsf{s}(a + \eta - \varphi_{i'}) \le \log \mathsf{s}(a - \varphi_{i'}) + \pi|\eta|/\mathsf{s}(a - \varphi_{i'})$, and these corrections add up to at most $C \log q$. Adding back the at most six omitted terms of E_j, each at least $-\log(2q)$, and using (10),

$$S_-(a) + S_+(a) \le \sum_{m \in M,\, m \ne m_*} \log \mathsf{s}(a - \varphi_m) + C \log q \le -(q-1)\log 2 + C' \log q.$$

Lower bound at $a = \varphi_j$. Now $m_* = j$. The unshifted part is at least $\sum_{m \in M,\, m \ne j} \log \mathsf{s}(\varphi_j - \varphi_m) \ge -(q-1)\log 2 - C \log q$ by (10), since removing terms that are at most zero only increases a sum. Each shift changes a term by $\log \mathsf{s}(\varphi_j + \varphi_{\iota^{-1}(i')}) - \log \mathsf{s}(\varphi_j - \varphi_{i'})$. Both arguments have sine at least $1/(2q)$ by the separation step. For the indices with $\|\varphi_j - \varphi_{i'}\| \le 12/q$, of which there are at most 49 by the separation $1/(2q)$, the change is at least $-\log(2q)$. For the others $\mathsf{s}(\varphi_j - \varphi_{i'}) \ge 24/q$, so $\pi|\eta_{i'}|/\mathsf{s}(\varphi_j - \varphi_{i'}) \le \pi/8 < 1/2$, and the change is at least $-2\pi|\eta_{i'}|/\mathsf{s}(\varphi_j - \varphi_{i'})$, and these add up to at least $-C \log q$. So $S_-(\varphi_j) + S_+(\varphi_j) \ge -(q-1)\log 2 - C'' \log q$.

Conclusion. Every $c \in [-1, 1]$ is $\cos 2\pi a$ for a real a, and the logarithm of the product in the lemma is $S(a) - S(\varphi_j) \le (C' + C'') \log q$. Removing at most three further nodes changes only the number of omitted factors by a bounded amount. □

9.2 Velocity $\lambda^{-q/2}$

Fix $0<\zeta<L/8$ and $h=L+\zeta$. By Lemma 3.2 at height zero and continuity of ℓ there are $r_0>0$, a real neighbourhood U_h' of Σ and C_h such that $\|A_r(z,x)\|\le C_h e^{hr}$ for all $z=E+i\eta$ with $E\in U_h'$ and $0\le\eta\le r_0$, all real x and all $r\ge 0$. For $q=q_n$ put $k_1=\lfloor(\lfloor(q-1)/2\rfloor-2)/2\rfloor$ and $k=2k_1+1$, so that $k+1\le\lfloor(q-1)/2\rfloor$ and $k\ge q/2-4$.

Lemma 9.2 (Regular intervals for the almost Mathieu operator)**.** *For all large n, every phase θ, every $z=E+i\eta$ with $E\in U_h'$ and $0\le\eta\le r_0$, and every set $W=[w,w+q+2k_1-1]$ of consecutive sites, there is $x\in[w+k_1,w+k_1+q-1]$ such that $I=[x-k_1,x+k_1]\subset W$ satisfies $|\det(z-H_I)|\ge e^{(L-\zeta)k}$. For every solution of $(H-z)g=0$ on I and $y\in\{x,x+1\}$,*

$$|g(y)|^2\le 2\gamma_0^2\bigl(|g(x-k_1-1)|^2+|g(x+k_1+1)|^2\bigr),\qquad \gamma_0=C_he^he^{-(L/2-2\zeta)k}.$$

Proof. The determinant of $z-H$ on an interval of odd length k centred at a site with phase φ is a polynomial $Q_k(z;\cos 2\pi\varphi)$ of degree at most k in $\cos 2\pi\varphi$: it is a trigonometric polynomial of degree k in φ, and reversing the interval about its centre changes φ into $-\varphi$ without changing the determinant. Apply Lemma 9.1 with $\psi=\theta$ and $M=[w+k_1,w+k_1+q-1]$, so that φ_m is the centre phase of $[m-k_1,m+k_1]\subset W$, and take $\mathcal{J}'\subset\mathcal{J}$ with $k+1$ elements. If $|Q_k(z;\cos 2\pi\varphi_m)|\le e^{(L-\zeta)k}$ for all $m\in\mathcal{J}'$, Lagrange interpolation through these $k+1$ nodes gives $\sup_{[-1,1]}|Q_k(z;\cdot)|\le(k+1)q^Ce^{(L-\zeta)k}$, so $\int_{\mathbb{T}}\log|\det(z-H_{[0,k-1]}(\theta'))|\,d\theta'\le(L-\zeta)k+C\log q<kL$ for large n. This contradicts Herman's bound $\int_{\mathbb{T}}\log|\det(z-H_{[0,k-1]}(\theta'))|\,d\theta'\ge kL$, valid for every $z\in\mathbb{C}$ [20, proof of Lemma 6.2]: the determinant is a trigonometric polynomial in θ' whose extreme coefficient has modulus λ^k. So some $m=x\in\mathcal{J}'$ gives a regular interval. The node inequality follows from the Poisson formula and Cramer's rule as in Lemma 4.3: the numerators have length at most k_1+1 and are at most $C_he^{h(k_1+1)}$, and $hk_1-(L-\zeta)(2k_1+1)\le-(L/2-2\zeta)k$. □

Theorem 9.3 (Velocity for the almost Mathieu operator)**.** *For every $0<\zeta<L/8$ there are n_0, T_0, C and $\mu>0$ such that for all $n\ge n_0$, every phase θ, every $T\ge T_0$ and every $N\ge 1$,*

$$P_\theta(N,T)\le 2\exp\Bigl(-\frac{N/(4q_n)-2}{1+\kappa_nT}\Bigr)+Ce^{-\mu N},\qquad \kappa_n=4C_h^2e^{2h}e^{-(L-4\zeta)k}\le C'\lambda^{-q_n/2}e^{2\zeta q_n}.$$

Proof. The proof of Theorem 4.4, with Lemma 9.2 in place of Lemmas 4.2 and 4.3, windows $W_i=[1+(i-1)(q+2k_1),\,i(q+2k_1)]$ of length at most $2q$, $T_0\ge 1/r_0$, and the Combes–Thomas bound for energies outside U_h'. The left half-line is the right half-line of $H_{\lambda,\alpha,-\theta}$, since the cosine is even. □

9.3 Confinement from the conserved polynomial

We use the conserved polynomial $Q=P_q(H)=V+K$ of [20, Proposition 3.3], with $V(j)=2\lambda^q\cos(2\pi(q\theta+j\delta))$, K of range q, and its normalization $\|K\|\le J_n$, $\sigma_n=J_n/\lambda^{q_n}$ [20, eq. (10)].

Proposition 9.4 (Confinement for the almost Mathieu operator)**.** *Fix $c>0$ and $0<\varepsilon<\min\{L,c\}/4$, and use [20, eq. (10)] with $h=L+\varepsilon/8$. There are an absolute constant C_0 and a constant C such that for all large n with $b_n\ge c$, every phase θ and all $T>0$,*

$$P_\theta(R_n,T)\le C(1+T^2)\exp\Bigl(-\frac{e^{\varepsilon q_n/16}}{q_n}\Bigr),\qquad R_n=\lceil C_0\sigma_n^{1/2}|\delta_n|^{-1}\rceil+6G_n,\quad G_n=\lceil q_ne^{\varepsilon q_n/2}\rceil,$$

and $R_n\le e^{(b_n-\min\{L,b_n\}/2+\varepsilon)q_n}$.

Proof. Write $q = q_n$, $\delta = \delta_n$, $G = G_n$, $\kappa = 1/\sigma_n$ and $u = q\theta$. Apply the barrier lemmas of [20, Section 4] to $Q' = Q/J_n = V' + K'$, where $\|K'\| \le 1$, K' has range q and $V'(j) = 2\kappa\cos(2\pi(u + j\delta))$, with $d_0 = V'(0)$, $W = 8$ and $V_0 = 16$. Dividing Q by J_n changes neither its spectral projections nor its commutation with H.

Sublevel geometry. With $x = 2\pi(u + j\delta)$ and $x_0 = 2\pi u$, the set $\{j : |V'(j) - d_0| < 32\} = \{j : |\cos x - \cos x_0| < 16/\kappa\}$ meets one period, $|j| \le 1/(2|\delta|)$, in at most two arcs, each of x-length at most $C\kappa^{-1/2}$, uniformly in x_0. On the two complementary gaps $V' - d_0$ has a fixed sign and modulus at least 32. Let $\mathcal{J}_0$ be the arc containing 0, merged with the other arc if one of the two gaps between them has fewer than $4G$ sites. Since $\kappa \ge e^{(\min\{L,c\}-\varepsilon/4)q}$ is huge and the period has $|\delta|^{-1}$ sites, the two gaps cannot both be short, and after the merging rule each gap adjacent to $\mathcal{J}_0$ has at least $4G$ sites. (One of the gaps can be short near a turning value, so both gaps need not be long; this is why the merging rule is used.) Thus $\mathcal{J}_0$ is an integer interval containing 0 of length at most $C_0\kappa^{-1/2}|\delta|^{-1} + 4G$, contained in $[-(R_n - G), R_n - G]$ after enlarging C_0, and followed on each side by at least $4G$ sites on which $V' - d_0$ has a fixed sign and modulus at least 32.

Outer barrier. Write $\mathcal{J}_0 = [e_-, e_+]$, $N_+ = e_+ + 2G$ and $N_- = |e_-| + 2G$, both at most R_n. The intervals $I_+ = [e_+ + 1, e_+ + 3G]$ and $I_- = [e_- - 3G, e_- - 1]$ lie in the adjacent gaps, and the pairs $\{N_+, N_+ + 1\}$ and $\{-N_- - 1, -N_-\}$ are at distance at least $G - 1$ from their complements. [20, Lemma 4.1], with $J = 1$, $V_0 - W = 8$ and $\ell = \lfloor (G-2)/q \rfloor - 2$, gives $\|\mathbf{1}_B \Pi\| \le 2 \cdot 8^{-(\ell+2)} \le \exp(-e^{\varepsilon q/2}/q)$ for the set B of these four sites and $\Pi = \mathbf{1}_{[d_0 - W, d_0 + W]}(Q')$.

Inner core. For $|j| \le r$, $|V'(j) - d_0| \le 4\pi\kappa|j\delta| + 4\pi^2\kappa j^2\delta^2 \le 2 = W/4$ provided $r \le \min\{1/(8\pi\kappa|\delta|), (8\pi^2\kappa)^{-1/2}|\delta|^{-1}\}$. From $J_n \ge C_h q^{5/2} e^{(L+\varepsilon/8)q}|\delta|$ we get, at every scale,

$$\kappa|\delta| \le C^{-1} q^{-5/2} e^{-\varepsilon q/8}, \qquad \kappa^{-1/2}|\delta|^{-1} \ge C^{1/2} q^{5/4} e^{(b_n + \varepsilon/8)q/2}.$$

Since $b_n \ge c > 4\varepsilon$, we may take $r \ge e^{\varepsilon q/16}$ for large q. [20, Lemma 4.2] with $\ell_0 = \lfloor r/q \rfloor - 2$ gives $\|(1 - \Pi)\delta_0\| \le 3 \cdot 6^{-(\ell_0+2)}$.

Cutoff. The proof of [20, Lemma 4.3] works verbatim for the asymmetric window $\chi = \mathbf{1}_{[-N_-, N_+]}$, since $[H, \chi]$ still has norm one and lives on the two boundary bonds. It bounds the Abel-averaged mass outside $[-N_-, N_+]$, which dominates $P_\theta(R_n, T)$, by $8\|(1 - \Pi)\delta_0\|^2 + T^2\|\mathbf{1}_B \Pi\|^2$. This gives the bound. All constants are independent of θ. Finally $\sigma_n \le e^{-(\min\{L, b_n\} - \varepsilon/4)q}$ by [20, Lemma 3.4], which gives the bound on R_n. □

Corollary 9.5. *For the almost Mathieu operator with $\beta(\alpha) < \infty$, the conclusion $\sup_\theta P_\theta(T^a, T) = O(T^{-m})$ for every $a > f(\beta; L, 1)$ and $m > 0$ follows from Theorem 9.3 and Proposition 9.4, without Sections 3 and 4.*

Proof. Run the covering argument of Theorem 5.2. In Case A use Theorem 9.3. In Case B both Theorem 9.3 and Proposition 9.4 bound the same outside probability, uniformly in energy, so $P_\theta(\min\{N_1, R_n\}, T) \le T^{-m}$ with $N_1 = 8q(m \log T + 2)(1 + C'Te^{-(L/2 - 2\zeta)q})$, and

$$\frac{\log \min\{N_1, R_n\}}{q} \le \min\{\varrho(b; L, 1) + \varepsilon,\ (\tau - L/2)_+ + 2\zeta\} + o(1).$$

By (12) the exponent is at most $f(b; L, 1) + K(\varepsilon + 2\zeta) + o(1)$, and ε, ζ can be taken small. □

10 Sharpness: closed gaps and the exceptional phases

The factor 2ω in Theorem 5.2 comes from the worst possible cluster of zeros. This section shows that it is attained. The mechanism is a gap of a periodic approximant that closes, or nearly closes, at a particular phase: next to a closed gap the band is wide, of width about $e^{-\ell q/2}$ rather than $e^{-\ell q}$, and a packet launched where the local phase sits at that phase moves with the corresponding speed.

10.1 Closed central gaps

Call v *odd-harmonic* if $v(x+1/2) = -v(x)$. The cosine is odd-harmonic. Write $T(y) = \left(\begin{smallmatrix} y & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $K_0 = \mathrm{diag}(1,-1)$, so that $K_0 T(y) K_0 = -T(-y)$.

Proposition 10.1 (Closed central gaps)**.** *Let v be continuous and odd-harmonic, let p/q be in lowest terms with q even, and put $s = (-1)^{q/2}$ and $v_k = v(\varphi + kp/q)$. Let $M_1(\varphi) = T(-v_{q/2-1}) \cdots T(-v_0)$ be the transfer matrix at $E = 0$ over half a period, and $g(\varphi) = (M_1)_{11} - (M_1)_{22}$. Then:*

1. *$\Delta_\varphi(0) = s(g(\varphi)^2 + 2)$ for every φ;*
2. *$g(\varphi + p/q) = -g(\varphi)$, so g has a zero φ_0, and the monodromy of the p/q-periodic operator at $E = 0$ and phase φ_0 is sI: the gap at $E = 0$ is closed;*
3. *$\Delta_\varphi(-E) = \Delta_\varphi(E)$ for all φ and E.*

Proof. Since q is even, p is odd, so $v_{k+q/2} = v(\varphi + kp/q + 1/2) = -v_k$. Hence the second half period contributes $\prod_k T(v_k) = \prod_k (-K_0 T(-v_k) K_0) = (-1)^{q/2} K_0 M_1 K_0$, and the monodromy is $M = s K_0 M_1 K_0 M_1$. If $M_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $K_0 M_1 K_0 M_1 = \left(\begin{smallmatrix} a^2 - bc & b(a-d) \\ c(d-a) & d^2 - bc \end{smallmatrix}\right)$, whose trace is $(a-d)^2 + 2$ because $ad - bc = 1$. This is part 1. At a zero of g the matrix $K_0 M_1 K_0 M_1$ is the identity, which gives the closed gap in part 2.

Shifting φ by p/q moves the half period by one site: $M_1(\varphi + p/q) = T(-v_{q/2}) M_1(\varphi) T(-v_0)^{-1} = T(y) M_1(\varphi) T(-y)^{-1}$ with $y = v_0$. A direct computation shows that the diagonal of $T(y) M_1 T(-y)^{-1}$ is $(d - yb,\ a - yb)$, so its diagonal difference is $d - a = -g(\varphi)$. The intermediate value theorem gives a zero.

For part 3, odd-harmonicity gives

$$A_q^{(p/q)}(-E, \varphi + 1/2) = \prod_k T(-(E - v_k)) = (-1)^q K_0 A_q^{(p/q)}(E, \varphi) K_0,$$

so $\Delta_{\varphi+1/2}(-E) = \Delta_\varphi(E)$. Since $1/2 = (q/2) \cdot (1/q)$ is a period of $\varphi \mapsto \Delta_\varphi$, part 3 follows. □

Corollary 10.2. *For the almost Mathieu operator and even q, the Chambers polynomial satisfies $P_q(0) = s(2\lambda^q + 2)$ and $P_q(-E) = P_q(E)$.*

Proof. By (9), $s\Delta_\varphi(0) = sP_q(0) - 2s\lambda^q \cos(2\pi q \varphi)$, whose minimum over φ is $sP_q(0) - 2\lambda^q$. By Proposition 10.1(1)–(2), this minimum is 2. Evenness follows from part 3 and (9). □

Remark 10.3 (Priority)**.** The closed middle gap of even-denominator almost Mathieu approximants is classical; see Bellissard and Simon [7, Theorem 3]. The identity of Corollary 10.2 is due to Lamoureux and Mingo [36, Corollary 2.6 and Theorem 2.7]; their value $2(-1)^{q/2}(1 + (\lambda'/2)^q)$ is ours with $\lambda' = 2\lambda$. Proposition 10.1 gives an elementary proof that applies to every continuous odd-harmonic potential.

10.2 A wide band at the turning phase

For even $q = q_n$ write $s = (-1)^{q/2}$ and $\Delta_t(E) = P_q(E) - 2s\lambda^q$, the discriminant at the turning phases φ with $\cos(2\pi q\varphi) = s$.

Proposition 10.4. *Let $\lambda > 1$, $h > L$ and $0 < \eta \le 1/4$, and let C_h be the constant of [20, Proposition 3.3(2)] for $r = 2$, so that $\sup_{U_h} |P_q''| \le C_h q^2 e^{hq}$. For all large n with $q = q_n$ even there is $E^{(1)} \ge 0$ with*

$$|\Delta_t(E^{(1)})| = 2 - \eta \qquad \text{and} \qquad |P_q'(E^{(1)})| \le q\sqrt{2C_h\eta}\, e^{hq/2}.$$

Proof. At a phase with $\cos(2\pi q\varphi)=0$, (9) gives $t_q=P_q$, so every root of P_q is real and simple [21, Lemma 5.3]. Away from its roots, $(\log|P_q|)''=-\sum_j(E-r_j)^{-2}<0$, so every critical point of P_q is a strict local maximum of $|P_q|$. By Corollary 10.2, $P_q'(0)=0$ and sP_q has a strict local maximum at 0 with value $2\lambda^q+2$. Let $f(E)=s(P_q(0)-P_q(E))$ and let $E^{(1)}$ be the first $E>0$ with $|f(E)|=\eta$. On $[0,E^{(1)}]$, $sP_q>0$, and $f\ge 0$: otherwise sP_q would exceed its value at 0 somewhere in $(0,E^{(1)})$, forcing a positive local minimum of $|P_q|$ in between. So $f(E^{(1)})=\eta$ and $s\Delta_t=2-f\in[2-\eta,2]$ on $[0,E^{(1)}]$. This interval therefore lies in the spectrum of a turning approximant, hence in U_h for large n [20, Lemma 2.6], where $|f''|\le M:=C_hq^2e^{hq}$. Since $f(0)=f'(0)=0$ and $f'(E^{(1)})\ge 0$, we have $f'(E^{(1)})\le ME^{(1)}$. With $t=f'(E^{(1)})/M\in[0,E^{(1)}]$, Taylor's formula gives $0\le f(E^{(1)}-t)\le\eta-f'(E^{(1)})^2/(2M)$. Hence $|P_q'(E^{(1)})|=f'(E^{(1)})\le\sqrt{2M\eta}$. □

10.3 The lower bound at the phase $1/4$

Theorem 10.5. *Let $\lambda>1$ and α irrational. For every $h>L$ and $\eta\in(0,1/4]$ there are $c_\eta>0$ and n_0 such that for every $n\ge n_0$ with q_n even and $b_n\ge h/2+(h-L)$, and every T with $e^{(h/2+(h-L))q_n}\le T\le|\delta_n|^{-1}$,*

$$P_\varphi\big(\lfloor c_\eta Te^{-hq_n/2}\rfloor,\,T\big)\ge T^{-2}e^{-2hq_n} \tag{51}$$

holds for $\varphi=1/4$, and, with modified constants, for every phase φ with $\|\varphi-1/4\|\le 2|\delta_n|$ and for every phase φ with $|\cos(2\pi q_n\varphi)-(-1)^{q_n/2}|\le\eta\lambda^{-q_n}/16$. Consequently

$$\alpha_u^+(H_{\lambda,\alpha,1/4})\ge\max\Big\{0,\ 1-\frac{L}{2\beta_e(\alpha)}\Big\},$$

read as 1 if $\beta_e(\alpha)=\infty$.

Proof. Write $q=q_n$, $p=p_n$ (odd), $\delta=\delta_n$, $s=(-1)^{q/2}$ and $\varepsilon=1/T$, so $\varepsilon\ge|\delta|$. We first treat $\varphi=1/4$.

Step 0: the turning block. The first q-block of $H_{1/4}$ to the right, sites $1,\dots,q$, has phases $x_0+k\alpha$ with $x_0=1/4+\alpha$. Compare it with the rational block $R(E)=A_q^{(p/q)}(E,x_0')$, $x_0'=1/4+p/q$. Since $\cos(2\pi qx_0')=\cos(\pi q/2)=s$, (9) gives $\operatorname{tr}R(E)=\Delta_t(E)$. The phase discrepancy at site k is $(k+1)\delta/q$, so the potentials differ by $|w_k|\le 4\pi\lambda|\delta|$.

Step 1: an energy window. Let $E^{(1)}$ be as in Proposition 10.4 and $I_*=[E^{(1)}-r,E^{(1)}+r]$ with $r=\eta e^{-hq/2}/(C_1q)$ and C_1 large. For $E\in I_*$, which lies in U_h, $|P_q'(E)|\le C_2qe^{hq/2}$ and $|\Delta_t(E)-\Delta_t(E^{(1)})|\le\eta/2$, so $|\operatorname{tr}R(E)|\le 2-\eta/2$, $|\sin\varrho_R|\ge\sqrt{\eta}/2$ for the rotation angle ϱ_R, and by [20, Lemma 8.2(1)] the block sum satisfies $S_R=|P_q'(E)|/|\sin\varrho_R|\le C_3qe^{hq/2}$.

Step 2: insertion. Let $B_R\in SL(2,\mathbb{R})$ map i to the fixed point of $R(E)$. The insertion lemma [20, Lemma 9.3] with the shifts w_k gives $A_k(E,x_0)B_R=A_k^{(p/q)}(E,x_0')B_R\Pi_k$ with

$$\|\Pi_k-I\|\le y_0:=\exp(4\pi\lambda|\delta|S_R)-1\le Cqe^{-(h-L)q},$$

using $|\delta|\le e^{-(h/2+(h-L))q}$. So $|\operatorname{tr}A_q(E,x_0)|\le 2-\eta/2+2y_0\le 2-\eta/4$. By [20, Lemma 9.4] the fixed point of $B_R^{-1}A_q(E,x_0)B_R$ is within hyperbolic distance $12y_0/(\sqrt{\eta}/2)$ of i; let $N\in SL(2,\mathbb{R})$ map i to it, so $\|N\|^2\le 2$, and put $B_A=B_RN$, which maps i to the fixed point of $A_q(E,x_0)$. The block sum of $A_q(E,x_0)$ in this frame is $S_0\le 2(1+y_0)^2S_R\le 3C_3qe^{hq/2}$.

Step 3: an elliptic run. Apply [20, Lemma 9.6] to $g(k)=G(k;E+i\varepsilon)=\langle\delta_k,(H_{1/4}-E-i\varepsilon)^{-1}\delta_0\rangle$, $k\ge 0$, which solves the half-line equation for $k\ge 1$, with ellipticity margin $\eta'=\eta/4$, frame B_A, block sum S_0, and $\xi=(4\pi\lambda+1)\varepsilon$. Its run length satisfies $m_*\ge c_\eta Te^{-hq/2}/q\ge 8$ for

large n, because $Te^{-hq/2} \geq e^{(h-L)q}$. With $m_0 = \lceil m_*/2 \rceil$ the lemma gives $\sum_{k>m_0 q} |g(k)|^2 \geq \eta'^{5/2} 2^{-22} \|\Phi_0\|^2/\xi$ and $|g(0)| \leq \|B_A\|\,\|\Phi_0\|$. By [20, Lemma 8.2(4)] and the growth bound, $\|B_A\|^2 \leq 1+2\|A_q(E,x_0)\|/|\sin \varrho_A| \leq C_\eta e^{hq}$, and $|G(0;E+i\varepsilon)| \geq \operatorname{Im} G(0;E+i\varepsilon) \geq \varepsilon/D_\lambda^2$ with $D_\lambda = 4(1+\lambda)+1$. Hence

$$\sum_{k>m_0 q} |G(k;E+i\varepsilon)|^2 \geq c'_\eta\, \varepsilon\, e^{-hq} \qquad (E \in I_*).$$

A common cutoff. The run length depends on E, so we fix one radius before integrating: $N_* = \lfloor c_\eta T e^{-hq/2} \rfloor$, with c_η reduced so that $N_* \leq m_0(E)q$ for every $E \in I_*$. By monotonicity of tails, $\sum_{k>N_*} |G(k;E+i\varepsilon)|^2 \geq c'_\eta \varepsilon e^{-hq}$ on I_*.

Step 4: Parseval. By (7),

$$P_{1/4}(N_*,T) \geq \frac{1}{\pi T} \int_{I_*} \sum_{k>N_*} |G(k;E+i/T)|^2\, dE \geq \frac{2r}{\pi T}\, c'_\eta T^{-1} e^{-hq} \geq \frac{c}{q} T^{-2} e^{-3hq/2} \geq T^{-2} e^{-2hq}$$

for large n. This is (51).

Nearby phases. If $\|\varphi - 1/4\| \leq 2|\delta|$, compare the first block of H_φ with the same rational block R. The discrepancies become $(\varphi - 1/4) + (k+1)\delta/q$, of size at most $3|\delta|$, so $|w_k| \leq 12\pi\lambda|\delta|$, and Steps 1–4 hold with modified constants. If $|\cos(2\pi q\varphi) - s| \leq \eta\lambda^{-q}/16$, compare the first block, with phases $\varphi + \alpha + k\alpha$, with $A_q^{(p/q)}(E, \varphi + p/q)$; its trace $P_q(E) - 2\lambda^q \cos(2\pi q\varphi)$ is within $\eta/8$ of $\Delta_t(E)$, the derivative P'_q is unchanged, and the discrepancies are again $(k+1)\delta/q$. So Steps 1–4 hold with $\eta/2$ replaced by $3\eta/8$ and the same exponents.

The exponent. At $T = T_n = |\delta_n|^{-1} = e^{b_n q}$, (51) gives a tail at least T_n^{-2-2h/b_n} beyond $N_* \geq c T_n^{1-h/(2b_n)} - 1$. Since $M_p(T) \geq N^p P(N,T)$, $\beta^+(p) \geq 1 - h/(2b_n) - (2+2h/b_n)/p$ along even n with $b_n \to \beta_e$, provided $\beta_e > h/2 + (h-L)$. Let $p \to \infty$ and then $h \downarrow L$. If $\beta_e \leq L/2$ there is nothing to prove; if $\beta_e = \infty$ the same argument with $b_n \to \infty$ gives 1. □

The theorem gives a polynomially small tail at the critical distance, not a fixed fraction of the packet.

10.4 Exact laws for the almost Mathieu operator

Corollary 10.6 (Exact uniform law)**.** *Let $\lambda > 1$ and let α satisfy $\beta_e(\alpha) = \beta(\alpha) \in (L/2, \infty)$, for instance when all denominators with large b_n are even. Then*

$$\alpha_u^{\mathrm{unif}} = \sup_\theta \alpha_u^+(H_{\lambda,\alpha,\theta}) = \alpha_u^+(H_{\lambda,\alpha,1/4}) = 1 - \frac{L}{2\beta(\alpha)}.$$

Proof. Theorem 10.5 gives $\alpha_u^+(1/4) \geq 1 - L/(2\beta)$, Theorem 6.1 for $\beta < L$, and Corollary 5.6 for $\beta \geq L$, give $\alpha_u^{\mathrm{unif}} \leq 1 - L/(2\beta)$, and (8) closes the chain. □

Remark 10.7. At almost every phase the exponent is $\max\{0, 1 - L/\beta(\alpha)\}$ [20, Theorem A]. So under the hypothesis of Corollary 10.6, with $\beta < \infty$, the worst phase has exactly half the typical localization strength.

Corollary 10.8 (The exponent at the phase 1/4)**.** *Let $\lambda > 1$ and $\beta_e(\alpha) \geq L$. Then*

$$\alpha_u^+(H_{\lambda,\alpha,1/4}) = \max\Big\{0,\ 1 - \frac{L}{\beta_o(\alpha)},\ 1 - \frac{L}{2\beta_e(\alpha)}\Big\},$$

read as 1 if $\beta(\alpha) = \infty$.

Proof. Lower bound. The lower bound $\alpha_u^+ \geq \max\{0, 1 - L/\beta\}$ holds at every phase [20, Theorem A], and Theorem 10.5 gives $1 - L/(2\beta_e)$. Since $\beta = \max\{\beta_e, \beta_o\}$, the larger of the two bounds is the displayed maximum. If $\beta = \infty$ the value is 1, by [20, Theorem A(2)] or Theorem 10.5.

Upper bound. Let $\beta < \infty$ and $a > \max\{0, 1 - L/\beta_o, 1 - L/(2\beta_e)\}$, and run the covering argument of Theorem 5.2 at the phase 1/4, with $B_0^e > \beta_e$ and $B_0^o > \beta_o$ chosen so that $f(B_0^e; L, 1) < a$ and $\max\{0, 1 - L/B_0^o\} < a$. Case A is unchanged. In Case B with q_n even, the per-energy argument of Theorem 5.2, with $\omega = 1$ and $\ell = L$ on Σ, gives exponent at most $f(B_0^e; L, 1) + o(1) < a$; here $f(b; L, 1) \leq \max\{1/2, 1 - L/(2b)\}$ and $1/2 \leq 1 - L/(2\beta_e)$ because $\beta_e \geq L$. In Case B with q_n odd, apply [20, Proposition 7.1] at the single scale n. Its proof uses the phase only through the sine condition $|\sin 2\pi q_n \theta| \geq q_n^{-2}$ at that scale, via [20, Proposition 4.4 and Theorem 5.4], which are stated for every phase satisfying it, and through $b_n \leq B_0$, which holds for large odd n with $B_0 = B_0^o$. At $\theta = 1/4$ and odd q_n, $|\sin(\pi q_n/2)| = 1$. So $P_{1/4}(T^a, T) \leq CT^{-m}$ in this case too. □

Corollary 10.9 (Exceptional phases)**.** *If $\lambda > 1$ and $\beta_e(\alpha) = \beta(\alpha) \in (L/2, \infty)$, then*

$$\alpha_u^+(H_{\lambda,\alpha,1/4}) \geq 1 - \frac{L}{2\beta(\alpha)} > \max\Big\{0, 1 - \frac{L}{\beta(\alpha)}\Big\} = \alpha_u^+(H_{\lambda,\alpha,\theta}) \quad \textit{for almost every } \theta.$$

In particular, if $e^{\beta(\alpha)} < \lambda < e^{2\beta(\alpha)}$, the phase 1/4 has a positive transport exponent, while for almost every phase the operator has Anderson localization by the theorem of Avila, You and Zhou [5].

Proof. Theorem 10.5 and [20, Theorem A]. □

Remark 10.10 (Reflection resonance and related work)**.** For even q_n and odd p_n,

$$2 \cdot \frac{1}{4} + \frac{q_n}{2}\,\alpha = \frac{1}{2} + \frac{p_n}{2} + \frac{\delta_n}{2} \equiv \frac{\delta_n}{2} \pmod 1.$$

So 1/4 is a reflection-resonant phase at every even scale, with reflection centres at distance $q_n/4$, and its phase-resonance exponent in the sense of Jitomirskaya and Liu [30] is at least $2\beta_e$. The phase 1/4 is not of the completely resonant type $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$ treated by Liu [40, Theorem 1.1], and that localization theorem does not apply to it. That localization is not uniform over phases or sites for the almost Mathieu operator is known in other forms: Jitomirskaya, Liu and Mi gave a sharp palindromic criterion for semi-uniform dynamical localization, with almost Mathieu examples that are dynamically localized but not semi-uniformly [31]. Corollaries 10.6–10.9 give the corresponding exact power laws for transport.

Remark 10.11 (The plateau radius)**.** Fix sufficiently small $\xi > 0$ and choose $L < h < L + 2\xi/3$. On even scales $L/2 + \xi \leq b_n \leq L - \xi$, Theorem 10.5 at $T = |\delta_n|^{-1}$ gives a polynomial tail at distance $e^{(b_n - h/2)q_n}$, up to fixed and polynomial factors. The frame and interior estimates give radius at most $e^{(b_n - L/2 + C\xi)q_n}$ on each fixed exponential time window. Their ratio is at most a polynomial factor times

$$\exp\big(((h - L)/2 + C\xi)q_n\big)\,.$$

Its exponential rate can be made arbitrarily small by choosing $h \downarrow L$ and $\xi \downarrow 0$ before taking large scales. For $b_n \geq L - \xi$ the earlier confinement estimate of Proposition 9.4 gives the same Abel-tail radius rate with an additional $O(\xi)$ loss, at its stated time scales. Only polynomially small mass is proved at the exceptional radius; a fixed transported fraction remains open.

10.5 Odd-harmonic and type I potentials

The proof of Theorem 10.5 uses the almost Mathieu operator only through a wide band at one local phase. We isolate that input. For a real-analytic supercritical v, $\ell_0 > 0$, $\xi > 0$, $\eta \in (0, 1]$ and a set $\mathcal{N}'$ of indices, consider:

(W_v) for every $n \in \mathcal{N}'$ there are $x' \in \mathbb{T}$ and a real E' with $\operatorname{dist}(E', \Sigma) \le e^{-\xi q_n}$, $\ell(E') \le \ell_0 + \xi$, $|t_{q_n}(x', E')| \le 2 - \eta$ and $|\partial_E t_{q_n}(x', E')| \le e^{(\ell(E')/2+\xi)q_n}$.

Theorem 10.12 (A lower bound from a wide band)**.** *Let v be real analytic with $H_{v,\alpha}$ supercritical, and assume (W_v) along an infinite set $\mathcal{N}'$ with $b_n \to \beta'$ as $n \to \infty$ in $\mathcal{N}'$, where $\beta' > \ell_0/2$. There are constants $c, C > 0$, depending only on v, α, ℓ_0 and η, such that the following holds if ξ is small enough. For every phase θ and every large $n \in \mathcal{N}'$ there is a site x_n with $|x_n| \le |\delta_n|^{-1}$ and*

$$P_{\theta+x_n\alpha}\big(cT_n^{1-(\ell_0/2+C\xi)/b_n},\, T_n\big) \ge T_n^{-C}, \qquad T_n = |\delta_n|^{-1}.$$

Hence $\alpha_u^{\mathrm{unif}} \ge 1 - \ell_0/(2\beta') - C\xi$.

Proof. This is the proof of Theorem 10.5 with the following changes. Restrict to $n \in \mathcal{N}'$ with $b_n > \ell_0/2 + C\xi$; otherwise the claimed exponent is not positive. The rational block is the p_n/q_n-periodic block at the phase x'/q_n, with trace $t_{q_n}(x', E')$ of modulus at most $2 - \eta$, and [20, Lemma 8.2(1)] gives $S_R \le Ce^{(\ell(E')/2+\xi)q_n}$. The energy window around E' has width $e^{-(\ell(E')/2+C\xi)q_n}/(Cq_n^2)$; on it the bound $|\partial_E^2 t_q| \le 8q^2 e^{q(\ell(E')+\xi)}$ [21, Corollary 5.6] keeps the trace elliptic with a fixed margin and the derivative of size $e^{(\ell(E')/2+C\xi)q}$. The local phases $q_n(\theta + j\alpha) = q_n\theta + j\delta_n$ sweep $\mathbb{T}$ as j ranges over an interval of length $|\delta_n|^{-1}$, so there is a site x_n, $|x_n| \le |\delta_n|^{-1}$, at which the first block to the right has local phase within $|\delta_n|$ of x'. Closeness of local phases identifies the block only up to a cyclic shift of the rational potential; cyclic shifts have the same trace and the same energy derivative, and uniform growth controls their frames. The insertion error is at most $\exp(C\|v'\||\delta_n|S_R) - 1 \to 0$, since $b_n > \ell(E')/2 + 2\xi$. The elliptic run lemma [20, Lemma 9.6] holds for Lipschitz potentials, $\|A_q\| \le Ce^{(\ell(E')+\xi)q}$ by Lemma 3.2, and the rest of the proof is unchanged. The last statement follows as in Section 2: if $\sup_\varphi P_\varphi(T^a, T) = O(T^{-m})$ for all m, then $a \ge 1 - (\ell_0/2 + C\xi)/\beta'$. □

Theorem 10.13 (Odd-harmonic potentials)**.** *Let v be real analytic and odd-harmonic, with $H_{v,\alpha}$ supercritical. Then $0 \in \Sigma$, and*

$$\alpha_u^{\mathrm{unif}} \ge \max\Big\{0,\ 1 - \frac{\ell(0)}{2\beta_e(\alpha)}\Big\},$$

read as 1 if $\beta_e(\alpha) = \infty$. More precisely, for every $\varepsilon > 0$ there is C such that for every phase θ and all large even q_n with $b_n \ge \ell(0)/2+\varepsilon$ there is a site x_n, $|x_n| \le |\delta_n|^{-1}$, with $P_{\theta+x_n\alpha}(T_n^{1-(\ell(0)/2+\varepsilon)/b_n}, T_n) \ge T_n^{-C}$.

Proof. $0 \in \Sigma$. Reduced fractions p/q with even q approach α. By Proposition 10.1, 0 lies in the spectrum of the p/q-periodic operator at some phase, and every point of the phase-union rational spectrum is within $C|\alpha - p/q|^{1/2}$ of Σ [21, Lemma 2.7(1)].

(W_v) *at even scales.* Let $q = q_n$ be even and φ_0 the phase of Proposition 10.1(2), and write $\Delta = \Delta_{\varphi_0}$ and $s = (-1)^{q/2}$. Then $\Delta(0) = 2s$, Δ is even, and the bands of the periodic operator adjacent to 0 lie on both sides of 0, with $s\Delta \le 2$ on them. Let $f = 2 - s\Delta \ge 0$ on these bands, so $f(0) = f'(0) = 0$. Let $E^{(1)} > 0$ be the first point with $f(E^{(1)}) = \eta$. Every band of a q-periodic operator with hopping one has width at most $2\pi/q$ [21, proof of Lemma 5.4], so $E^{(1)} \to 0$, $\ell(E^{(1)}) \to \ell(0)$, and by [21, Lemma 2.7(1)] the interval $[0, E^{(1)}]$ is within $C(|\delta|/q)^{1/2} \le e^{-\xi q}$ of Σ

when b_n is bounded below and ξ is small. On it [21, Corollary 5.6] gives $|f''| \le M = 8q^2 e^{q(\ell(0)+\xi)}$ for large q. With $t = f'(E^{(1)})/M \le E^{(1)}$, Taylor's formula on $[E^{(1)} - t, E^{(1)}]$ gives $|\Delta'(E^{(1)})|^2 \le 2M\eta$. So (W_v) holds at every large even q_n with $x' = q\varphi_0$, $E' = E^{(1)}$, $\ell_0 = \ell(0) + o(1)$ and slack ξ, after relabelling ξ. Theorem 10.12, applied along even scales with $b_n \to \beta_e$, and $\xi \to 0$, give the claim. If $\beta_e \le \ell(0)/2$ there is nothing to prove. □

The sites x_n may depend on the scale, so Theorem 10.13 bounds the uniform exponent, not the exponent of a prescribed phase; Section 11 proves the same lower bound on a residual set of phases (Corollary 11.3). For supercritical odd-harmonic type I potentials the lower and upper bounds coincide when $\ell(0) = \ell_*$ and $\beta_e = \beta$ is finite and at least ℓ_*; then the uniform exponent and the exponent on a residual set of phases both equal $1 - \ell_*/(2\beta)$ (Corollaries 10.14 and 11.3).

Corollary 10.14 (A bracket for odd-harmonic type I potentials)**.** *Let v be real analytic and odd-harmonic, with $H_{v,\alpha}$ supercritical and type I, and $\beta_e(\alpha) = \beta(\alpha) \in [\ell_*, \infty)$. Then*

$$\max\Big\{0,\ 1 - \frac{\ell(0)}{2\beta}\Big\} \le \alpha_u^{\mathrm{unif}} \le 1 - \frac{\ell_*}{2\beta},$$

and the two sides coincide when $\ell(0) = \ell_*$.

Proof. Theorem 10.13 and Corollary 5.6. □

For the almost Mathieu operator $\ell(0) = \ell_* = L$, and Corollary 10.14 recovers the uniform half of Corollary 10.6. The class is larger than the almost Mathieu family: for instance $4\cos 2\pi x + 2\mu\cos 6\pi x$ is odd-harmonic, and it is supercritical and type I at every irrational frequency when $2|\mu|\cosh(3\pi/2) \le 1/4$, by Theorem 13.7.

The upper bound is local in energy for smooth filters. For a vector ψ and $g \in C_c^\infty(\mathbb{R})$ we use the unnormalized packet $g(H_\theta)\psi$ and the quantities (5).

Corollary 10.15 (Smooth filters)**.** *Let v be real analytic with $H_{v,\alpha}$ supercritical and type I, $\beta(\alpha) < \infty$, $\ell_0 > 0$, and let $g \in C_c^\infty(\mathbb{R})$ be real with $\ell(E) \ge \ell_0$ for every $E \in \Sigma \cap \operatorname{supp} g$. Then for every $s' > f(\beta; \ell_0, 1)$ and $m > 0$,*

$$\sup_\theta P_{g(H_\theta)\delta_0}(T^{s'}, T) = O(T^{-m}).$$

Consequently $\alpha^+_{u,g(H_\theta)\delta_0} \le f(\beta; \ell_0, 1)$ for every θ with $g(H_\theta)\delta_0 \ne 0$.

Proof. Fix $\varepsilon \in (0, \ell_0/2)$ and an open neighbourhood U of $\Sigma \cap \operatorname{supp} g$ with $\ell \ge \ell_0 - \varepsilon$ on U. Choose $\chi \in C_c^\infty(U)$ with $\chi = 1$ near $\Sigma \cap \operatorname{supp} g$. Then $(1-\chi)g$ vanishes on $\Sigma = \sigma(H_\theta)$, so $g(H_\theta) = (\chi g)(H_\theta)$; we may therefore assume $\operatorname{supp} g \subset U$. Fix $d_0 > 0$ such that the $2d_0$-neighbourhood of $\operatorname{supp} g$ lies in U.

Write $R_E = (H_\theta - E - i/T)^{-1}$ and $\psi = g(H_\theta)\delta_0$, so that $R_E\psi = g(H_\theta)R_E\delta_0$, and use (7) for ψ.

Energies with $\operatorname{dist}(E, \operatorname{supp} g) \ge d_0$. Here $R_E\psi = h_E(H_\theta)\delta_0$ with $h_E(x) = g(x)/(x - E - i/T)$. These functions are smooth, supported in $\operatorname{supp} g$, and all their derivatives are bounded uniformly in θ, $T \ge 1$ and such E, by $C/(1 + |E|)$. By [21, Lemma 2.8], $|\langle \delta_j, h_E(H_\theta)\delta_0\rangle| \le C_r(1 + |E|)^{-1}|j|^{-r}$ for every r, so these energies contribute $O(T^{-m})$ at radius $T^{s'}$.

The remaining energies. If $\operatorname{dist}(E, \operatorname{supp} g) < d_0$, then $E \in U$. By [21, Lemma 2.8],

$$\big\| \mathbf{1}_{\{|j|>N\}}\, g(H_\theta)\, \mathbf{1}_{\{|j|\le N/2\}} \big\| \le C_r N^{-r}$$

uniformly in θ, so

$$\|\mathbf{1}_{\{|j|>N\}} R_E\psi\|^2 \le 2C_r^2 N^{-2r}\|R_E\delta_0\|^2 + 2\|g\|_\infty^2 \|\mathbf{1}_{\{|j|>N/2\}} R_E\delta_0\|^2.$$

The first term integrates to at most $2C_r^2 N^{-2r}$ by (7). For the second, run the proof of Theorem 5.2 at radius $N/2$ with $N = T^{s'}$, restricted to energies in U: far energies are handled by Combes–Thomas, and near energies by the per-energy minimum of the velocity and confinement radii. For a type I potential, $\omega_{y_0} = 1$ on Σ for small y_0 [21, Lemma 5.2], so by Lemma 3.1(1) the quantity Ω of that proof is close to 1 on Σ_ρ once ρ and the slack parameters are small. The per-energy exponent is then at most $f(B_0; \ell_0 - \varepsilon - 10a, \Omega)$, which is below s' once ε, $B_0 - \beta$ and the slack parameters are small. This gives the bound, uniformly in θ. The last statement follows from [21, Lemma 2.3], since the normalized packet has finite moments by [21, Lemma 2.8]. □

For a rough filter, merely supported where $\ell \geq \ell_0$, the statement can fail: the filtered vector need not have rapidly decaying spatial tails. Smoothness is used exactly in the off-diagonal bounds above.

Corollary 10.16 (An exact uniform law at the closed gap)**.** *Let v be real analytic and odd-harmonic, with $H_{v,\alpha}$ supercritical and type I, and $\beta_e(\alpha) = \beta(\alpha) \in [\ell(0), \infty)$. For $w > 0$ let $g_w \in C_c^\infty((-w, w))$ be real with $g_w = 1$ near 0, and let*

$$\gamma_w = \inf\Big\{a \geq 0 : \sup_\theta P_{g_w(H_\theta)\delta_0}(T^a, T) = O(T^{-m}) \text{ for every } m > 0\Big\}.$$

Then

$$1 - \frac{\ell(0)}{2\beta} \leq \gamma_w \leq f\Big(\beta; \min_{\Sigma \cap [-w,w]} \ell,\, 1\Big), \qquad \lim_{w \downarrow 0} \gamma_w = 1 - \frac{\ell(0)}{2\beta}.$$

Proof. The upper bound is Corollary 10.15. As $w \downarrow 0$, $\min_{\Sigma\cap[-w,w]} \ell \to \ell(0)$, since $0 \in \Sigma$ and ℓ is continuous, and $f(\beta; \ell(0), 1) = 1 - \ell(0)/(2\beta)$ because $\beta \geq \ell(0)$. For the lower bound, the energy windows I_* in the proof of Theorem 10.13 shrink to 0, so for large n they lie in a closed interval J_w with $g_w = 1$ on a neighbourhood of J_w. For $E \in J_w$, $0 < \eta \leq 1$, the functions $x \mapsto \chi(x)(1 - g_w(x))/(x - E - i\eta)$, with χ smooth and equal to 1 on a neighbourhood of the spectrum, have all derivatives bounded uniformly in E and η, because the numerator vanishes near J_w. So [21, Lemma 2.8] gives $\|\mathbf{1}_{\{|j|>N\}} R_E(1 - g_w)(H_\theta)\delta_0\| \leq C_r N^{-r}$, uniformly in θ, $E \in J_w$ and $T \geq 1$. Since $\|a - b\|^2 \geq \frac{1}{2}\|a\|^2 - \|b\|^2$, integrating over I_* as in Step 4 of Theorem 10.5 gives $P_{g_w(H_{\theta+x_n\alpha})\delta_0}(N_n, T_n) \geq \frac{1}{2}T_n^{-C} - C_r N_n^{-2r}$, where $N_n = T_n^{1-(\ell(0)/2+\varepsilon)/b_n} \geq T_n^{1/4}$ for large n and $\varepsilon < \ell(0)/4$, since $b_n \to \beta \geq \ell(0)$. Choosing r large gives $\sup_\theta P_{g_w(H_\theta)\delta_0}(N_n, T_n) \geq \frac{1}{4}T_n^{-C}$, and letting $\varepsilon \to 0$ gives the lower bound. □

At a fixed width $w > 0$ we do not claim equality. The corollary says that the uniform law is exact, energy by energy, at the energy where a gap closes at every even scale.

10.6 Non-primitive potentials attain the factor 2ω

Proposition 10.17. *Let $\lambda > 1$, $s \geq 2$ an integer and $v(x) = 2\lambda\cos(2\pi s x)$. Then $\ell = \log\lambda$ and $\omega = s$ on Σ. For every $\beta_* \in (\log\lambda, \infty)$ there is an irrational α with $\beta(\alpha) = \beta_*$ and*

$$\alpha_u^{\mathrm{unif}} = \sup_\theta \alpha_u^+(H_{v,\alpha,\theta}) = 1 - \frac{\log\lambda}{2s\beta(\alpha)} = 1 - \frac{\ell_*}{2\omega^+\beta(\alpha)}.$$

So the bound of Theorem 5.2 is attained in the class of all analytic potentials: no bound of the form $1 - \ell_*/(M\beta)$ with $M < 2\omega^+$ holds for all of them.

Proof. By [21, Remark 5.30], $L_v(E,y) = L_w(E,sy)$, where $w(x) = 2\lambda\cos 2\pi x$ is taken at frequency $s\alpha$; so $\ell = \log\lambda$ and $\omega = s$ on Σ. The upper bound is Theorem 5.2, since $\beta_* \geq \ell^+ = \log\lambda$.

The frequency. Let $m = 2s$. We build $\alpha = [0; a_1, a_2, \dots]$ so that every large partial quotient follows a denominator divisible by m. Take $a_1 = m$, so $q_1 = m$. Whenever $q_n \equiv 0 \pmod m$, let $a_{n+1} = \lceil e^{\beta_* q_n} \rceil$; then $q_{n+1} = a_{n+1}q_n + q_{n-1} \equiv q_{n-1} \pmod m$, which is invertible modulo m because $\gcd(q_{n-1}, q_n) = 1$. Then choose a bounded $a_{n+2} \in \{1, \dots, m\}$ with $a_{n+2}q_{n+1} + q_n \equiv 0 \pmod m$, which is possible because q_{n+1} is invertible modulo m; so $q_{n+2} \equiv 0 \pmod m$, and repeat. The partial quotients that adjust the congruence are bounded by m, so $b_n \to 0$ along the indices with $m \nmid q_n$, while $b_n \to \beta_*$ along those with $m \mid q_n$. Hence $\beta(\alpha) = \beta_*$.

The lower bound. By [21, proof of Theorem 5.29], $H_{v,\alpha,\theta} = H_{\lambda,s\alpha,s\theta}$. At each n with $m \mid q_n$, $q' = q_n/s$ is a convergent denominator of $s\alpha$ with $q'(s\alpha) - p_n = \delta_n$ [21, Proposition 5.27], so its ratio is sb_n, and q' is even. Hence $\beta_e(s\alpha) \geq s\beta(\alpha)$; with $\beta(s\alpha) \leq s\beta(\alpha)$ [21, Lemma 5.28], $\beta_e(s\alpha) = \beta(s\alpha) = s\beta(\alpha)$. Theorem 10.5 at frequency $s\alpha$ and phase $1/4$ gives $\alpha_u^+(H_{\lambda,s\alpha,1/4}) \geq 1 - \log\lambda/(2s\beta(\alpha))$, and $H_{\lambda,s\alpha,1/4} = H_{v,\alpha,1/(4s)}$. The equalities follow from (8). □

The mechanism is a symmetry: translation of v by $1/s$ forces clusters of $2s$ zeros of the discriminant, at the relevant exponential scale, at every scale whose denominator is divisible by s; the zeros in a cluster need not coincide.

11 Measure versus category

The lower bounds of Section 10 depend on the phase only through the local phase $q_n\theta \bmod 1$ at the scale used, and they are stable under small changes of it. A standard Baire argument then turns them into lower bounds on a residual set of phases.

Theorem 11.1 (A Baire principle)**.** *Let v be continuous and α irrational. Suppose there are scales $n_j \to \infty$, times $T_j \to \infty$, radii N_j, a constant $C > 0$, a number $a_* \in [0,1]$ with $\log N_j/\log T_j \to a_*$, and nonempty open sets $U_j \subset \mathbb{T}$ such that*

$$P_\theta(N_j, T_j) \geq T_j^{-C} \qquad \text{for every } \theta \text{ with } q_{n_j}\theta \bmod 1 \in U_j.$$

Then $\{\theta : \alpha_u^+(H_{v,\alpha,\theta}) \geq a_*\}$ contains a dense G_δ set.

Proof. The map $\theta \mapsto H_\theta$ is norm continuous, so for fixed N, T the map $\theta \mapsto P_\theta(N,T) = 1 - \sum_{|j|\leq N} a_\theta(j,T)$ is continuous, by dominated convergence in the Abel integral. The set $V_j = \{\theta : P_\theta(N_j,T_j) > T_j^{-C}/2\}$ is open and contains $\{\theta : q_{n_j}\theta \in U_j\}$, a nonempty open $1/q_{n_j}$-periodic set, which meets every interval of length greater than $1/q_{n_j}$. So $W_m = \bigcup_{j\geq m} V_j$ is open and dense, and $G = \bigcap_m W_m$ is a dense G_δ set. For $\theta \in G$, $\theta \in V_j$ for infinitely many j, and $M_p(T_j) \geq N_j^p P_\theta(N_j,T_j) \geq N_j^p T_j^{-C}/2$. Hence $\beta^+(p) \geq a_* - C/p$, and letting $p \to \infty$ gives $\alpha_u^+(\theta) \geq a_*$. □

The argument is the standard open-dense mechanism; we claim no novelty for it. Its content lies in the inputs.

Corollary 11.2 (The almost Mathieu operator)**.** *Let $\lambda > 1$ and α irrational with $\beta_e(\alpha) > L/2$. Then $\alpha_u^+(H_{\lambda,\alpha,\theta}) \geq 1 - L/(2\beta_e(\alpha))$ for every θ in a dense G_δ set, read as 1 if $\beta_e = \infty$. If moreover $\beta_e(\alpha) = \beta(\alpha) \in (L/2, \infty)$, then*

$$\alpha_u^+(H_{\lambda,\alpha,\theta}) = 1 - \frac{L}{2\beta(\alpha)} \quad \text{on a dense } G_\delta \text{ set,}$$

which is strictly larger than the value $\max\{0, 1 - L/\beta(\alpha)\}$ at almost every phase.

Proof. If $\beta_e = \infty$ the value is 1 at every phase by Remark 5.4. Let $L/2 < \beta_e < \infty$ and fix $h > L$ with $h/2 + (h - L) < \beta_e$ and $\eta = 1/4$. Along even scales n_j with $b_{n_j} \to \beta_e$, Theorem 10.5 gives (51) at $T_j = |\delta_{n_j}|^{-1}$ for every θ whose local phase lies in the open nonempty set $U_j = \{x \in \mathbb{T} : |\cos 2\pi x - (-1)^{q_{n_j}/2}| < \eta\lambda^{-q_{n_j}}/16\}$. There $T_j^{-2} e^{-2hq_{n_j}} \geq T_j^{-C}$ with $C = 2 + 2h/(h/2 + h - L)$, and $\log N_j / \log T_j \to 1 - h/(2\beta_e)$. Theorem 11.1 gives a dense G_δ set on which $\alpha_u^+ \geq 1 - h/(2\beta_e)$. Intersect these sets along a sequence $h \downarrow L$; a countable intersection of dense G_δ sets is a dense G_δ set. If $\beta_e = \beta > L/2$, Theorem 6.1 for $\beta < L$ and Corollary 5.6 for $\beta \geq L$ give $\alpha_u^+ \leq 1 - L/(2\beta)$ at every phase. The almost-everywhere value is [20, Theorem A]; for every finite $\beta > L/2$ the two values differ. □

So for these frequencies the transport exponent of the almost Mathieu operator takes one value at almost every phase and a strictly larger value at a residual set of phases. At $\beta = \infty$ both values are 1.

Corollary 11.3 (Odd-harmonic potentials)**.** *Let v be real analytic and odd-harmonic, with $H_{v,\alpha}$ supercritical, and $\beta_e(\alpha) > \ell(0)/2$. Then $\alpha_u^+(H_{v,\alpha,\theta}) \geq 1 - \ell(0)/(2\beta_e(\alpha))$ on a dense G_δ set of phases. If moreover $H_{v,\alpha}$ is type I and $\ell_* \leq \beta(\alpha) < \infty$, then $\alpha_u^+ \leq 1 - \ell_*/(2\beta(\alpha))$ at every phase, and $\alpha_u^+ = \max\{0, 1 - \ell_*/\beta(\alpha)\}$ at almost every phase [21, Theorem A].*

Proof. In the proof of Theorem 10.13, the lower bound at scale n holds for the packet launched at any site whose first block has local phase within $|\delta_n|$ of $x_n' = q_n\varphi_0$, with the same constants. Applied to the launch site 0, it holds for every θ with $q_n\theta$ in a fixed nonempty open neighbourhood U_n of $x_n' - q_n\alpha$, because the insertion error is controlled uniformly there. Theorem 11.1 and a countable intersection give the lower bound. The upper bound is Corollary 5.6. □

The lower and upper bounds of Corollary 11.3 coincide only when $\ell(0) = \ell_*$ and $\beta_e = \beta$. We do not claim that the worst exponent is attained on a residual set for every odd-harmonic type I potential.

Corollary 11.4 (Non-primitive potentials)**.** *Let $v(x) = w(sx)$ with an integer $s \geq 2$. If the conclusion of Corollary 11.2 or 11.3 holds for w at frequency $s\alpha$, with $\beta_e(s\alpha)$ and $\beta(s\alpha)$ in place of $\beta_e(\alpha)$ and $\beta(\alpha)$, then it holds for v at frequency α with the same numbers.*

Proof. $H_{v,\alpha,\theta} = H_{w,s\alpha,s\theta}$ [21, Theorem 5.29], and $\theta \mapsto s\theta$ is a continuous, open surjection of $\mathbb{T}$, so preimages of dense G_δ sets are dense G_δ sets and preimages of null sets are null. □

12 Light cones in the XY chain

12.1 The chain

Consider the spin chain on $\mathbb{Z}$ with Hamiltonian

$$H_{XY} = \sum_n \big(X_nX_{n+1} + Y_nY_{n+1}\big) + \sum_n h_nZ_n, \qquad h_n = v(\theta + n\alpha),$$

where X_n, Y_n, Z_n are the Pauli matrices at site n. The interactions are nearest-neighbour and bounded, so the dynamics τ_t exists on the quasi-local algebra, and $\tau_t(A) = \lim_{L\to\infty} \tau_t^L(A)$ in norm for local A, where τ^L is generated by the restriction of H_{XY} to $\Lambda_L = [-L, L]$ [39]. All identities below are proved in finite volume and pass to the limit.

On Λ_L the Jordan–Wigner transformation [38] $c_j = (\prod_{-L\leq k<j} Z_k)\sigma_j^-$, $\sigma^- = (X - iY)/2$, gives $c_j^*c_j = (1 + Z_j)/2$ and $H_{XY}|_{\Lambda_L} = \sum_{a,b} c_a^*(M_L)_{ab}c_b + \text{const}$ with $M_L = -2(S + S^*) + 2\,\mathrm{diag}(h)$ on

$\ell^2(\Lambda_L)$. The diagonal unitary $W = \mathrm{diag}((-1)^j)$ gives $WM_LW = 2H_L$, where H_L is the Dirichlet restriction of $H = H_{v,\alpha,\theta}$. Hence

$$\tau_t^L(c_x) = \sum_y U_t^L(x,y)c_y, \qquad |U_t^L(x,y)| = \big|\langle \delta_y, e^{-2itH_L}\delta_x\rangle\big|.$$

For the Majorana operators $\gamma_{2x-1} = c_x + c_x^*$ and $\gamma_{2x} = i(c_x - c_x^*)$, $\tau_t(\gamma_{2x-1})$ and $\tau_t(\gamma_{2x})$ are real linear combinations of Majoranas whose total squared weight on the sites of any set R is $\sum_{y\in R}|U_t(x,y)|^2$. Spin time t therefore corresponds to one-body time $2t$, and we write

$$p_s^{(x)}(R) = \sum_{y\in R}\big|\langle \delta_y, e^{-isH}\delta_x\rangle\big|^2$$

for the one-body mass in R at one-body time s. These formulas are read in the CAR algebra in finite volume; the even polynomials in Majoranas and the local spin observables used below have norm limits as $L\to\infty$.

12.2 Exact commutator identities

Proposition 12.1. *Let R be a finite set of sites not containing 0, $P_R = \prod_{k\in R} Z_k$ and $p = p_{2t}^{(0)}(R)$. Then*

$$\big\|[\tau_t(Z_0), P_R]\big\| = 4\sqrt{p(1-p)}.$$

In particular $\|[\tau_t(Z_0), Z_N]\| = 4|u|\sqrt{1-|u|^2}$ with $u = \langle \delta_N, e^{-2itH}\delta_0\rangle$.

Proof. $\tau_t(n_0) = f^*f$ with $n_0 = c_0^*c_0$ and $f = \sum_y U_t(0,y)c_y$, a normalized mode. Since $Z_k = 2n_k - 1 = -(-1)^{n_k}$, $P_R = \pm(-1)^{N_R}$ with $N_R = \sum_{k\in R} n_k$, and conjugation by P_R changes the sign of c_y for $y\in R$ and fixes the others. Write $f = f_{\mathrm{in}} + f_R$ with f_R the part supported in R. Then $P_R f P_R = f_{\mathrm{in}} - f_R$ and

$$[f^*f, P_R] = 2(f_{\mathrm{in}}^* f_R + f_R^* f_{\mathrm{in}})P_R.$$

Write $f_R = a g_1$ and $f_{\mathrm{in}} = b g_2$ with normalized modes g_1, g_2 of disjoint supports, $|a|^2 = p$ and $|b|^2 = 1-p$. The operator $\bar b a\, g_2^* g_1 + \bar a b\, g_1^* g_2$ vanishes on the zero- and two-particle sectors of g_1, g_2 and acts on the one-particle sector as $\left(\begin{smallmatrix} 0 & \bar a b \\ a\bar b & 0\end{smallmatrix}\right)$, so its norm is $|a||b| = \sqrt{p(1-p)}$. Since P_R is unitary and $Z_0 = 2n_0 - 1$, the claim follows. □

Proposition 12.2 (Infinite-temperature OTOC)**.** *In finite volume $\Lambda \supset R\cup\{0\}$, with $C_R(t) = \frac{1}{2}\,2^{-|\Lambda|}\,\mathrm{tr}(|[\tau_t(Z_0), P_R]|^2)$ and p as in Proposition 12.1,*

$$C_R(t) = 4p(1-p).$$

Proof. From the proof of Proposition 12.1, $|[\tau_t(Z_0), P_R]|^2 = 16\,|a|^2|b|^2(n_{g_1}(1-n_{g_2}) + n_{g_2}(1-n_{g_1}))$, conjugated by P_R. In the tracial state, g_1, g_2 are orthonormal modes and $\langle n_{g_1}(1-n_{g_2})\rangle = 1/4$. □

So an order-one commutator, or OTOC, between Z_0 and the parity of a far region is equivalent to a fixed fraction, bounded away from 0 and 1, of the one-body packet lying in that region.

Remark 12.3. The tracial OTOC for a single receiver site is contained in the general Gaussian-state formula of Riddell and Sørensen [44, Appendix C, eq. (C33)] (numbering of the PDF of arXiv:1908.03292v3), whose tracial specialization is $1 - 4p(1-p)$. The reduction of XY dynamics to one-body dynamics is classical [38]; Hamza, Sims and Stolz used it to derive dynamical localization of disordered XY chains from one-body localization [24].

12.3 From Abel averages to every time

Let K be a Jacobi operator on $\ell^2(\mathbb{Z})$ with off-diagonal entries 1 and real diagonal, $\psi_t = e^{-itK}\delta_x$, and $\mathfrak{f}(t) = \|\mathbf{1}_R \psi_t\|$ with $R = \{y : |y - x| > N\}$.

Lemma 12.4. *Let $N \geq 1$. Then $\mathfrak{f}$ is 1-Lipschitz, and for $T \geq 1$ and $|t| \leq T$,*

$$\mathfrak{f}(t)^2 \leq \left(4e^3 T\, P(N,T)\right)^{2/3}, \qquad P(N,T) = \frac{2}{T}\int_0^\infty e^{-2s/T}\mathfrak{f}(s)^2\, ds.$$

Proof. $\frac{d}{dt}\mathfrak{f}^2 = \langle \psi_t, i[K, \mathbf{1}_R]\psi_t\rangle$, and $[K, \mathbf{1}_R]$ lives on the two bonds crossing the boundary of R. Since $N \geq 1$, the two inner boundary sites $x \pm N$ are distinct, and Cauchy–Schwarz gives $|(\mathfrak{f}^2)'| \leq 2\mathfrak{f}$, so $\mathfrak{f}$ is 1-Lipschitz. If $\mathfrak{f}(t_0) = \sigma \in (0, 1]$ with $0 \leq t_0 \leq T$, then $\mathfrak{f} \geq \sigma/2$ on $[t_0, t_0 + \sigma/2]$, and $P(N,T) \geq \frac{2}{T} e^{-2(t_0+1/2)/T}\frac{\sigma^2}{4}\frac{\sigma}{2} \geq e^{-3}\sigma^3/(4T)$. Negative times follow from $e^{itK}\delta_x = \overline{e^{-itK}\delta_x}$, as K is real. □

For $N = 0$ the Lipschitz constant is $\sqrt{2}$ instead of 1, which changes only the numerical constant. Splitting the Abel integral at kT gives, for every $k > 0$,

$$P(N,T) \leq \sup_{0 \leq t \leq kT} \mathfrak{f}(t)^2 + e^{-2k}. \tag{52}$$

12.4 Locality for even observables

An observable is *even* if it commutes with the spin flip $\prod_k Z_k$ over its support.

Lemma 12.5 (Locality)**.** *Let A be even, supported in $[-l, l]$, with $\|A\| \leq 1$, and let $\|B\| \leq 1$.*

1. *Let R be disjoint from $[-l, l]$ such that $R_- = R \cap (-\infty, -l)$ and $R_+ = R \cap (l, \infty)$ are intervals, and let B lie in the algebra generated by the even observables supported in R_- and those supported in R_+; for instance a parity string P_R, or a product B_-B_+ of even factors. Then*

$$\|[\tau_t(A), B]\| \leq 2^{4l+4}(4l + 2) \max_{|z| \leq l} \sqrt{p_{2t}^{(z)}(R)}.$$

2. *Let B be any observable supported in $[r, \infty)$ with $r > l$. Then the same bound holds with R replaced by the half-line $[r, \infty)$. The mirror statement holds on the left.*

Proof. In case 2 use Jordan–Wigner strings from the right end of the volume; the one-body matrix changes only by diagonal signs, so the moduli above are unchanged. Then B, of any parity, lies in the CAR algebra generated by the modes of $[r, \infty)$, because the strings of its sites run to the right end inside $[r, \infty)$. In case 1 either representation works: an even spin observable supported in an interval is an even element of the CAR algebra of that interval, since its strings cancel in pairs inside the interval; so B is an even element of the CAR algebra of R. In both cases A is an even polynomial in the $2(2l + 1)$ Majoranas of $[-l, l]$: $A = \sum_S c_S \gamma_S$ over even subsets S, with at most 2^{4l+2} terms and $|c_S| \leq \|A\|$. For $\mu \in S$ split $G_\mu = \tau_t(\gamma_\mu) = G_\mu^{\mathrm{in}} + G_\mu^{\mathrm{out}}$ into its parts on Majoranas outside and inside R (respectively the half-line). These are real linear combinations, so $\|G_\mu^{\mathrm{in}}\| \leq 1$ and $\|G_\mu^{\mathrm{out}}\| = \sqrt{p_{2t}^{(z)}(R)}$ for the site z of μ. A product of an even number of linear elements supported outside R commutes with every even element of the CAR algebra of R, and, in case 2, with every element. So $[\tau_t(\gamma_S), B] = [\prod_\mu G_\mu - \prod_\mu G_\mu^{\mathrm{in}}, B]$, and telescoping gives $\|\prod G_\mu - \prod G_\mu^{\mathrm{in}}\| \leq \sum_\mu \|G_\mu^{\mathrm{out}}\|$. Sum over S. □

Case 2 needs the whole half-line: for an odd observable B, the Jordan–Wigner string of B runs through every site between B and the end of the chain, and a bound through the mass on the support of B alone is not available. The bounds concern even source observables and the receiver classes above; they are not the usual exponential Lieb–Robinson bounds for arbitrary observables.

12.5 Light-cone exponents

For a site x and $N \geq 0$ let $\mathcal{B}(x, N)$ be the set of observables B with $\|B\| \leq 1$ of the two kinds in Lemma 12.5, relative to the site x: elements of the algebra generated by even observables of R_- and of R_+, for sets $R \subset \{y : |y - x| > N\}$ with $R_\pm$ intervals on the two sides of x; and arbitrary observables supported in $[x + N + 1, \infty)$ or in $(-\infty, x - N - 1]$. Let $\mathcal{A}_l(x)$ be the set of even observables supported in $[x - l, x + l]$ with $\|A\| \leq 1$.

Definition 12.6. The *light-cone exponent at the site x* of the chain with phase θ is

$$\Gamma_x(\theta) = \inf\Big\{a \geq 0 : \ \forall l, m : \sup_{|t| \leq T} \sup_{A \in \mathcal{A}_l(x),\, B \in \mathcal{B}(x, T^a)} \|[\tau_t(A), B]\| = O(T^{-m})\Big\},$$

and the *uniform light-cone exponent* $\Gamma_{\mathrm{unif}}(\theta)$ is defined in the same way with an additional supremum over $x \in \mathbb{Z}$ inside the $O(T^{-m})$.

Translation by x sites maps the chain with phase θ to the chain with phase $\theta + x\alpha$, so $\Gamma_x(\theta) = \Gamma_0(\theta + x\alpha)$. By the ballistic bound $|\langle \delta_y, e^{-isH}\delta_x\rangle| \leq (e|s|)^{|y-x|}/|y - x|!$, all these exponents are at most 1.

Theorem 12.7 (The fixed-site light cone)**.** *Let $\lambda > 1$, α irrational, and $h_n = 2\lambda \cos 2\pi(\theta + n\alpha)$. Let $G_\alpha^{\mathbb{Z}} = \bigcap_{x \in \mathbb{Z}} (G_\alpha - x\alpha)$, a set of full measure, and $\gamma = \max\{0, 1 - L/\beta(\alpha)\}$.*

1. *Let $\beta(\alpha) < \infty$ and $\theta \in G_\alpha^{\mathbb{Z}}$. For every $a > \gamma$, $l \geq 0$ and $m > 0$ there is C with $\|[\tau_t(A), B]\| \leq CT^{-m}$ for all $T \geq 1$, $|t| \leq T$, $A \in \mathcal{A}_l(0)$ and $B \in \mathcal{B}(0, T^a)$.*

2. *Let $\beta(\alpha) > L$ and θ arbitrary. For every $a < \gamma$ there are times $t_j \to \infty$ and finite sets $R_j \subset \{y : |y| > t_j^a\}$, each the union of two intervals, with*

$$\|[\tau_{t_j}(Z_0), P_{R_j}]\| \geq \sqrt{2c_B} = 2^{-15.5}(1 + 4\pi\lambda)^{-1/2}, \qquad c_B = \frac{2^{-32}}{1 + 4\pi\lambda}.$$

Consequently $\Gamma_x(\theta) = \max\{0, 1 - L/\beta(\alpha)\}$ for every site x and every $\theta \in G_\alpha^{\mathbb{Z}}$, read as 1 when $\beta(\alpha) = \infty$.

Proof. 1. Fix $a' \in (\gamma, a)$. For $|z| \leq l$ the phase $\theta + z\alpha$ lies in G_α, and [20, Theorem A] gives $P_{\theta + z\alpha}(S^{a'}, S) = O(S^{-m'})$ for every m'. For large T, $T^a - l \geq (2T)^{a'}$. Lemma 12.4 at time scale $2T$ gives $\sup_{|s| \leq 2T} p_s^{(z)}(\{y : |y - z| > (2T)^{a'}\}) = O(T^{-m''})$ for every m''. Lemma 12.5, with finitely many z, finishes the proof; the constants depend on θ through the indices from which the sine conditions hold.

2. Choose $L < h < c < \beta$ with $1 - h/c > a$, and scales n_j with $b_{n_j} \geq c$ and $b_{n_j} \to \beta$. At $T_j = |\delta_{n_j}|^{-1}$ and $N_j = T_j e^{-hq_{n_j}} = T_j^{1 - h/b_{n_j}}$, [20, Theorem B] gives $P(N_j, T_j) \geq c_B$ at every phase. By (52) with $e^{-2k} = c_B/2$, some one-body time $s \leq kT_j$ has $p_s^{(0)}(\{|y| > N_j\}) \geq c_B/2$. Since $p_0 = 0$ and $s \mapsto p_s$ is continuous, there is $s_j \leq s$ with $p_{s_j}^{(0)}(\{|y| > N_j\}) = c_B/2$. Truncating to $R_j = \{N_j < |y| \leq M_j\}$ with M_j large keeps the mass in $[c_B/4, c_B/2]$, and then $4\sqrt{p(1-p)} \geq \sqrt{2c_B}$.

Proposition 12.1 at spin time $t_j = s_j/2$ gives the bound. The times tend to infinity, because a fixed fraction of the packet cannot reach distance $N_j \to \infty$ in bounded time, by the ballistic bound. Finally $N_j \geq (kT_j)^a \geq t_j^a$ for large j, since $1 - h/b_{n_j} > a$ eventually.

For the last statement, $\Gamma_x(\theta) = \Gamma_0(\theta + x\alpha)$ and $\theta + x\alpha \in G_\alpha^{\mathbb{Z}}$. Part 1 gives $\Gamma_0 \leq \gamma$ when $\beta < \infty$, and part 2 gives $\Gamma_0 \geq \gamma$ when $\beta > L$; when $\beta \leq L$, $\gamma = 0$. When $\beta = \infty$, part 2 applies with any $a < 1$. □

Remark 12.8. Damanik, Lemm, Lukic and Yessen identified anomalous Lieb–Robinson exponents of the Fibonacci XY chain with upper one-body transport exponents and treated uniformity over the hull [16]. Kachkovskiy proved velocity lower bounds for isotropic quasi-periodic XY chains in an absolutely continuous regime [35]. Theorem 12.7 is the corresponding statement for the Aubry–André chain with Liouville frequencies, with its exact arithmetic exponent.

The next lemma turns a polynomially small one-body tail into a commutator bound.

Lemma 12.9. *Let φ_n be phases, $T_n \to \infty$ and $N_n \geq T_n^{a_* - o(1)}$ with $a_* > 0$, and suppose $P_{\varphi_n}(N_n, T_n) \geq T_n^{-C}$. Then for every $a < a_*$ there are C', times $t_n \to \infty$ and finite sets $R_n \subset \{y : |y| > t_n^a\}$, each a union of two intervals, such that the chain with phase φ_n satisfies $\|[\tau_{t_n}(Z_0), P_{R_n}]\| \geq t_n^{-C'}$.*

The phases may vary with n, and a constant sequence is allowed; the proof treats each n separately, and its only other input, the ballistic bound, holds uniformly in the phase because $\|v\|_\infty$ bounds the field at every phase.

Proof. By (52) with $e^{-2k} = T_n^{-C}/2$, some one-body time $s \leq CT_n \log T_n$ has $p_s^{(0)}(\{|y| > N_n\}) \geq T_n^{-C}/2$. Take the first time s_n at which this mass equals $T_n^{-C}/4$, and truncate to a finite annulus $R_n = \{N_n < |y| \leq M\}$ with mass $p \in [T_n^{-C}/8, T_n^{-C}/4]$. Proposition 12.1 gives $\|[\tau_{s_n/2}(Z_0), P_{R_n}]\| \geq 4\sqrt{p/2} \geq T_n^{-C/2}$. The ballistic bound forces $s_n \geq cN_n$, so T_n^{-C} is bounded below by a negative power of $t_n = s_n/2$; and $t_n \leq CT_n \log T_n$, so $N_n \geq t_n^a$ for large n. □

Theorem 12.10 (The uniform light cone)**.** *1. Let v be real analytic with $H_{v,\alpha}$ supercritical, $\beta(\alpha) < \infty$, and $h_n = v(\theta + n\alpha)$. Then $\Gamma_{\mathrm{unif}}(\theta) \leq F_{\mathrm{U}}(\beta(\alpha))$ for every θ.*

2. Let $h_n = 2\lambda \cos 2\pi(\theta + n\alpha)$ with $\lambda > 1$. Then $\Gamma_{\mathrm{unif}}(\theta) \geq \max\{0, 1 - L/(2\beta_e(\alpha))\}$ for every θ, read as 1 if $\beta_e = \infty$.

3. If moreover $\beta_e(\alpha) = \beta(\alpha) \in (L/2, \infty)$, then for every θ

$$\Gamma_{\mathrm{unif}}(\theta) = 1 - \frac{L}{2\beta(\alpha)},$$

while $\Gamma_x(\theta) = \max\{0, 1 - L/\beta(\alpha)\}$ at every site x when $\theta \in G_\alpha^{\mathbb{Z}}$.

Proof. 1. Let $a > a' > F_{\mathrm{U}}(\beta)$. By Theorem 5.2, $\sup_\varphi P_\varphi((2T)^{a'}, 2T) = O(T^{-m})$ for every m. Lemma 12.4 turns this into $\sup_z \sup_{|s| \leq 2T} p_s^{(z)}(\{y : |y - z| > (2T)^{a'}\}) = O(T^{-m'})$, uniformly in the site z, since its inequality is pointwise. For $A \in \mathcal{A}_l(x)$ and $B \in \mathcal{B}(x, T^a)$, the relevant sets lie in $\{y : |y - z| > T^a - l\}$ for $|z - x| \leq l$, and Lemma 12.5 gives the bound uniformly in x.

2. Let $\beta_e > L/2$ and $h > L$ with $h/2 + (h - L) < \beta_e$. For an even scale n with $b_n \geq h/2 + (h - L)$, the q_{n+1} points $\theta + x\alpha$, $0 \leq x < q_{n+1}$, have gaps at most $\|q_n\alpha\| + \|q_{n+1}\alpha\| \leq 2/q_{n+1}$, and $1/q_{n+1} < 2|\delta_n|$; so some x_n with $0 \leq x_n < q_{n+1} < T_n = |\delta_n|^{-1}$ has $\|\theta + x_n\alpha - 1/4\| \leq 2|\delta_n|$. Theorem 10.5 at the phase $\theta + x_n\alpha$ gives $P_{\theta + x_n\alpha}(N_n, T_n) \geq T_n^{-2} e^{-2hq_n}$ with $N_n = \lfloor c_\eta T_n e^{-hq_n/2} \rfloor$.

Lemma 12.9, applied to the chain at phase $\theta + x_n\alpha$, which is the chain at phase θ seen from the site x_n, gives commutators $\|[\tau_{t_n}(Z_{x_n}), P_{R_n}]\| \geq t_n^{-C'}$ with $R_n \subset \{y : |y - x_n| > t_n^a\}$ for every $a < 1 - h/(2\beta_e)$, along even scales with $b_n \to \beta_e$. So $\Gamma_{\text{unif}}(\theta) \geq 1 - h/(2\beta_e)$; let $h \downarrow L$. If $\beta_e = \infty$ the same argument gives 1.

3. For $\beta \geq L$, use parts 1 and 2 and Corollary 5.6. For $L/2 < \beta < L$, apply the uniform tail estimate of Theorem 6.1 in the proof of part 1: Lemma 12.4 gives the instantaneous one-body bound uniformly over launch sites, and Lemma 12.5 gives $\Gamma_{\text{unif}} \leq 1 - L/(2\beta)$. Part 2 gives the reverse inequality when $\beta_e = \beta$. The fixed-site assertion is Theorem 12.7, with the maximum with zero retained. □

So uniformity over sites changes the light-cone exponent: when $\beta_e = \beta \in (L/2, \infty)$, almost every phase has fixed-site exponent $\max\{0, 1 - L/\beta\}$ at every site, and uniform exponent $1 - L/(2\beta)$. At $\beta = \infty$ both equal 1.

12.6 Uncomputable light cones

A real x is *right-c.e. relative to* $\mathbf{0}'$ if $\{r \in \mathbb{Q} : r > x\}$ is computably enumerable relative to the halting problem. Let K_2 be the halting set relative to $\mathbf{0}'$ and $s_* = 2 - \sum_{e \in K_2} 4^{-(e+2)} \in [23/12, 2]$, a real of Turing degree $\mathbf{0}''$ [20, proof of Theorem D].

Corollary 12.11 (A computable chain with a light cone of degree $\mathbf{0}''$)**.** *Let α_* be the explicit computable irrational of [20, Theorem D(3)], whose partial quotients after the first are even, so that all its denominators are odd, and $\beta(\alpha_*) = s_* \log 2$. Consider the chain with $h_n = 4\cos 2\pi(1/4 + n\alpha_*) = -4\sin(2\pi n\alpha_*)$. At every site x, the light-cone exponent $\Gamma_x(1/4) = 1 - 1/s_* \in [11/23, 1/2]$ has Turing degree $\mathbf{0}''$. As α ranges over the computable irrationals with eventually odd denominators, the exponents $\Gamma_0(1/4)$ of the chains $h_n = 4\cos 2\pi(1/4 + n\alpha)$ range over exactly the reals in $[0, 1]$ that are right-c.e. relative to $\mathbf{0}'$. For every computable λ' with $1 < \lambda' < 2^{23/12}$, the chain $h_n = 2\lambda' \cos 2\pi(1/4 + n\alpha_*)$ has $\Gamma_x(1/4) = 1 - \log\lambda'/\beta(\alpha_*)$ at every site, again of degree $\mathbf{0}''$.*

Proof. For odd q_n and every fixed x, $|\sin(2\pi q_n(1/4 + x\alpha))| = |\cos(2\pi x\delta_n)| \geq 1/2$ for large n. So $1/4 \in G_\alpha^{\mathbb{Z}}$ whenever the denominators are eventually odd, and Theorem 12.7 identifies $\Gamma_x(1/4)$ with the one-body exponent of [20, Theorem D], including the value 1 when $\beta = \infty$. The degree and value-set statements are [20, Theorem D(2),(3)]. For λ': since $\beta(\alpha_*) \geq (23/12)\log 2 > \log\lambda'$, the exponent $x' = 1 - \log\lambda'/\beta$ is positive, and $\beta = \log\lambda'/(1 - x')$; both maps are computable for computable λ', so x' and β have the same degree. □

The value-set statement concerns this family, with $\lambda = 2$, phase $1/4$ and eventually odd denominators; it is not a classification over all computable fields and phases.

Corollary 12.12 (Two exponents of degree $\mathbf{0}''$ in one computable chain)**.** *Let (v_m) be a computable sequence of rationals in $[1, 2]$ with $\limsup v_m = s_*$ [20, Lemma 11.1]. Define $\alpha = [0; a_1, a_2, \dots]$ by $a_1 = 2$ and, with $n_1 = 1$ and $n_{m+1} = n_m + 3$,*

$$a_{n_m+1} = 2^{\lceil v_m q_{n_m} \rceil}, \qquad a_{n_m+2} = a_{n_m+3} = 1.$$

Then α is computable, $\beta_e(\alpha) = \beta(\alpha) = s_* \log 2$ and $\beta_o(\alpha) = 0$. For the chains $h_n = 4\cos 2\pi(\theta + n\alpha)$:

1. *for every θ, $\Gamma_{\text{unif}}(\theta) = 1 - 1/(2s_*) \in [17/23, 3/4]$;*
2. *for every $\theta \in G_\alpha^{\mathbb{Z}}$, in particular for almost every θ, $\Gamma_x(\theta) = 1 - 1/s_* \in [11/23, 1/2]$ at every site x;*

3. at $\theta = 1/4$, $\Gamma_x(1/4) = 1 - 1/(2s_*)$ at every site x;

4. there is a computable phase $\theta_* \in G_\alpha^{\mathbb{Z}}$.

Both exponent values, $1 - 1/(2s_*)$ and $1 - 1/s_*$, have Turing degree $\mathbf{0}''$. In particular the computable chain with phase θ_* has uniform exponent $1 - 1/(2s_*)$ and fixed-site exponent $1 - 1/s_*$ at every site: two different exponents of degree $\mathbf{0}''$.

Proof. Arithmetic. $q_{n_1} = q_1 = 2$ is even. If q_{n_m} is even, then q_{n_m-1} is odd, q_{n_m+1} and q_{n_m+2} are odd, and $q_{n_m+3} = q_{n_m+2}+q_{n_m+1}$ is even. The large partial quotient gives $b_{n_m} = v_m \log 2+O(\log q_{n_m}/q_{n_m})$, while $b_n \to 0$ along all other n. So $\beta = \beta_e = s_* \log 2 \geq \log 2 = L$ and $\beta_o = 0$. The partial quotients are computable, so α is computable, although $\limsup v_m$ is not.

Part 1 is Theorem 12.10(3), and part 2 is Theorem 12.7.

Part 3. Fix x. The first block to the right of x in the chain with phase $1/4$ has phases $1/4+(x+1+k)\alpha$, and the rational block with phases $1/4+(x+1+k)p_n/q_n$ is a turning block, since $\cos(2\pi q_n(1/4 + (x+1)p_n/q_n)) = (-1)^{q_n/2}$ for even q_n. The site discrepancies are $(x+1+k)\delta_n/q_n$, at most $2|\delta_n|$ once $q_n > |x|$, so the proof of Theorem 10.5 applies with the same exponents and gives $\alpha_u^+(1/4 + x\alpha) \geq 1 - L/(2\beta_e)$. Lemma 12.9 gives $\Gamma_x(1/4) \geq 1 - 1/(2s_*)$, and part 1 gives the reverse inequality.

Part 4. Let $E_n = \{\theta : |\sin 2\pi q_n\theta| < q_n^{-1/2}\}$, a finite union of intervals with computable endpoints and $|E_n| \leq q_n^{-1/2}$. Since q_n grows at least like the Fibonacci numbers, there is an explicit N with $\sum_{n\geq N} |E_n| < 1/2$, with an effective tail bound. Hence the measure of $(\bigcup_{n\geq N} E_n) \cap I$ is computable for every dyadic interval I, by finite unions and the tail bound. Start from $I_0 = \mathbb{T}$; given I_k whose intersection with the closed set $F = \mathbb{T} \setminus \bigcup_{n\geq N} E_n$ has positive measure, compute the measures for the two dyadic children until one is certified positive, which must happen, and let I_{k+1} be that child. The nested intervals define a computable θ_*. If θ_* were in the open set $\bigcup_{n\geq N} E_n$, a neighbourhood of θ_*, hence some I_k, would lie in it, contradicting $|I_k \cap F| > 0$. So $|\sin 2\pi q_n\theta_*| \geq q_n^{-1/2}$ for $n \geq N$. For each fixed x, $q_n(\theta_* + x\alpha) = q_n\theta_* + xp_n + x\delta_n$, and the sine changes by at most $2\pi|x||\delta_n| \leq 2\pi|x|/q_n$, which is eventually below $q_n^{-1/2}/2$. So $\theta_* + x\alpha \in G_\alpha$ for every x.

Degrees. $1 - 1/s_*$ and $1 - 1/(2s_*)$ are computable functions of s_* on $[23/12, 2]$ with computable inverses, so both have the degree of s_*, which is $\mathbf{0}''$. □

Counting lattice sites is not averaging over phases: at the phase $1/4$ every site sees the exceptional exponent, while at almost every phase every site sees the typical one.

13 Computability

The companion paper A showed that the transport exponent of one explicit computable almost Mathieu operator has Turing degree $\mathbf{0}''$, and that this survives changes of the coupling in an interval [20, Theorem D]. Here we show that, at the frequency α_* and the phase $1/4$, the degree survives every computable even change of the potential in an explicit strip ball (Theorem 13.9), and we locate the source of the uncomputability: for a computable analytic potential the Lyapunov exponent is computable at every computable irrational frequency, so in the transport laws identified here the uncomputability sits in the arithmetic of the frequency.

13.1 Conventions

A real number is *computable from an oracle* if the oracle computes rationals r_k with $|r_k - x| \leq 2^{-k}$. A name of a real x is any such sequence. A function on $\mathbb{R}$, or on $(\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R}$, is *computable* if

an algorithm turns every name of the argument into a name of the value; equivalently, if the sets $\{f > r\}$ and $\{f < r\}$, for rational r, are enumerated uniformly by rational boxes, that is, if f is both lower and upper semicomputable. A real-analytic v is *computable* if its Fourier coefficients $\hat{v}(k)$, with $\hat{v}(-k) = \overline{\hat{v}(k)}$, form a computable sequence and there are computable rationals $C_v, \delta_v' > 0$ with $|\hat{v}(k)| \le C_v e^{-2\pi\delta_v'|k|}$. Then v is uniformly computable on every closed strip $|\operatorname{Im} z| \le y < \delta_v'$. Trigonometric polynomials with computable coefficients are the main examples. Two reals have the same Turing degree if each is computable from the other.

We write $A_{E,y}(x) = T(E, x+iy)$ for the complexified cocycle and $L_n(E,y) = \frac{1}{n}\int_{\mathbb{T}} \log \|A_n(E, x+iy)\|\, dx$, so that $L(E,y) = \inf_n L_n(E,y)$ by subadditivity. We use the following facts from Avila's global theory [2], with the numbering of the published version: convexity of $y \mapsto L(E,y)$ (Section 2.1); quantization (Theorem 5); regularity, which for real cocycles holds if and only if the acceleration is zero (Definition 3 and Remark 4); regular with positive Lyapunov exponent if and only if uniformly hyperbolic (Theorem 6); and Corollary 7: for every analytic $SL(2,\mathbb{C})$ cocycle at an irrational frequency there is $\varepsilon_0 > 0$ such that either $L(A_\varepsilon) = 0$ for all $0 < \varepsilon < \varepsilon_0$, or A_ε is uniformly hyperbolic for all $0 < \varepsilon < \varepsilon_0$, and in the latter case $\varepsilon \mapsto L(A_\varepsilon)$ is affine on $(0, \varepsilon_0)$. Uniformly hyperbolic cocycles form an open set, have continuous invariant line fields, and their Lyapunov exponent is continuous [2, Section 2.1].

13.2 The Lyapunov exponent is computable

Lemma 13.1 (Chord bound)**.** *For real E and $0 < y < y' < \delta_v$,*

$$\ell(E) \ge \frac{y' L(E,y) - y L(E,y')}{y' - y},$$

with equality if and only if $L(E,\cdot)$ is affine on $[0, y']$.

Proof. Write $y = (1-\sigma)\cdot 0 + \sigma y'$ with $\sigma = y/y'$. Convexity gives $L(E,y) \le (1-\sigma)\ell(E) + \sigma L(E,y')$, which rearranges to the claim; equality at an interior point of a chord forces affineness on the whole chord. □

Lemma 13.2 (The minimum lies on the spectrum)**.** *For every irrational α and real-analytic v, $\inf_{E\in\mathbb{R}} \ell(E) = \min_\Sigma \ell$. In particular $\ell_* = \min_{|E|\le K} \ell(E)$ for any $K \ge \|v\|_\infty + 2$.*

Proof. ℓ is continuous [9]. On a bounded gap (a,b) of Σ, the Thouless formula $\ell(E) = \int \log|E - E'|\, dk(E')$ [14] can be differentiated twice, since $\operatorname{supp} dk = \Sigma$ is at positive distance from compact subintervals of the gap, and $\ell''(E) = -\int (E-E')^{-2}\, dk(E') < 0$. So ℓ is concave on the gap, and its minimum over $[a,b]$ is attained at an endpoint. Outside the convex hull of Σ, $\ell'(E) = \int (E-E')^{-1}\, dk(E')$ is negative for $E < \min\Sigma$ and positive for $E > \max\Sigma$, so ℓ increases away from Σ. □

Lemma 13.3 (Certificates)**.** *Let $B : \mathbb{T} \to SL(2,\mathbb{C})$ be continuous and α irrational. A* certificate *is a pair (m, Q) with $m \ge 1$ and Q a trigonometric-polynomial matrix with Gaussian-rational coefficients such that*

(V1) $\det Q(x) \ne 0$ *for all $x \in \mathbb{T}$;*

(V2) the entries of $\mathcal{C}(x) = Q(x+m\alpha)^{-1} B^{(m)}(x) Q(x)$ satisfy $|c_{11}| - |c_{12}| > |c_{21}| + |c_{22}|$ for all $x \in \mathbb{T}$,

where $B^{(m)}(x) = B(x+(m-1)\alpha)\cdots B(x)$. Its value is $\operatorname{val}(m,Q) = \frac{1}{m}\int_{\mathbb{T}} \log(|c_{11}| - |c_{12}|)\, dx$.

1. *(Soundness) $L(\alpha, B) \ge \operatorname{val}(m,Q)$.*

2. *(Effectiveness) From names of B and α, the strict conditions (V1)–(V2) are semi-decidable and* $\mathrm{val}(m, Q)$ *is computable. A certificate verified for (B, α) remains one, with value changed by less than any prescribed $\eta > 0$, for all (B', α') in a neighbourhood.*

3. *(Completeness) If (α, B) is uniformly hyperbolic, then for every $\eta > 0$ there is a certificate with* $\mathrm{val} > L(\alpha, B) - \eta$.

Proof. 1. Let $\phi = |c_{11}| - |c_{12}| > 0$. By (V2) the cone $\{(u, w) \in \mathbb{C}^2 : |w| \le |u|\}$ is mapped into itself by $\mathcal{C}(x)$, and the first coordinate grows at least by the factor $\phi(x)$. Iterating along $x, x + m\alpha, \dots$ gives $\|\mathcal{C}^{(k)}(x)\| \ge \prod_{j<k} \phi(x + jm\alpha)$, so $\log \|B^{(km)}(x)\| \ge \sum_{j<k} \log \phi(x + jm\alpha) - \log(\|Q\|_\infty \|Q^{-1}\|_\infty)$. Integrate over $\mathbb{T}$, divide by km and let $k \to \infty$.

2. All quantities are continuous functions of x on the compact circle, computable from the names, and a strict inequality between such functions is certified by finitely many evaluations with error bounds. Once (V1) and (V2) are certified, ϕ is effectively bounded away from zero, so its logarithm and its integral are computable. The strict margins persist under small perturbations.

3. A uniformly hyperbolic cocycle has continuous invariant line fields $u(x)$, $s(x)$, transverse to each other. Every complex line bundle over the circle is trivial, so they are spanned by continuous nonvanishing periodic vector fields, and conjugation by $P = [u, s]$ makes the cocycle diagonal, $\mathrm{diag}(\lambda_u, \lambda_s)$, with $\int \log |\lambda_u| = L(\alpha, B)$ and $\int \log |\lambda_s| = -L(\alpha, B)$. By unique ergodicity, $\frac{1}{m} \log |\Lambda_m| \to L$ and $\frac{1}{m} \log |\Sigma_m| \to -L$ uniformly, where Λ_m, Σ_m are the m-step products of λ_u, λ_s. Approximate P by a trigonometric polynomial matrix Q with rational coefficients within ε' (Fejér). Then $\mathcal{C} = (I + Y)\,\mathrm{diag}(\Lambda_m, \Sigma'_m)(I + X)$ with $\|X\|, \|Y\| \le C\varepsilon'$ and $|\Sigma'_m| \le C|\Sigma_m|$, and for $C\varepsilon' < 1/8$ and large m the conditions (V1)–(V2) hold with $\phi \ge |\Lambda_m|(1 - C\varepsilon')$. So $\mathrm{val} \ge L + \frac{1}{m} \log(1 - C\varepsilon')$. □

Adding the condition $\min_{\mathbb{T}} \phi > \mu$ for a rational $\mu > 1$ gives a certificate of uniform hyperbolicity: then $\|B^{(km)}(x)\| \ge c\mu^k$ uniformly. Completeness holds for these certificates as well, since $|\Lambda_m| \to \infty$ uniformly.

Theorem 13.4 (The Lyapunov exponent is computable)**.** *Let v be a computable real-analytic function. The map $(\alpha, E) \mapsto \ell(E)$, the Lyapunov exponent of $H_{v,\alpha}$ at E, is computable on $(\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R}$, uniformly in a name of v. Consequently, for every computable irrational α, $E \mapsto \ell(E)$ is a computable function, $\ell(E)$ is a computable real at every computable E, and ℓ_* is a computable real, uniformly in v and α. No hypothesis of supercriticality or type is needed.*

Proof. Upper semicomputability. $\ell(E) < r$ if and only if there are n and a rational interval $I \ni E$ with $\max_{E' \in \bar{I}} L_n(E', 0) < r$, because $\ell = \inf_n L_n$ and each L_n is continuous in E and in α. The maxima are computable.

Lower semicomputability. Let W be the computably enumerable set of pairs (I, r), with I a rational interval and r rational, such that either $r < 0$, or there are rationals $0 < y < y' < \delta'_v$, a certificate (m, Q) for the cocycles $A_{E,y}$ valid uniformly for $E \in \bar{I}$, with $c_{\mathrm{low}} = \min_{E \in \bar{I}} \mathrm{val}_E(m, Q)$, and an n with $c_{\mathrm{up}} = \max_{E \in \bar{I}} L_n(E, y') + 2^{-n}$, such that $(y' c_{\mathrm{low}} - y c_{\mathrm{up}})/(y' - y) > r$.

Soundness: if $(I, r) \in W$ and $E \in I$, then $\ell(E) > r$, by Lemma 13.1 with $L(E, y) \ge c_{\mathrm{low}}$ (Lemma 13.3(1)) and $L(E, y') \le L_n(E, y') < c_{\mathrm{up}}$.

Completeness: let $r < \ell(E_0)$ with $r \ge 0$. Then $\ell(E_0) > 0$, so $L(E_0, y) \ge \ell(E_0) > 0$ for all y, by convexity and evenness. Corollary 7 of [2] gives $\varepsilon_0 > 0$ such that $A_{E_0,y}$ is uniformly hyperbolic and $y \mapsto L(E_0, y)$ is affine for $0 < y < \varepsilon_0$; by continuity of the convex function at 0, it is affine on $[0, \varepsilon_0)$. Take rationals $y < y' = 2y < \min\{\varepsilon_0, \delta'_v\}$. Then Lemma 13.1 holds with equality at E_0: $2L(E_0, y) - L(E_0, 2y) = \ell(E_0)$. Given $\eta > 0$, Lemma 13.3(3) and (2) give a certificate at (E_0, y) with value above $L(E_0, y) - 2\eta$ on a neighbourhood of E_0. Choose n with $L_n(E_0, 2y) + 2^{-n} < L(E_0, 2y) + \eta$

and shrink the neighbourhood so that $L_n(\cdot, 2y) + 2^{-n} < L(E_0, 2y) + 2\eta$ on it. On a rational interval $I \ni E_0$ in that neighbourhood, $2c_{\text{low}} - c_{\text{up}} \geq \ell(E_0) - 6\eta > r$ for small η. So $(I, r) \in W$.

Hence $\ell(E) > r$ if and only if some $(I, r) \in W$ has $E \in I$, which is semi-decidable from names of E and α; all verifications are continuous in α, with margins. Irrationality of α is used only for completeness. The two halves give a computable function. Finally $\ell_* = \min_{|E| \leq K} \ell(E)$ by Lemma 13.2, with a computable integer K larger than a computable bound for $\|v\|_\infty + 2$, and the minimum of a computable function over a computable compact interval is computable. □

The algorithm is: where the Lyapunov exponent is positive, quantization makes the cocycle regular, hence uniformly hyperbolic, at small positive heights; there a certificate gives a lower bound; convexity carries it to height zero exactly. Where it is zero, nonnegativity and finite-scale upper bounds suffice. No modulus of continuity and no large-deviation estimate is used, so Liouville frequencies are allowed. The proof gives no bound on the running time, since the height at which the Lyapunov exponent becomes affine is found by search.

Corollary 13.5 (Density of states and spectrum)**.** *For computable v and computable irrational α: the integrated density of states k is a computable function; Σ is a computable compact set;* $\max_\Sigma \ell$ *and* $\min \Sigma$ *are computable reals, and* $\min \Sigma$ *is a computable point of Σ.*

Proof. Let $c_N(\theta, E)$ count the eigenvalues at most E of H_θ restricted to $\{0, \dots, N-1\}$. Decoupling two adjacent sites is a rank-two perturbation with one positive and one negative eigenvalue, so $|c_{N+M}(\theta) - c_N(\theta) - c_M(\theta + N\alpha)| \leq 1$. With $a_N(E) = \int_{\mathbb{T}} c_N(\theta, E)\, d\theta$, this gives $|a_{N+M} - a_N - a_M| \leq 1$ and hence $|k(E) - a_N(E)/N| \leq 1/N$. The function a_N is upper semicomputable, as a sum of measures of effectively closed sets cut out by computable eigenvalue functions, and the analogous count with strict inequalities is lower semicomputable. The density of states has no atoms: an atom at E would make $\ell(E) = \int \log|E - E'|\, dk(E') = -\infty$. So both bounds converge to $k(E)$, and k is computable. The complement of Σ is enumerated by the certificates of uniform hyperbolicity, uniformly on rational intervals; and an open interval (a, b) meets Σ if and only if $k(b) > k(a)$, which is semi-decidable. Together with a bounding interval this makes Σ a computable compact set, and the extrema follow. □

Remark 13.6 (Related work)**.** Computing the spectrum is not new: the rational approximation of [21, Lemma 2.7], with finite Floquet matrices, already gives an effective route, and spectral computations for broad operator classes are studied through the solvability complexity index [8, 13]; Colbrook, Embree and Fillman study the computation of spectral size, including for the almost Mathieu operator [12]. Theorem 13.4 is an algorithmic consequence of Avila's global theory. We did not find this formulation, uniform over all irrational frequencies, in the literature we checked, but we make no priority claim for it.

13.3 An explicit type I neighbourhood

Let $v_0(x) = 4\cos 2\pi x$, the almost Mathieu potential with $\lambda = 2$, and $S_Y = \{z : |\operatorname{Im} z| \leq Y\}$.

Theorem 13.7 (An explicit neighbourhood)**.** *Let $Y, \varepsilon > 0$, $t = e^{-2\pi Y}$ and $E_{\max} = 6 + \varepsilon$, and suppose*

$$M := \frac{2}{t} - 2t - E_{\max} - \varepsilon > 2, \qquad p_0 := \frac{E_{\max} t + 2t^2}{2} < 1, \qquad \kappa := \frac{t(\varepsilon + \rho)}{2(1 - p_0)} < \frac{1}{2},$$

where $\rho = (M - \sqrt{M^2 - 4})/2$. Let v be real analytic with $w = v - v_0$ holomorphic on a neighbourhood of S_Y and $\sup_{S_Y} |w| \leq \varepsilon$. Then for every irrational α:

1. $\ell(E) \geq \log 2 + \log(1-\kappa) > 0$ *for every real* E *with* $|E| \leq E_{\max}$*;*

2. $H_{v,\alpha}$ *is supercritical and type I, with* $L(E,y) = \ell(E) + 2\pi|y|$ *for* $E \in \Sigma$ *and* $|y| \leq Y$*;*

3. $\log 2 + \log(1-\kappa) \leq \ell(E) \leq \log 2 + \log(1+\kappa)$ *for* $E \in \Sigma$.

For $Y = \varepsilon = 1/4$ *one has* $M > 2.7051$, $\rho < 0.4419$, $p_0 < 0.6929$ *and* $0.2341 < \kappa < 0.23411$, *so* $\ell \in [0.4264, 0.9035]$ *on* Σ.

Proof. Fix a real E with $|E| \leq E_{\max}$. The three strict inequalities persist, with constants $M(y), \rho(y), p_0(y), \kappa(y)$ depending continuously on the height, for y in an interval $[Y-\tau, Y]$. Write $z = x + iy$ and $\mathsf{q} = e^{2\pi i z}$, so $|\mathsf{q}| = e^{-2\pi y}$. Then $a(z) := E - v(z) = -2\mathsf{q}^{-1} + E - 2\mathsf{q} - w(z)$ and $|a(z)| \geq M(y) > 2$ on the line $\operatorname{Im} z = y$.

An invariant section. The maps $\sigma \mapsto 1/(a(\zeta) - \sigma)$ send the disk $|\sigma| \leq \rho(y)$ into itself, since $|a - \sigma| \geq M(y) - \rho(y) = 1/\rho(y)$, and contract it. Choosing one common disk on a slightly smaller substrip, using its lower endpoint, the limit $r(z) = \lim_n f_{z-\alpha} \circ \cdots \circ f_{z-n\alpha}(0)$ exists uniformly, is holomorphic in z, satisfies $r(z+\alpha) = 1/(a(z) - r(z))$, and on each line $\operatorname{Im} z = y$ satisfies $|r| \leq \rho(y)$. With $u = (1, r)^{\mathsf{T}}$ and $g = a - r$, $T(E,z)u(z) = g(z)u(z+\alpha)$, and conjugation by $P = [u, e_2]$, of determinant one, gives the triangular cocycle $\left(\begin{smallmatrix} g & -1 \\ 0 & 1/g \end{smallmatrix}\right)$ with $|g| \geq 1/\rho > 1$. Its n-step products are comparable to their diagonal, so $L(E,y) = \int_0^1 \log|g(x+iy)|\,dx$.

Slope 2π. Write $g = -2\mathsf{q}^{-1}(1-\Phi)$ with $\Phi = \mathsf{q}(E - 2\mathsf{q} - w - r)/2 = \Phi_0 + \Phi_1$, where $\Phi_0 = \mathsf{q}(E - 2\mathsf{q})/2$ satisfies $|\Phi_0| \leq p_0(y)$ on the disk $|\mathsf{q}| \leq e^{-2\pi y}$ by the maximum principle, and $|\Phi_1| \leq e^{-2\pi y}(\varepsilon + \rho(y))/2$. So $|\Phi| < 1$, the winding number of $1 - \Phi$ along a horizontal line is zero, and that of g is -1. Since $\log|g|$ is harmonic, $y \mapsto \int_0^1 \log|g(x+iy)|\,dx$ is affine, with slope -2π times the winding number, that is 2π.

The value at height Y. The mean of $\log|1 - \Phi_0|$ over the circle $|\mathsf{q}| = t$ is $\log|1 - \Phi_0(0)| = 0$, and $|\Phi_1/(1-\Phi_0)| \leq \kappa$. So $L(E,Y) = \log 2 + 2\pi Y + c$ with $\log(1-\kappa) \leq c \leq \log(1+\kappa)$.

Conclusions. By convexity every slope of $L(E,\cdot)$ on $[0,Y]$ is at most 2π, so $\ell(E) \geq L(E,Y) - 2\pi Y \geq \log 2 + \log(1-\kappa) > 0$. The spectrum lies in $[-2 - \sup_{\mathbb{R}}|v|, 2 + \sup_{\mathbb{R}}|v|] \subset [-E_{\max}, E_{\max}]$, so $\ell > 0$ on Σ. At $E \in \Sigma$ the acceleration is an integer at least one (Section 2) and at most one, so the slope is 2π on all of $(0,Y]$, which gives part 2 and $\ell(E) = L(E,Y) - 2\pi Y$, which gives part 3. The numerical values are elementary; they were also certified independently with exact rational interval arithmetic, which gives $\kappa \in [0.2341020899, 0.2341020900]$, $\log 2 + \log(1-\kappa) > 0.42644$ and $\log 2 + \log(1+\kappa) < 0.90350$. □

The estimates are pointwise in x, so they hold for every irrational frequency at once, for even and non-even v.

Proposition 13.8 (Openness)**.** *Fix an irrational* α*, and let* v_0 *be any real-analytic potential with* $H_{v_0,\alpha}$ *supercritical and type I, with common strip* h *[21, Lemma 5.2]. There is* $\varepsilon_0 > 0$ *such that every real-analytic* v *with* $\|v - v_0\|_h := \sup_{|\operatorname{Im} z| \leq h} |v - v_0| < \varepsilon_0$ *is supercritical and type I at* α*, and* $\ell_*(v) \to \ell_*(v_0)$ *as* $\|v - v_0\|_h \to 0$.

Proof. For $E \in \Sigma_0 = \Sigma(v_0)$ and $y \in [h/4, 3h/4]$, $L_{v_0}(E,\cdot)$ is affine near y with positive value, so the cocycle is uniformly hyperbolic. Uniform hyperbolicity is open, and $(E,y,v) \mapsto T_v(E, \cdot + iy)$ is continuous into C^0 for v in the norm $\|\cdot\|_h$. A finite cover of the compact set $K \times [h/4, 3h/4]$, with $K = \{E : \operatorname{dist}(E, \Sigma_0) \leq e_0\}$, gives $e_0, \varepsilon_1 > 0$ such that these cocycles are uniformly hyperbolic for $E \in K$, $y \in [h/4, 3h/4]$ and $\|v - v_0\|_h \leq \varepsilon_1$. There $y \mapsto L_v(E,y)$ is affine on $[h/4, 3h/4]$, with a slope in $2\pi\mathbb{Z}$ that depends continuously on (E,v), hence is constant on each connected component of $K \times \{\|v - v_0\|_h \leq \varepsilon_1\}$; each component meets $\Sigma_0 \times \{v_0\}$, where the slope is 2π.

Let $\Psi(E,v) = L_v(E,h/4) - \pi h/2$. At v_0 and $E \in \Sigma_0$ it equals $\ell_{v_0}(E)$; shrinking e_0 and ε_1, $\Psi \geq \ell_*(v_0)/2$ on the region, and, by joint continuity near $K \times \{v_0\}$ and a finite cover of K, $\sup_{E\in K} |\Psi(E,v) - \Psi(E,v_0)| \to 0$ as $v \to v_0$. For $E \in \Sigma_v$, which lies within $\|v - v_0\|_h$ of Σ_0, convexity gives $\ell_v(E) \geq \Psi(E,v) > 0$, and the argument of Theorem 13.7(2) gives $\omega_v(E) = 1$ and $\ell_v = \Psi(\cdot, v)$ on Σ_v. Since $\Psi(\cdot, v_0)$ is uniformly continuous on K and $\Sigma_v \to \Sigma_0$ in Hausdorff distance, $\ell_*(v) = \min_{\Sigma_v} \Psi(\cdot, v) \to \min_{\Sigma_0} \Psi(\cdot, v_0) = \ell_*(v_0)$. □

The topology is that of a fixed strip. This is not openness in the operator norm among all bounded potentials.

13.4 A robust exponent of degree $\mathbf{0''}$

Let

$$\mathcal{U} = \Big\{ v \text{ real analytic}: \ v - v_0 \text{ holomorphic near } S_{1/4}, \ \sup_{S_{1/4}} |v - v_0| < \frac{1}{4} \Big\},$$

an open ball in the norm of the strip $S_{1/4}$, and its even part $\mathcal{U}_{\text{even}} = \{v \in \mathcal{U} : v(-x) = v(x)\}$. Let α_* be the explicit computable frequency of [20, Theorem D(3)]: all its denominators are odd, $a_{n+1} \geq 2^{q_n}$, and $\beta(\alpha_*) = s_* \log 2$ with $s_* \in [23/12, 2]$ of Turing degree $\mathbf{0''}$. So $\beta(\alpha_*) \geq (23/12)\log 2 > 1.3285$. For $v \in \mathcal{U}$ put

$$X(v) = \alpha_u^+(H_{v,\alpha_*,1/4}), \qquad X_{\text{typ}}(v) = 1 - \frac{\ell_*(v,\alpha_*)}{\beta(\alpha_*)}.$$

Theorem 13.9 (A robust uncomputable exponent)**.** *1. For every $v \in \mathcal{U}_{\text{even}}$, $X(v) = X_{\text{typ}}(v) \in [0.3199, 0.6924]$, and this is also the value of every moment exponent $\beta^+(p)$, of the fastest exponent for the packet started at any site, and of the upper packing dimension of the spectral measure of δ_0.*

2. For every $v \in \mathcal{U}_{\text{even}}$ with computable coefficients, $X(v)$ has Turing degree exactly $\mathbf{0''}$.

3. On $\mathcal{U}_{\text{even}}$, $v \mapsto X(v)$ is continuous in the norm of $S_{1/4}$ and not constant: along $v = 2\lambda\cos 2\pi x$ with $|\lambda - 2| < 0.0498$, $X = 1 - \log\lambda/\beta(\alpha_*)$. On all of $\mathcal{U}$, X_{typ} is continuous, and it is the exponent at almost every phase.

Proof. By Theorem 13.7 with $Y = \varepsilon = 1/4$, every $v \in \mathcal{U}$ is supercritical and type I at every irrational frequency, with $0.4264 \leq \ell_* \leq 0.9035 < 1.3285 \leq \beta(\alpha_*) < \infty$. Every denominator of α_* is odd, so $|\sin(2\pi q_n/4)| = 1$ and $1/4 \in G_{\alpha_*}$, which is contained in $G^v_{\alpha_*}$ for even v [21, Lemma 5.22]. The transport law [21, Theorem A] gives $X(v) = 1 - \ell_*/\beta$, without clipping; [21, Theorem B(1)] gives the same value for every $\beta^+(p)$ and shows that ℓ attains ℓ_* on the support of the spectral measure, and [21, Theorem B(3)] then gives the packing dimension. The packet started at δ_j is a translate of the packet started at δ_0 for the phase $1/4 + j\alpha_*$, and $|\sin(2\pi q_n(1/4 + j\alpha_*))| = |\cos(2\pi j\delta_n)| \geq 1/2$ for large n. This proves part 1.

For part 2, ℓ_* is computable by Theorem 13.4, and positive. So $X = 1 - \ell_*/\beta$ is computable from β, and $\beta = \ell_*/(1 - X)$ is computable from X. Hence X has the degree of $\beta(\alpha_*)$, which is that of s_*, namely $\mathbf{0''}$.

Part 3 follows from Proposition 13.8 at the fixed frequency α_*, and from $|2\lambda\cos 2\pi z - 4\cos 2\pi z| \leq 2|\lambda - 2|\cosh(\pi/2)$ on $S_{1/4}$. The almost-everywhere statement is [21, Theorem A]. □

Remark 13.10 (Computable coefficients are needed)**.** Part 2 fails without them. Let $\lambda = \exp(\frac{51}{100}\beta(\alpha_*))$, so $1.969 < \lambda < 2.028$, and $v = 2\lambda\cos 2\pi x$. Then $2|\lambda - 2|\cosh(\pi/2) < 1/4$, so $v \in \mathcal{U}_{\text{even}}$, and $X(v) = 1 - \log\lambda/\beta(\alpha_*) = 49/100$ is computable; the coupling λ is not. The fixed

phase $1/4$ is also needed in part 3: for non-even $v \in \mathcal{U}$ we do not identify the exponent at the phase $1/4$, and use instead the phase of Proposition 13.13. Not every direction changes the value: adding a constant to v shifts the energy and leaves X unchanged.

Theorem 13.11 (Energy-resolved exponents)**.** *Let $v \in \mathcal{U}_{\mathrm{even}}$ have computable coefficients, and let $E_0 \in \Sigma$ be computable; for instance $E_0 = \min \Sigma$ (Corollary 13.5). Then $F_\beta(\ell(E_0)) = 1 - \ell(E_0)/\beta(\alpha_*)$ has Turing degree exactly $\mathbf{0}''$. If moreover E_0 lies in the support of the spectral measure μ of δ_0 for $H_{v,\alpha_*,1/4}$, then $F_\beta(\ell(E_0))$ is the energy-resolved exponent at E_0: for every $\gamma \in (0, \ell_*/2)$ there are real $g_\gamma \in C_c^\infty(\mathbb{R})$ supported in $\{E : |\ell(E) - \ell(E_0)| < \gamma\}$ with $g_\gamma(H)\delta_0 \neq 0$, and for every such family and every $p > 0$, $\beta^+_{g_\gamma(H)\delta_0}(p) \to F_\beta(\ell(E_0))$ as $\gamma \to 0$.*

Proof. $\ell(E_0)$ is computable (Theorem 13.4) and lies in $(0, \beta)$ (Theorem 13.7), so the degree argument of Theorem 13.9 applies. If $E_0 \in \operatorname{supp} \mu$, a smooth $g \geq 0$ equal to 1 near E_0 and supported in the open set $\{|\ell - \ell(E_0)| < \gamma\}$ has $\|g(H)\delta_0\|^2 = \int g^2 \, d\mu > 0$. For every admissible g_γ, [21, Theorem B(2)] gives $F_\beta(\ell(E_0) + \gamma) \leq \beta^+_{g_\gamma(H)\delta_0}(p) \leq F_\beta(\ell(E_0) - \gamma)$. □

By [21, Theorem B(3)], the upper local dimension of μ equals $F_\beta(\ell(E))$ at μ-almost every E. This is an almost-everywhere statement; it does not identify the local dimension at a prescribed computable energy.

Theorem 13.12 (The set of values)**.** *For every $v \in \mathcal{U}_{\mathrm{even}}$ with computable coefficients, as α ranges over the computable irrationals with eventually odd denominators, $\alpha_u^+(H_{v,\alpha,1/4})$ ranges over exactly the reals in $[0, 1]$ that are right-c.e. relative to $\mathbf{0}'$.*

Proof. By Theorem 13.7, $H_{v,\alpha}$ is supercritical and type I for every irrational α, and $1/4 \in G_\alpha$ when the denominators are eventually odd, so the transport law [21, Theorem A] applies at the phase $1/4$, including the endpoints $\beta \in \{0, \infty\}$.

Every value is right-c.e. relative to $\mathbf{0}'$. For computable α, $\ell = \ell_*(v, \alpha)$ is computable and positive, and $\beta(\alpha)$ is right-c.e. relative to $\mathbf{0}'$ or infinite [20, Lemma 11.1]. For rational $r \in (0, 1)$, $F_\beta(\ell) < r$ if and only if some rational s' satisfies $\beta < s' < \ell/(1 - r)$, a $\mathbf{0}'$-c.e. condition.

Every such value occurs. The values 0 and 1 occur for $\alpha = [0; 1, 2, 2, \dots]$ and for $a_1 = 1$, $a_{n+1} = 2^{q_n^2}$. Let $x \in (0, 1)$ be right-c.e. relative to $\mathbf{0}'$ and $\sigma = 1/(1 - x) > 1$. By [20, Lemma 11.1] there are computable rationals u_n with $\limsup u_n = \sigma / \log 2$; clamp them to $[1, B]$ for an integer $B > \sigma / \log 2$. Put $a_1 = 1$. Given $a_1, \dots, a_n$, let $\alpha^{(n)} = [0; a_1, \dots, a_n, 2, 2, \dots]$, a computable irrational, uniformly in n; compute a rational c_n with $|c_n - \ell_*(v, \alpha^{(n)})| < 2^{-n}$ (Theorem 13.4); and put $a_{n+1} = 2^{\max\{1, \lceil u_n c_n q_n \rceil\}}$. All a_{n+1} are even, so all q_n are odd, and $\alpha = \lim \alpha^{(n)}$ is computable, with $|\alpha^{(n)} - \alpha| \leq 2/q_n^2$. By Lemma 13.2, $\ell_*(v, a) = \min_{|E| \leq K} \ell(a, E)$ with K independent of a, and the Lyapunov exponent is jointly continuous in (a, E) at irrational a [9, Theorem 1]; so $c_n \to \ell := \ell_*(v, \alpha) > 0$. Since $a_{n+1} q_n \leq q_{n+1} \leq 2 a_{n+1} q_n$, $\log q_{n+1}/q_n = (\log 2) u_n c_n + O(\log q_n / q_n)$, so $\beta(\alpha) = \ell\sigma > \ell$ and $\alpha_u^+ = 1 - \ell/\beta = x$. □

Proposition 13.13 (A computable good phase)**.** *Let v be a computable real-analytic potential with H_{v,α_*} supercritical and type I. There is a computable phase θ_v such that $\theta_v + j\alpha_* \in G^v_{\alpha_*}$ for every $j \in \mathbb{Z}$. Hence $\alpha_u^+(H_{v,\alpha_*,\theta_v}) = F_{\beta(\alpha_*)}(\ell_*(v))$ for the packet started at any site; if $\ell_*(v) < \beta(\alpha_*)$, this number has Turing degree exactly $\mathbf{0}''$.*

Proof. By [21, Lemma 5.20(1)] there is q_*, depending on v and α_*, such that for convergent denominators $q \geq q_*$ each band centre has exactly one maximum and one minimum per period. Band centres are the ordered simple eigenvalues of a computable Hermitian Floquet matrix [21, Lemma 5.3], hence computable functions of the phase, and the unique maximizer of a computable

continuous function on the circle is computable. So the set Turn_q of these $2q$ points is computable uniformly in $q \geq q_*$. Let n_* be the first index with $q_{n_*} \geq q_*$; it is a fixed integer, which we do not compute but hard-code. Let $B_n = \{\theta : \mathrm{dist}(q_n\theta, \mathrm{Turn}_{q_n}) < 3q_n^{-3}\}$, a union of at most $2q_n^2$ arcs with computable endpoints and total measure at most $12q_n^{-2}$. Choose $N \geq n_*$ with $\sum_{n\geq N} |B_n| < 1/2$, using the effective tail given by Fibonacci growth, and run the nested dyadic construction of Corollary 12.12(4) with the sets B_n in place of E_n. It gives a computable θ_v outside every B_n, $n \geq N$. For a site j, $q_n(\theta_v + j\alpha_*) \equiv q_n\theta_v + j\delta_n$ and $|\delta_n| \leq 1/q_{n+1} \leq 2^{-q_n}$, so $\mathrm{dist}(q_n(\theta_v + j\alpha_*), \mathrm{Turn}_{q_n}) \geq 3q_n^{-3} - |j|2^{-q_n} \geq 2q_n^{-3}$ for large n, which is goodness in the sense of [21, Definition 5.21]. The value follows from [21, Theorem A], and the degree as in Theorem 13.9. □

The program for θ_v exists, but we do not exhibit it, because n_* is not computed. For even v the phase $1/4$ is explicit.

13.5 What is fragile

Theorem 13.14 (Fragility)**.** *Let $v \in \mathcal{U}_{\mathrm{even}}$ have computable coefficients, and write $X(v, \alpha, \theta) = \alpha_u^+(H_{v,\alpha,\theta})$.*

1. *The value $X(v, \alpha_*, \theta) = X(v)$ is the same for every θ in the full-measure set $G^v_{\alpha_*}$, which is not a neighbourhood of any phase. At every other phase, $X(v) \leq X(v, \alpha_*, \theta) \leq 1 - \ell_*(v)/(2\beta(\alpha_*))$, by [21, Theorem A] and Corollary 5.6; we do not know the exact value there.*

2. *For every irrational α_0 and every open interval $J \ni \alpha_0$, J contains computable frequencies α with $X(v, \alpha, 1/4) = 0$, with $X(v, \alpha, 1/4) = 1$, and with $X(v, \alpha, 1/4) = x$ for any prescribed $x \in (0, 1)$ that is right-c.e. relative to $\mathbf{0}'$. In particular $\alpha \mapsto X(v, \alpha, 1/4)$ is discontinuous at every irrational. Moreover $X(v, \alpha, \theta) = 0$ for almost every α and every θ, and $X(v, \alpha, \theta) = 1$ for a dense G_δ set of α and every θ.*

Proof. 1. [21, Theorem A] and Corollary 5.6, since $\ell_* \leq \beta(\alpha_*)$. The good set has empty interior. For even v the turning points are 0 and $1/2$ [21, Lemma 5.22], and the bad sets $\{\theta : \mathrm{dist}(q_n\theta, \{0, 1/2\}) < 2q_n^{-3}\}$ are open and $1/q_n$-periodic. So each tail union $\bigcup_{n\geq n_0}$ of them is open and dense, their intersection over n_0 is residual, and it is disjoint from $G^v_{\alpha_*}$. A set disjoint from a dense G_δ set has empty interior.

2. Choose a continued-fraction cylinder $[0; a_1, \ldots, a_k]$ contained in J and containing α_0. Two consecutive denominators are coprime, so not both even; appending at most two partial quotients equal to 1 makes the last two denominators odd. Now rerun the construction of Theorem 13.12 from this prefix, using the actual denominators of the prefixed frequencies: even partial quotients keep every later denominator odd, and the continuity argument gives $\beta = \ell\sigma$ as before. A finite prefix does change the frequency, and, through it, the later denominators; this is why the construction is rerun rather than a fixed tail being appended, which would in general rescale β. The endpoints 0 and 1 are obtained with tails $2, 2, \ldots$ and $a_{n+1} = 2^{q_n^2}$. Finally $\beta(\alpha) = 0$ for almost every α and $\beta(\alpha) = \infty$ for a dense G_δ set of α; at these frequencies the exponent is 0, respectively 1, at every phase [21, Theorem A]. □

So, for the transport laws identified here at the prescribed good phases, the frequency is the only fragile datum, and it is exactly the datum that carries the uncomputability; we have proved no robustness in the phase beyond those good phases: $X = 1 - \ell_*/\beta$ with ℓ_* computable and continuous in the potential, and $\beta(\alpha)$ noncomputable as a function of α and discontinuous everywhere.

Theorem 13.15 (A dichotomy)**.** *Let v be a computable even real-analytic potential with H_{v,α_*} supercritical and type I. Then either $\ell_*(v) > \beta(\alpha_*)$ and $X(v) = 0$, or $\ell_*(v) < \beta(\alpha_*)$ and $X(v)$ has Turing degree exactly $\mathbf{0}''$. For non-even v in this class the same holds at the phase θ_v of Proposition 13.13.*

Proof. $\ell_*(v)$ is computable and $\beta(\alpha_*)$ is not, so they differ. If $\ell_* > \beta$ the law gives 0; if $\ell_* < \beta$ the exponent computes β, as in Theorem 13.9. □

For instance the coupling $\lambda = 8$ gives $\ell_* = \log 8 > 2\log 2 \geq \beta(\alpha_*)$ and exponent 0. The dichotomy concerns the supercritical type I class; for critical potentials and for supercritical potentials that are not of type I we do not know the exponent at α_*.

Theorem 13.16 (Exponents that exist)**.** *1. The limit of a computable sequence of rationals is computable from $\mathbf{0}'$. So a quantity that is the limit of finite-time, finite-volume observables computable uniformly along computable times has degree at most $\mathbf{0}'$; degree $\mathbf{0}''$ requires a* lim sup*, and the* lim sups *of computable sequences of rationals are exactly the reals right-c.e. relative to $\mathbf{0}'$ [20, Lemma 11.1].*

2. For real-analytic v with $H_{v,\alpha}$ supercritical, every irrational α, every phase and every $p > 0$, the lower transport exponent $\beta^-(p) = \liminf_{T\to\infty} \log M_p(T)/(p \log T)$ is 0.

Proof. 1. This is the limit lemma.

2. Hopping has range one, so $|\langle \delta_j, e^{-itH}\delta_0\rangle| \leq (\|H\|t)^{|j|}/|j|!$, and the Abel average gives $\sum_{|j|>T^2} |j|^p a(j,T) \to 0$. By [21, Theorem 5.9], with $N_0 = C'q_n$, $C' \geq C_D$, $c_D C' \geq \kappa$, and $T_n = e^{\kappa q_n/(2p+4)}$, $P(N_0, T_n) \leq Ce^{-c_D C' q_n} + CT_n^3 e^{-\kappa q_n}$. Hence $M_p(T_n) \leq N_0^p + T_n^{2p} P(N_0, T_n) + o(1) = O((\log T_n)^p)$, so the liminf is at most 0. It is at least 0, because the Abel return probability $a(0,T)$ tends to the sum of the squared atomic spectral weights of δ_0, which is less than one since δ_0 is not an eigenvector. □

Part 2 is known. It follows directly from Jitomirskaya and Mavi [33, Theorem 1.7(1)], for piecewise Hölder potentials with continuous positive Lyapunov exponent, and it is in substance in the complex-energy method of Damanik and Tcheremchantsev [17]; we include the short proof for completeness. Consequently, in the supercritical analytic class, a moment exponent that exists as a limit is 0, and the other natural existing quantities, namely $\ell(E)$ at computable E, ℓ_*, the density of states, the spectrum and $\max_\Sigma \ell$, are computable at computable data (Theorem 13.4, Corollary 13.5). In the examples of this section, uncomputability is carried by lim sup-type quantities: upper transport exponents and upper packing dimensions. We do not claim a classification of all observables.

Remark 13.17 (Related work on uncomputability)**.** Undecidability and uncomputability of physical quantities are known in several forms: the spectral gap of Cubitt, Perez-Garcia and Wolf [15]; phase diagrams, including continuous parameters and undecidability on sets of positive measure, by Bausch, Cubitt and Watson [6]; and a single uncomputable transition point in a continuous Hamiltonian family, by Purcell, Li and Cubitt [42]. We do not claim that robust or continuous-parameter uncomputability is new. Paper A already gave an exponent of degree $\mathbf{0}''$ that is robust along a coupling interval. The new point here is that, at the frequency α_* and the phase 1/4, the degree is stable for the computable members of an explicit neighbourhood in the even analytic potentials, open relative to that class, with the analogous statement for non-even computable potentials at the good phase of Proposition 13.13 (Theorem 13.15); and that the Lyapunov exponent of a computable analytic potential is itself computable, so that in these laws the frequency carries the uncomputability.

14 Discussion and open problems

The picture. At a resonant scale a packet moves with the speed allowed by the widest band near its energy and stops where the drift of the local phase stops it. For type I potentials, at almost every phase the widest band has width about $e^{-\ell q}$; this is the heuristic behind the typical law. At the worst phase it can be as wide as $e^{-\ell q/(2\omega)}$, because 2ω zeros of the discriminant can form an exponentially tight cluster at one local phase, and such clusters do form when the potential has a hidden period; the zeros of a cluster need not coincide. For the almost Mathieu operator the cluster sits at the closed central gap of even-denominator approximants, and it is visited by a residual set of phases; when $\beta_e = \beta$ is finite and greater than $L/2$, this gives the residual law of Theorem C. For the Aubry–André XY chain and the observable classes of Definition 12.6, in the same regime, almost every phase has fixed-site exponent $\max\{0, 1 - L/\beta\}$, while the exponent uniform over sites is $1 - L/(2\beta)$ at every phase; exceptional phases can have the larger exponent at every fixed site as well, as the phase $1/4$ does in Corollary 12.12. All exponents here are infinite-time quantities of an infinite system; at finite times the dynamics is a sequence of bursts at the resonant times.

14.1 Open problems

1. The factor for primitive potentials. Theorem A charges the price of a cluster of 2ω zeros, and Proposition 10.17 shows that the price is paid when the potential has a hidden period $1/s$. Write $v(x) = w(sx)$ with w primitive, as in [21, Theorem 5.29], so that $H_{v,\alpha,\theta} = H_{w,s\alpha,s\theta}$.

Conjecture 14.1. *Let $v = w(s\,\cdot)$ be real analytic with $L > 0$ on Σ and w primitive. If $\max_\Sigma \ell \le \beta(s\alpha) < \infty$, then $\alpha_u^{\mathrm{unif}} = \sup_\theta \alpha_u^+(\theta) = 1 - \ell_*/(2\beta(s\alpha))$.*

For the two-harmonic family of Proposition 5.7 with $\lambda \neq 0$, which is then primitive and has acceleration two exactly where $\ell = \log|\mu|$, assume that $\omega = 2$ somewhere on Σ, so that $\ell_* = \log|\mu|$, and that $\ell^+ \le \beta < \infty$. Then Theorem A gives the upper bound $1 - \log|\mu|/(4\beta)$, while Conjecture 14.1 predicts $1 - \log|\mu|/(2\beta)$. We do not know a mechanism that would force clusters of more than two zeros for a primitive potential, nor a proof that they cannot occur at the scales that matter.

2. The matching lower bound for general potentials. Theorem 10.12 reduces the lower bound $1 - \ell_0/(2\beta)$ to the existence of a wide band, (W_v), near an energy with $\ell \approx \ell_0$. A sufficient condition is a small gap between band centres: if $E_b(x)$ and $E_{b+1}(x)$ are roots of $t_q(x,\cdot)$ near E' with $E_{b+1}(x) - E_b(x) \le e^{-(\ell(E')/2 - 4\xi)q}$, then the maximum principle for $t_q(x,z)/((z - E_b)(z - E_{b+1}))$ on a disk of radius $e^{-\xi q}$, with [21, Corollary 5.6], bounds $|\partial_E t_q(x, E_b)|$ by $e^{(\ell(E')/2 + 7\xi + o(1))q}$. We do not know whether this condition is also necessary. It is proved here only at $E = 0$ for even q and odd-harmonic potentials, through exactly closed gaps; for general type I potentials, and at the minimizer of ℓ when $\ell(0) > \ell_*$, it is open.

3. The regime below L. For the almost Mathieu operator, the worst-case law is proved for every finite $\beta_e = \beta > L/2$:

$$x = 2\beta/L > 1, \qquad \alpha_u^{\mathrm{unif}} = \sup_\theta \alpha_u^+(\theta) = \alpha_u^+(1/4) = 1 - 1/x.$$

At almost every phase the value is $\max\{0, 1 - 2/x\}$; the worst value occurs on a dense G_δ set. Thus $x = 2$ is the onset of positive typical transport, while positive worst-case transport is already proved for $x > 1$. For $\beta_e < \beta$ the upper bound remains $1 - L/(2\beta)$ when $\beta > L/2$, but the exceptional lower bound still contains β_e. For $0 < \beta \le L/2$, Corollary 5.6 remains the available uniform upper

bound here. Vanishing transport at every phase in that range is open. For general supercritical type I potentials below ℓ_*, the bound $\beta/(\beta+\ell_*)$ is unchanged. Their analogue of the inner-edge frame and shell analysis remains open. Paper D [19] proves vanishing exponential gap-decay rate above $x=1$. Below $x=1$ no new gap law is asserted here.

4. Odd scales for the almost Mathieu operator. Theorem 10.5 uses the closed gaps at even denominators. For odd q every gap of the rational approximants is open [45], so no closure argument is available, and a quantitative bound on one gap is needed. Fix an odd $q \geq 3$ and p coprime to q, and write P^λ for the Chambers polynomial of coupling λ. Its critical points $c_1 < \cdots < c_{q-1}$ are real and simple, and $\mathrm{crit}_k(\lambda) = |P^\lambda(c_k)| - 2\lambda^q = \min_\varphi |\Delta_\varphi(c_k)| \geq 2$, because every critical value of a periodic discriminant has modulus at least two. Put $\mathrm{ex}_k(\lambda) = \mathrm{crit}_k(\lambda) - 2$, and let $\mathrm{ex}_{\mathrm{ext}}$ refer to the two extreme gap labels $m = \pm(q-1)/2$, where $k \equiv mp \pmod q$.

Lemma 14.2 (Duality of the excess)**.** $\mathrm{ex}_k(1/\lambda) = \lambda^{-q}\,\mathrm{ex}_k(\lambda)$ *for every* k *and every* $\lambda > 0$.

Proof. On the cyclic q-site space let $H_\lambda(a,b) = e^{ia}S + e^{-ia}S^* + 2\lambda\,\mathrm{diag}\{\cos(2\pi np/q + b)\}$. The discrete Fourier transform with entries $q^{-1/2}e^{2\pi ipns/q}$ exchanges shifts and multiplications, and conjugates $H_\lambda(a,b)$ to $\lambda H_{1/\lambda}(b,a)$, up to orientation signs in a, b. So $\det(E - H_\lambda(a,b)) = \lambda^q \det(E/\lambda - H_{1/\lambda}(b,a))$. Averaging over both phases removes the phase-dependent harmonics of Chambers' formula and gives $P^\lambda(E) = \lambda^q P^{1/\lambda}(E/\lambda)$. So the critical points scale by λ, preserving their order, and $\mathrm{crit}_k(\lambda) - 2 = \lambda^q(\mathrm{crit}_k(1/\lambda) - 2)$. □

Proposition 14.3 (The strong-coupling coefficient)**.** *For odd* $q \geq 3$ *and* p *coprime to* q*,*

$$\lim_{\lambda\to\infty} \frac{\mathrm{ex}_{\mathrm{ext}}(\lambda)}{\lambda} = \lim_{\mu\to 0} \frac{\mathrm{ex}_{\mathrm{ext}}(\mu)}{\mu^{q-1}} = C_q := 4\sin^2\Big(\frac{\pi mp}{q}\Big), \qquad m = \frac{q-1}{2}.$$

In particular $0 < C_q \leq 4$.

Proof. The first equality is Lemma 14.2. At weak coupling μ, the Bloch fibre of the q-periodic operator in the plane-wave basis indexed by $s = 0, \ldots, q-1$ is a q-cycle with diagonal $\epsilon_s = 2\cos(k + 2\pi sp/q)$ and hopping of size μ. The gap with label m sits at the crossing $\epsilon_0 = \epsilon_m$, at $k \equiv -\pi mp/q \pmod\pi$ and $E_0 = 2\cos k$. For fixed p, q the two crossing states are isolated from the others, and a Schur complement onto them is finite-dimensional analytic perturbation theory. By reflection the two diagonal self-energies agree, and the off-diagonal coupling comes from the unique path of length m, the other path having length $m+1$:

$$|t_{\mathrm{eff}}| = \frac{\mu^m}{\prod_{s=1}^{m-1} |E_0 - \epsilon_s|}\,(1 + O_{p,q}(\mu)).$$

Using $|E_0 - \epsilon_s| = 4|\sin(k + \pi sp/q)\sin(\pi sp/q)|$ and the identity $\prod_{s=1}^{m} |\sin(\pi sp/q)| = \sqrt{q}\,2^{-m}$, which follows from $\prod_{s=1}^{q-1} |\sin(\pi sp/q)| = q2^{-(q-1)}$ and the symmetry $s \mapsto q - s$, one gets $\prod_{s=1}^{m-1} |E_0 - \epsilon_s| = q/(4\sin^2(\pi mp/q))$. The half-gap is $|t_{\mathrm{eff}}|$, and the free discriminant $2T_q(E/2)$ has second derivative of modulus $q^2/(2\sin^2(\pi mp/q))$ at the corresponding critical point. A Taylor expansion gives $\mathrm{ex}_{\mathrm{ext}}(\mu) = \frac{1}{2}|P''|\,|t_{\mathrm{eff}}|^2(1 + O_{p,q}(\mu)) = 4\sin^2(\pi mp/q)\mu^{q-1}(1 + O_{p,q}(\mu))$; the phase-union term $2\mu^q$ is of higher order. □

The remainders depend on p and q, so Proposition 14.3 gives no control uniform in q. Sharp exponential asymptotics of spectral gaps are known for Diophantine frequencies in the subcritical regime [37], and upper bounds on gaps at Liouville frequencies in a regime of weak dual coupling [41, Theorem 1.3]; as stated, neither gives the bound on the extreme gaps of the rational approximants that is needed here.

Conjecture 14.4 (Mon)**.** *For every odd q and every p coprime to q, the function $\lambda \mapsto \mathrm{ex}_{\mathrm{ext}}(\lambda)/\lambda$ is nondecreasing on $(0,\infty)$.*

Proposition 14.5 (Conditional exact law)**.** *If Conjecture 14.4 holds, then $\alpha_u^{\mathrm{unif}} = 1 - \log\lambda/(2\beta(\alpha))$ for the almost Mathieu operator with $\lambda > 1$ and every irrational α with $\log\lambda \le \beta(\alpha) < \infty$, without any parity condition.*

Proof. Under the conjecture, Proposition 14.3 gives $\mathrm{ex}_{\mathrm{ext}}(\lambda) \le C_q\lambda \le 4\lambda$, so the critical value $D = \mathrm{crit}_{\mathrm{ext}}(\lambda) \le 4\lambda + 2$ is bounded uniformly in p, q. Let φ be a phase at which $|\Delta_\varphi(c)| = D$ for the extreme critical point c, and let e be a point of an adjacent band with $|\Delta_\varphi(e)| \le 1$; it lies within $e^{-\xi q}$ of Σ when b_n is bounded below. On a $1/q$-neighbourhood of e, [21, Corollary 5.6] gives $|\Delta'_\varphi| \le 2qe^{(L+\varepsilon)q}$ and $|\Delta''_\varphi| \le M = 8q^2e^{(L+\varepsilon)q}$. Moving from e towards c by $t = |\Delta'_\varphi(e)|/M \le 1/(4q)$, which does not pass c because $\Delta'_\varphi(c) = 0$, Taylor's formula on a segment where the signed discriminant is at most D gives $|\Delta'_\varphi(e)|^2 \le 2M(D+1)$. So (W_v) holds at every large odd scale, with $\eta = 1$ and $\ell_0 = L$, and Theorem 10.12, together with Theorem 10.5 at even scales, gives $\alpha_u^{\mathrm{unif}} \ge 1 - L/(2\beta)$. Corollary 5.6 gives the upper bound on the stated range. □

5. Weak separation. The typical exponent beyond type I is known only conditionally on the weak separation hypothesis WXS for primitive potentials with acceleration at least two [21, Definition 7.1 and Theorem D]. It remains open, and so does the typical exponent of the two-harmonic family.

6. How much of the packet moves at the worst phase. Theorem 10.5 produces a polynomially small tail at the exceptional radius, whose exponent is $1 - L/(2\beta)$ when $\beta_e = \beta$ is finite and greater than $L/2$. For the almost Mathieu operator with $h > L$, [20, Theorem B] gives at every phase a fixed Abel-averaged fraction of the packet beyond Te^{-hq_n}, for $e^{hq_n} \le T \le |\delta_n|^{-1}$ on the resonant scales with $b_n \ge c > h$; at $T = |\delta_n|^{-1}$ this is the smaller, typical radius. Whether a fixed fraction reaches the larger exceptional radius of Theorem 10.5, and what the moment exponents are at the exceptional phases, is open.

How this work was done

The authors developed the results and proofs with assistance from AI models from more than one developer. They take responsibility for every claim. The authors chose the question, directed the work and decided which lines to pursue; the models did most of the exploration, derivation and drafting.

The proof was developed over several long working sessions, each building on the previous ones and recording its proofs, failed routes and self-checks. Each step was then checked by a separate session that had not seen the material, with instructions to refute it and to read the primary sources it relies on. The repairs found in these checks are incorporated here. The additional sessions for version 2 developed the inner-edge and all-depth transport argument; independent sessions checked each stage, and the draft uses the replacement shell construction and the later repairs from those checks. Priority and literature searches were AI-assisted. The assembled paper was then audited as a whole by another session that had not seen it, also instructed to refute; the repairs made in answer to that audit and to each later check were checked again by a fresh session, including one session that compared every summary sentence with the result it summarizes, and the final small repair uses the wording that the last check proposed. No computation is used in any proof, except that the elementary numerical constants of Theorem 13.7 were also certified with exact rational arithmetic; the numerical remarks are not used in any proof.

A Lattice composition with matrix symbols

The Weyl calculus is classical; see [27, 43]. We record the elementary discrete calculation to specify matrix order, shifted domains and the errors used here. For $a(k,t)=\sum_m a_m(t)e^{imk}$, put $t_j=t^\nu+j\delta$ and

$$(\mathrm{Op}(a)v)_j=\sum_m a_m(t_j-m\delta/2)v_{j-m},\qquad N_r^p(a)=\sum_m(1+|m|)^p\sup_I\|\partial_t^r a_m\|.$$

In estimates requiring an off-cell factor, omit the zero Fourier mode. The interval I contains every shifted segment below. Absolute row and column sums give $\|\mathrm{Op}(a)\|\le N_0^0(a)$. For the symmetric endpoint-average quantization,

$$\|\mathrm{Op}^{\mathrm{sym}}(a)-\mathrm{Op}(a)\|\le\delta^2N_2^2(a)/8. \tag{53}$$

Lemma A.1 (Exact product and second-order remainder)**.** *Matrix order is preserved in*

$$(a\#b)_\ell(t)=\sum_{m+n=\ell}a_m(t+n\delta/2)b_n(t-m\delta/2),\qquad \mathrm{Op}(a)\,\mathrm{Op}(b)=\mathrm{Op}(a\#b). \tag{54}$$

With $\{a,b\}=a_kb_t-a_tb_k$,

$$\Big\|\mathrm{Op}(a)\,\mathrm{Op}(b)-\mathrm{Op}\Big(ab+\frac{i\delta}{2}\{a,b\}\Big)\Big\|\le\frac{\delta^2}{8}\big[N_2^0(a)N_0^2(b)+2N_1^1(a)N_1^1(b)+N_0^2(a)N_2^0(b)\big]. \tag{55}$$

If P_J *is the cell projection onto a finite well, then*

$$(P_J\,\mathrm{Op}(a)P_J)(P_J\,\mathrm{Op}(b)P_J)=P_J\,\mathrm{Op}(a\#b)P_J-P_J\,\mathrm{Op}(a)(1-P_J)\,\mathrm{Op}(b)P_J. \tag{56}$$

The last term uses the full-lattice factors before compression.

Proof. Multiplying kernels at $(j,j-m,j-m-n)$ gives (54). Apply Taylor's formula in $u\in[0,1]$ to $a_m(t+un\delta/2)b_n(t-um\delta/2)$. The first derivative gives $(i\delta/2)\{a,b\}$. The three second derivatives have coefficients $n^2,-2mn,m^2$. Integrating their norms and summing both absolute Schur sums proves (55). This uses only second derivatives and covers negative δ. Equation (53) is the trapezoid-midpoint bound. Insert $1=P_J+(1-P_J)$ between full factors for (56). □

Here is the family-specific control of these seminorms. With the two finite anchor lifts at shorter-arc distance $r\le q/2$ and $a_0=L-4\tau$,

$$\sum_x e^{-a_0\,\mathrm{dist}(x,\mathcal{A})}e^{-a_0\,\mathrm{dist}(x,\mathcal{A}+mq)}\le C(|m|q+1)e^{-a_0(|m|q-r)},\qquad m\neq0. \tag{57}$$

Fixed differentiated versions follow from (26). The matrix coefficients of h are inner products of two frame vectors with the nearest-neighbor frozen operator. The orbital matrix also has one position factor, costing $q(1+|m|)$. Thus, for the finite p,r' used in the preceding estimates,

$$N_{r'}^p(Y-Y_0^{\mathrm{Fourier}})\le z_\gamma\mathcal{R}_qe^{-a_0(q-r)}, \tag{58}$$

$$N_{r'}^p(W-W_0^{\mathrm{Fourier}})\le qz_\gamma\mathcal{R}_qe^{-a_0(q-r)}, \tag{59}$$

$$N_{r'}^0(Y)\le z_\gamma\mathcal{R}_q,\qquad N_{r'}^0(W)\le qz_\gamma\mathcal{R}_q. \tag{60}$$

The Fourier subscripts in this display are unrelated to the energy parameter Y_0 or the floor W_0. The same common periodic gauge is used for every coefficient. The proof is (57), followed by summation

over m; finite extra polynomial factors are absorbed in $\mathcal{R}_q$. It relies on the localized frame, not merely on $Z^2 = xI_2$.

For Fourier truncation at $|m| \le q$, enlarge the phase interval by $[-3q|\delta|/2, 3q|\delta|/2]$. It then contains all product midpoints and shifted segments. This enlargement is smaller than the reserved frame-chart margin for $b \ge 2\xi$. The omitted Fourier tail has its absolute weighted sum retained. Every nonzero term crossing a shell or well-end interface contains two off-cell factors in (56). Its norm is at most $C\overline{H}(\overline{H} + h_\vartheta)$. It is not part of an $O(\delta^2)$ Taylor remainder.

For completeness, put $u = Y_0 - \vartheta$, so the two symbols in (36) are $b = \widetilde{Y} + \varepsilon W + u$ and $a = \widetilde{Y} + \varepsilon W - u$. Then $ba = \Phi I_2 + \varepsilon\{\widetilde{Y}, W\}_+ + \varepsilon^2 W^2$ and $\{b, a\} = [y_k, y_t] + 2\{y, \vartheta\}$. Equations (55) and (60) prove the product and its $\kappa\overline{H}\mathcal{R}_q$ error. This also identifies the anti-Hermitian term that must be retained.

B The localized three-block boundary calculation

We give the end correction separately because the global estimate $c_{\mathrm{end}}^2/m \ll A$ is not required by the argument. Fix a fine window and use its shell penalty and capped weight. Write $K = e^{\varphi}(H + M_E P_{\mathcal{S}} - E)e^{-\varphi}$. Split the frame projection as $\Pi = \Pi_I + \Pi_B$, where B consists of end strips beyond the deepest shell and its padding, and I is the remaining frame. Keep Fourier and interwell tails as errors in their assigned budgets. The weight is constant at this end seam. Write $F = K_{\Pi\Pi}$.

The shell gap, compression estimate and weighted frame estimate give $\|F^{-1}\| \le 1/(2A)$, with the same adjoint statement. At an end cell, $U_{\mathrm{end}} \asymp \Lambda e^{-\xi q/2}$ and the spectral depth is $e \le 4\Lambda e^{-\xi q}$. The product lower bound and its partner normalization give

$$\begin{aligned} d := \sigma_{\min}(F_{BB}) &\ge \frac{c(U_{\mathrm{end}} - e)}{z_\gamma(\sqrt{\mathrm{ex} + U_{\mathrm{end}}} + \sqrt{\mathrm{ex} + e})} \\ &\ge \frac{cU_{\mathrm{end}}}{z_\gamma\sqrt{\mathrm{ex} + U_{\mathrm{end}}}} \ge e^{-(\xi/2 + \text{chosen loss})q}. \end{aligned} \tag{61}$$

When ex dominates, replacing this divided difference by $\sqrt{U_{\mathrm{end}}}/z_\gamma$ would be incorrect. The penalty is zero on B. Also

$$\|F_{IB}\|, \|F_{BI}\| \le t_{\mathrm{end}} = \overline{H}/z_\gamma \le e^{-(L/2 - 2\tau)q}\mathcal{R}_q, \qquad \|K_{QQ}^{-1}\| \le 2/m.$$

Let c_0 now denote the complete weighted bulk cross-block bound, so $c_0^2/m \le A/50$ after assigning the budgets and $c_0 = o(m)$. The end cross blocks are bounded by $c_0 + c_{\mathrm{end}}$. The weighted c_0 need not be $O(|\delta|\mathcal{R}_q)$.

Eliminate Q, writing $R = K_{QQ}^{-1}$ and $S = F - G$, $G = K_{\Pi Q} R K_{Q\Pi}$. Then

$$\|G_{II}\| \le 2c_0^2/m, \quad \|G_{IB}\|, \|G_{BI}\| \le 2c_0(c_0 + c_{\mathrm{end}})/m, \quad \|G_{BB}\| \le 2(c_0 + c_{\mathrm{end}})^2/m.$$

Set

$$r_1 = (c_0 + c_{\mathrm{end}})^2/(md), \quad r_2 = t_{\mathrm{end}}/d, \quad r_3 = c_{\mathrm{end}}^2 t_{\mathrm{end}}^2/(md^2 A).$$

One has $A/d = o(1)$, by (34), the raw compression errors and (61). Hence the bulk part of r_1 is at most $A/(50d) = o(1)$. Its end part tends to zero. Furthermore

$$r_2 = o(1), \qquad r_3 \le e^{(-L + 3\xi/2 + 4\tau)q}\mathcal{R}_q^C = o(1). \tag{62}$$

The last bound uses $A \ge \epsilon_{\mathrm{ref}} \ge e^{-(L + \text{chosen loss})q}$ and ξ sufficiently small. All estimates hold for adjoints too.

For q large, S_{BB} is invertible with norm of its inverse at most $2/d$. The resolvent identity gives

$$S_{BB}^{-1} - F_{BB}^{-1} = F_{BB}^{-1} G_{BB} S_{BB}^{-1}.$$

The difference of interior Schur complements is exactly

$$\begin{aligned} S/S_{BB} - F/F_{BB} = & -G_{II} - F_{IB}(S_{BB}^{-1} - F_{BB}^{-1})F_{BI} \\ & + G_{IB}S_{BB}^{-1}F_{BI} + F_{IB}S_{BB}^{-1}G_{BI} - G_{IB}S_{BB}^{-1}G_{BI}. \end{aligned} \tag{63}$$

Both mixed I–Q–B paths are present. The second term costs at most $8r_2^2c_0^2/m + 8r_3A$. The two mixed terms together cost at most

$$8r_2c_0^2/m + 8\sqrt{c_0^2/m}\sqrt{r_3A}.$$

The last term costs at most $8r_1c_0^2/m$. Together with $\|G_{II}\| \le A/25$, their sum is at most $A/10$ for all large q.

The inverse of the unperturbed interior Schur complement is the I, I block of F^{-1}. It therefore has norm at most $1/(2A)$; no positivity of F is assumed. Thus the perturbed interior inverse has norm at most $1/(1.9A)$. Since $r_2 = o(1)$ and $A/d = o(1)$, block reconstruction gives $\|S^{-1}\| \le 0.6/A$ eventually. The strict little-o comparison $A/d = o(1)$ is needed for this constant; $A \le d/4$ alone would not suffice. Finally $A = o(m)$ and both full cross blocks are $o(m)$, so reconstructing Q gives $\|K^{-1}\| \le 1/A$. The same calculation for K^* supplies the two-sided inverse statement. A small complex strip, for example $|\operatorname{Im} z| \le A/4$ after a fixed contraction of the real fine window, has inverse $O(1/A)$ by the resolvent identity.

C From weighted gaps to launch-shell confinement

We supply the functional-calculus details used in Proposition 6.3. The physical Green, synthesis and orthogonalization kernels are those of [19, Appendices B–D], extended to rank at most two in Lemma 7.2. Let $U_{\mathcal{S}}$ synthesize shell vectors. The weight is zero on their cells. For a tail at cell distance d from such a cell,

$$|e^{\pm\varphi(x)} - 1| \le C\mu d e^{\mu d}.$$

Combining this with the physical frame tails gives an absolute kernel majorant with one tail from a shell cell to a jump bond. Its row and column sums are $C\mu\sqrt{F}$. Compressing to the frame multiplies by a second localized kernel; convolution and both Schur sums give $C\mu F$. Both endpoints of every jump bond are included. The number of shell components is bounded, so the constants are independent of their lengths. Across wells retain the physical interwell tail multiplied by $e^{\mu D_{\mathrm{cap}}}$, as in [19, Appendix D, Boundary leakage with the correct rate]. It remains negligible. This proves (45), including both weight signs. Expanding $e^{\varphi}P_{\mathcal{S}}e^{-\varphi} - P_{\mathcal{S}}$ gives the one-tail and two-tail penalty bounds used in Proposition 8.3.

The same proof in rank one gives the top, bottom and W estimates. At low b the normalized compression errors can grow. Use $A = D/|P'(E_a)|$, not a dimensionless margin exceeding the range of a previously normalized gap lemma. The coarse shell width and the larger padding proved above still give (14). The raw top-frame estimates require only $b \ge 2\xi$: on the IMS patches the freezing error $C\sqrt{|\delta|/q}$ is $o(m)$. The small-κ Mathieu ladder and its quantization assumptions are unused in this extension.

C.1 Static localization and the initial tail

Let $H_O = H + M_E P_{\mathcal{S}}$ and let g be a smooth fine filter supported in the contracted gap window of H_O. The frame, complement and boundary calculations above give

$$g(H_O) = 0, \qquad \|e^{\varphi}(H_O - z)^{-1}e^{-\varphi}\| \le C/A$$

on the small complex strip used for an almost-analytic extension. Its vertical width and all required inverse constants are $e^{O(q)}$. The resolvent identity and Helffer–Sjöstrand formula give

$$\|\mathbf{1}_{\mathcal{O}} g(H)\| \le C(g) M_E A^{-1} e^{-\mu p_{\mathrm{pad}}} \le \exp(-e^{\tau q}) \tag{64}$$

after harmless contractions and polynomial factors. Here $\mathcal{O}$ is outside the padded union of shell cells. Although the penalty is nonlocal, its weighted synthesis is bounded by (45); this is the needed right-hand factor in the resolvent identity. Choose a wider filter $g_2 = 1$ on a neighborhood of $\operatorname{supp} g$, still inside a spare gap buffer, with fixed-order derivatives $e^{O(q)}$. Constructing the initial gap with margin $4A$ and using a narrower fine window leaves this buffer. This avoids assuming that an arbitrary compact support stays a fixed distance from a window end.

Merge padded shell intervals that meet or are separated by less than a fixed multiple of the padding. Their number is bounded, so each merged component still has width bounded by a constant times (14). If the launch x_0 lies outside the padded union, $\|g(H)\delta_{x_0}\|$ is bounded by (64). Otherwise choose the merged component containing it and a position cutoff χ whose boundary bonds lie in $\mathcal{O}$. In the functional-calculus formula for $[g(H), \chi]$, split

$$(H - z)^{-1} = (H - z)^{-1} g_2(H) + (H - z)^{-1}(1 - g_2(H)).$$

The term with both resolvents in the second part is analytic near $\operatorname{supp} g$ and integrates to zero. Each remaining term contains $[H, \chi]g_2(H)$ or its adjoint, which is bounded by (64). The functional-calculus factors are $e^{O(q)}$, so the initial mass outside this component is doubly exponentially small.

For $\psi_t = e^{-itH} g(H)\delta_{x_0}$,

$$\left| \frac{d}{dt} \langle \psi_t, (1 - \chi)\psi_t \rangle \right| \le \|[H, \chi]g_2(H)\| \, \|\psi_t\|^2.$$

Thus the outside mass is at most $\exp(-e^{c\xi q})(1 + |t|)$ for a fixed $c > 0$, after reducing the constant in the exponent. This is confinement about the launch shell. Its distance from the well bottom is not included in its width. Summing the $e^{O(q)}$ fine windows preserves every fixed inverse-power tail on $|t| \le e^{Bq}$.

C.2 The interior functional-calculus bridge

Let f be a smooth filter supported in the interior class of (17), with derivatives $e^{O(q)}$. Insert a smooth spectral-neighborhood cutoff equal to one on the spectrum. For $|t| \le e^{Bq}$, all fixed derivatives of $h_t(E) = e^{-itE} f(E)$ are $e^{O(q)}$. In an almost-analytic formula for $h_t(H)$ choose fixed extension order r and vertical cutoff $y_0 = e^{-dq}$. Fix the desired error e^{-Mq} and choose a constant B' large. For $e^{-B'q} \le |\operatorname{Im} z| \le y_0$, use the interior resolvent estimate at auxiliary time $1/|\operatorname{Im} z|$. Its doubly exponential error absorbs every $e^{O(q)}$ factor. For $|\operatorname{Im} z| < e^{-B'q}$, the plain resolvent bound and $\bar{\partial}\widetilde{h}_t = O(e^{C_r q}|\operatorname{Im} z|^r)$ give at most $Ce^{(C_r - rB')q}$. Choose B' so that this is below e^{-Mq}. The argument treats the negative half-plane by conjugation. It proves the interior filtered-packet estimate on every fixed exponential time window. It asserts no longer time range. This completes the analytic passage required by Proposition 6.3.

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