One-shot zero-error communication with normal entanglement is Σ_2^0-complete

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Version 1.0.0, November 2026

Abstract

Can a finite classical channel require an infinite entangled resource to send a fixed number of messages without error? We classify the one-use feasibility problem for rational channels. With an arbitrary normal spatial resource it is Σ_2^0-complete. The same classification holds when the resource must have finite marginal entropy. A separate channel admits a normal zero-error code, while every normal code achieving its specified message count has infinite marginal entropy on both sides.

The proof extends Mančinska's finite-dimensional game-to-channel correspondence to trace-class operators. For the game's channel, a normal code attaining the clique bound (one message per local question) corresponds to perfect play with summable Schmidt amplitudes. Four successive constraints, each doubling a ratio of Schmidt weights, are critical and exclude such a code; five suffice for both reductions. As a corollary of Mančinska's finite-dimensional correspondence and $\mathsf{MIP}^* = \mathsf{RE}$, we obtain Σ_1^0-completeness for finite-dimensional resources. We also prove Π_1^0-completeness for commuting operators. Our capacity statements are upper computability bounds; the hardness results concern one use.

AI systems assisted with this work. The authors take responsibility for its claims; see "How this work was done." This preprint has not been peer reviewed.

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1 Introduction

A sender uses a finite noisy classical channel once. Beforehand, sender and receiver share an entangled state. The sender encodes one of m messages into the channel input, and the receiver must identify that message with certainty. We ask whether such a code exists, allowing an arbitrary normal state on a tensor product of Hilbert spaces. The channel and the message count remain finite; only the shared resource is unrestricted.

Zero-error communication depends on exact zeros, rather than on how closely a protocol approaches perfect transmission. This makes attainment decisive. We show that allowing a normal infinite-dimensional resource raises one-use feasibility from the first to the second existential level of the arithmetical hierarchy. We also construct a channel for which infinite marginal entropy is necessary to attain a specified message count.

For a finite rational stochastic matrix N and an integer $m \geq 1$, write $\mathrm{ZE}_t(N, m)$ for this feasibility problem. The model q uses finite-dimensional tensor products. The model qs uses a normal density operator on a spatial tensor product, with bounded finite-outcome POVMs. The model qc uses commuting local operator algebras and a state. Here normal means represented by a positive trace-class density operator of trace one; on a spatial tensor product this includes pure vectors and countable mixtures. The entropy restriction finS means that both resource marginals have finite von Neumann entropy. We use logarithms to base two, countably infinite Schmidt decompositions, and computable many-one reductions.

For readers unfamiliar with the hierarchy, a set is Σ^0_2 if membership can be written $\exists j\, \forall r\, R(e, j, r)$ with R decidable. Completeness means that the problem is itself of this form and every set of this form reduces computably to it. The classes Σ^0_1 and Π^0_1 use a single existential or universal number quantifier, respectively.

Theorem 1.1. *For finite rational classical channels,* $\mathrm{ZE}_{\mathrm{qs}}$ *and* $\mathrm{ZE}_{\mathrm{finS}}$ *are* Σ^0_2*-complete.*

Theorem 1.2. *There are a finite rational classical channel* N *and an integer* m *for which* $\mathrm{ZE}_{\mathrm{qs}}(N, m)$ *holds, and every normal spatial* m*-message zero-error resource has infinite marginal entropy on both sides.*

The main bridge to these results is a trace-class extension of Mančinska's correspondence. For any finite two-player game G, her graph $\Gamma(G)$ has one vertex per tagged local question–answer pair. Let k be the total number of local questions. We prove that the associated channel admits k messages with a normal resource exactly when G has a perfect pure spatial strategy with summable Schmidt amplitudes. This is stronger than mere normal attainment. The finite-dimensional statement, including the converse and the state transformation, is already [10, Theorem 6].

We apply this extension to the compiler of [6]. Its original four-bridge games lie exactly at the summability threshold: none of their channels admits a saturating normal code, even on YES inputs. Adding a fifth bridge changes the modular ratio and restores summability. The same five-bridge YES codes have finite marginal entropy. The threshold is visible in the four-letter tree density $h_{F,\lambda} = (1 - 4\lambda) \sum_w \lambda^{|w|} |w\rangle\langle w|$ of Section 3.3:

$$\operatorname{Tr} \sqrt{h_{F,\lambda}} = \sqrt{1 - 4\lambda} \sum_{\ell=0}^{\infty} (4\sqrt{\lambda})^{\ell}, \qquad \lambda = 2^{-n}.$$

There are four branches per level, so summability requires $n \geq 5$. A four-bridge version of the Catalan assembly of [5] instead makes the code's Catalan marginal factor exactly h_{Cat}, with infinite entropy. The resulting game H_{Cat} has finite-entropy perfect strategies: its Catalan factor k_{Cat} has entropy about 0.41 bits, and the remaining canonical factors are finite dimensional. Its channel forces infinite entropy at the specified message count. Thus the communication task itself creates the entropy demand through the normalized square-root map.

Finite and commuting resources. We also obtain that ZE_q is Σ_1^0-complete and $\mathrm{ZE}_{\mathrm{qc}}$ is Π_1^0-complete. The finite-dimensional lower bound is a direct corollary of Mančinska's Corollary 7 and the perfect-completeness form of $\mathsf{MIP}^* = \mathsf{RE}$ [10, 8]. We claim no new finite-dimensional game-to-channel reduction. For commuting resources we give the saturation argument for arbitrary codes, rather than assuming the code comes from a synchronous strategy.

Prior work and attribution. Cubitt, Leung, Matthews and Winter established the operator characterization of entanglement-assisted zero-error codes and the pseudo-telepathy construction [3]. Mančinska proved finite-dimensional clique saturation in Lemma 5, the converse and square-root state map in Theorem 6, and the entanglement-assisted independence-number characterization in Corollary 7 of [10]. Her tracial game value admits infinite-dimensional tracial strategies, but her zero-error results assume a finite-dimensional shared state. Her question count q is our k. Corollary 8 gives the fifty-input flagged-magic-square channel with ten entangled messages and at most nine messages from finite-dimensional maximally entangled resources. Our finite numerical checks reproduce that example as a regression check.

Mančinska, Roberson and Varvitsiotis studied perfect strategies and independent-set game reductions [12]; Mančinska and Roberson developed quantum graph homomorphisms [11]. Harris proved universality results for synchronous graph homomorphism games [7]. Those game parameters should be distinguished from codes with arbitrary resources. Kim, Paulsen and Schafhauser's equality of finite-dimensional and spatial synchronous correlation sets [9] does not apply to our nonsynchronous compiler.

Bhattacharyya, Mehta and Zhao proved undecidability for a restricted entanglement-assisted one-shot parameter of classical-to-quantum channels [2]. Their formal one-shot parameter uses maximal entanglement and projective measurements [2, Definition 2.3]. The finite row here removes these restrictions for classical channels and for a one-shot threshold. It does not resolve the unrestricted asymptotic capacity question. Slofstra proved Π_1^0-hardness for linear system games [16]; with the NPA bound this is Π_1^0-completeness, which Mousavi, Nezhadi and Yuen reprove for synchronous games [13, Theorem 6.4]. This is the source problem for our commuting row.

To our knowledge, the new statements developed here are the normal trace-class correspondence, the second-level normal complexity classification, the summability threshold of the compiled games, the infinite-entropy-only channel, and the commuting-code saturation reduction. The finite row is recorded as an explicitly attributed corollary. All claims about capacities in Section 7 are upper bounds on computability.

This priority statement is provisional. It rests on the cited hierarchy and correlation papers, their nearby results, and a targeted literature search. It is not an exhaustive bibliographic certificate.

Notation and glossary

symbol or term	meaning
N, m	finite rational channel and number of messages
$G, H_{\mathrm{Cat}};\ \Gamma(G)$	a game, the Catalan assembly game; its tagged confusability graph
$k;\ \mathcal{C}_c$	number of tagged local questions; a clique in their partition
$n, \lambda = 2^{-n}$	number of FMS bridges and creation covariance ratio
$d_{\mathrm{row}}, q_{\mathrm{clk}}$	finite row-ancilla dimension and clock order/dimension
$D, h = D^2$	positive Schmidt operator and pure-game marginal
ρ	channel-code marginal, equal to $D/\operatorname{Tr} D$ in the forward map
$h_0, h_{F,\lambda}, \sigma$	source density, four-letter tree density and spectator density
$h_{\mathrm{Cat}}, k_{\mathrm{Cat}}$	critical Catalan density and its normalized square
C_{Cat}	Catalan height counter; C_{ctr} is the computation counter
Ω_n, μ_n, ν_n	scaffold cover constant, scaffold density weights and their normalized square roots
$\mathcal{W}, \mathcal{W}_{\mathrm{Cat}};\ u(w)$	four-letter words, legal Catalan words; number of Catalan up-steps
$M_g;\ \mathrm{Mat}_r$	magic support of bridge g; scalar r-by-r matrix algebra
τ	tracial state on the commuting centralizer

Normal: represented by a trace-one positive trace-class operator. *Centralizer*: operators commuting with a density, or satisfying $\phi(XT) = \phi(TX)$ for a state ϕ. *Bridge*: an FMS test fixing the density ratio between two selected corners. *Scaffold*: the finite corner system joining those bridges. *Summable Schmidt amplitudes*: $\operatorname{Tr} D < \infty$. *Saturating code*: $m = k$ for the specified clique partition. *YES/NO*: an instance with/without the specified perfect strategy or code.

2 Normal zero-error codes and perfect play

2.1 Channels and trace-class assemblages

For channel inputs x, x', write $x \sim x'$ when some output y has $N(y \mid x)N(y \mid x') > 0$. This is a reflexive relation: it includes $x = x'$. Its restriction to distinct inputs is the confusability graph. Every finite graph X has a rational realizing channel N_{X}: a nonisolated vertex outputs a uniformly random incident edge, and an isolated vertex outputs its own private symbol.

We first extend the finite-dimensional assemblage characterization of [3]. All supports below are closed range supports.

For a Hilbert–Schmidt operator T, the double ket denotes vectorization in fixed orthonormal bases:

$$|T\rangle\!\rangle = \sum_{a,b} T_{ab}|a\rangle \otimes |b\rangle, \qquad \| |T\rangle\!\rangle \|^2 = \operatorname{Tr} T^*T.$$

Lemma 2.1 (Normal assemblages)**.** *An m-message normal zero-error code exists if and only if there are positive trace-class operators $R_{i,x}$ and a density ρ such that*

$$\sum_x R_{i,x} = \rho \quad (1 \le i \le m), \qquad R_{i,x}R_{j,x'} = 0 \quad (i \ne j,\ x \sim x'). \tag{1}$$

The assemblage is realizable with the pure resource $|\sqrt{\rho}\rangle\!\rangle$. The equivalence includes mixed resources.

Proof. Given a mixed resource ω and Alice's encoding effect $E_{i,x}$, define

$$R_{i,x} = \operatorname{Tr}_A[(E_{i,x}^{1/2} \otimes I)\omega(E_{i,x}^{1/2} \otimes I)].$$

The sum is Bob's marginal, independently of i. For output y, Bob must distinguish the positive operators $T_i^y = \sum_x N(y \mid x) R_{i,x}$ perfectly. If F_i^y is a decoding POVM, perfect decoding gives $F_i^y = I$ on $\operatorname{supp} T_i^y$ and $F_j^y = 0$ there for $j \neq i$. Thus these supports are orthogonal. Positivity of the sum puts $\operatorname{supp} R_{i,x}$ in $\operatorname{supp} T_i^y$ whenever $N(y \mid x) > 0$. This proves the product condition, including equal inputs.

Conversely restrict to $\operatorname{supp} \rho$, so ρ is faithful. Each $0 \leq R \leq \rho$ has a unique bounded effect G with

$$R = \rho^{1/2} G \rho^{1/2}, \qquad 0 \leq G \leq I. \tag{2}$$

Indeed Douglas' factorization lemma [4] applied to $R \leq \rho$ gives $R^{1/2} = \rho^{1/2} V_0$ for a contraction V_0, and $G = V_0 V_0^*$. Uniqueness follows from the dense range of $\rho^{1/2}$. The sum identities imply $\sum_x G_{i,x} = I$, by the same dense-range argument. No unbounded inverse of ρ is used.

In a spectral basis for ρ, let Alice measure $G_{i,x}^{\mathsf{T}}$ on the purification $|\sqrt{\rho}\rangle\!\rangle$. Bob's conditional operator is $R_{i,x}$. Equation (1) makes the T_i^y have orthogonal supports, so their support projections, with any unused subspace assigned to one message, form a decoding POVM. This constructs a normal pure code.

Alternatively, any normal mixed perfect code decomposes spectrally into pure perfect codes with the same measurements: every forbidden probability is nonnegative, so it vanishes on every positive-weight component. This observation will also be used for entropy. □

2.2 Saturation of a clique partition

Beigi's Lovász-theta bound implies the finite-dimensional clique-cover bound [1]. The following trace-class saturation statement extends [10, Lemma 5].

Lemma 2.2. *If X is partitioned into k cliques $\mathcal{C}_c$, every normal code has at most k messages. If it has exactly k, put*

$$B_{i,c} = \sum_{x \in \mathcal{C}_c} R_{i,x}, \qquad P_{i,c} = \operatorname{supp} B_{i,c}.$$

On $\operatorname{supp} \rho$ one has

$$B_{i,c} = \rho P_{i,c} = P_{i,c} \rho, \qquad \sum_i P_{i,c} = I, \qquad \sum_c P_{i,c} = I. \tag{3}$$

Each row and column of $(P_{i,c})$ is an orthogonal projection partition.

Proof. For fixed c, distinct i give orthogonal $B_{i,c}$, because every pair of inputs in $\mathcal{C}_c$ is confusable, including identical inputs. Since $B_{i,c} \leq \rho$,

$$\sum_i \operatorname{Tr} B_{i,c} \leq \sum_i \operatorname{Tr}(P_{i,c} \rho) \leq 1.$$

Summing over c gives $m \leq k$. If $m = k$, all inequalities are equalities. Faithfulness of ρ gives $\sum_i P_{i,c} = I$. For each i, c, positivity and $\operatorname{Tr}[P_{i,c}(\rho - B_{i,c})] = 0$ imply $(\rho - B_{i,c})^{1/2} P_{i,c} = 0$. Hence $\rho P_{i,c} = B_{i,c} = P_{i,c} \rho$. Summing over c yields $\rho \sum_c P_{i,c} = \rho$; injectivity gives the row sum. A finite family of projections summing to identity is pairwise orthogonal: compress the sum by one member and use positivity of all other compressed members. All traces used here are finite traces of positive trace-class operators. □

A code is *saturating* for this partition if its message count equals the clique bound, $m = k$.

2.3 The normal correspondence

Let G have question sets U, V, finite nonempty answer sets, and a Boolean losing predicate on the question pairs of positive sampling probability. Its graph $\Gamma(G)$ has vertices (L, u, a) and (R, v, b). Distinct answers to the same tagged question form a clique; cross edges are precisely losing tuples. There are no other edges. Set $k = |U| + |V|$.

For a pure strategy, use Schmidt form $|D\rangle\!\rangle$ with $D \geq 0$ and $\operatorname{Tr} D^2 = 1$. For bounded positive effects, zero probability is equivalent to $EDF^{\mathsf{T}} = 0$. To see this, zero probability is the vanishing of the Hilbert–Schmidt norm of $\sqrt{E}D\sqrt{F^{\mathsf{T}}}$. Multiplication by the square roots gives one implication; equality of the closures of the ranges of an effect and its square root gives the other.

Theorem 2.3 (Trace-class correspondence)**.** *For every finite game G,*

$$\mathrm{ZE}_{\mathrm{qs}}(N_{\Gamma(G)}, k) \iff G \text{ has a perfect pure spatial strategy with } \operatorname{Tr} D < \infty.$$

For a specified code marginal ρ, the reverse strategy has $D = \rho/\|\rho\|_2$. For a specified game strategy D with finite trace, the forward pure code has marginal $\rho = D/\operatorname{Tr} D$. No synchrony hypothesis is required.

Proof. In the forward construction, store Bob's effects in transposed form, and denote all local effects by $E_{c,a}$ for tagged questions c. Put $\rho = D/\operatorname{Tr} D$ and

$$R_{c,(c,a)} = \rho^{1/2} E_{c,a} \rho^{1/2},$$

with zero operators off the designated clique. Each message family sums to ρ. For cross losing edges, the perfect-game sandwich $E_{c,a} D E_{c',b} = 0$ gives the required zero product. The remaining products between distinct messages vanish by their clique assignments. Lemma 2.1 realizes the code. The trace condition is exactly what normalizes its marginal.

For the converse, as in [10, Theorem 6], apply Lemma 2.2 to an arbitrary saturating code, not only a code in the displayed form. Aggregate

$$T_{c,a} = \sum_i R_{i,(c,a)}.$$

Each family sums to ρ. If $(c, a), (c', b)$ is a cross losing edge, every product $R_{i,(c,a)} R_{j,(c',b)}$ is zero. For $i \neq j$ this is a code condition. For $i = j$ its two factors are supported on the orthogonal row projections $P_{i,c}, P_{i,c'}$. Thus $T_{c,a} T_{c',b} = 0$.

Use (2) to write $T_{c,a} = \rho^{1/2} E_{c,a} \rho^{1/2}$ with bounded POVMs. The product equality becomes

$$\rho^{1/2} E_{c,a} \rho E_{c',b} \rho^{1/2} = 0.$$

The injectivity and dense range of $\rho^{1/2}$ cancel its two outside factors, giving $E_{c,a} \rho E_{c',b} = 0$. Use the left-tagged effects on Alice and the transposes of the right-tagged effects on Bob. With $D = \rho/\|\rho\|_2$ these are a perfect spatial strategy. Since $\operatorname{Tr} \rho = 1$, $\operatorname{Tr} D = 1/\|\rho\|_2 < \infty$.

Only the coarse clique projections must commute with ρ. The fine outcomes need not do so; this is why the argument retains nonsynchronous games. Restricting every step to finite dimension recovers [10, Theorem 6]. □

Remark. For the game marginal $h = D^2$, the condition $\operatorname{Tr} D < \infty$ is finite Rényi entropy of order $1/2$, namely $2 \log_2 \operatorname{Tr} \sqrt{h}$. It is stronger than finite von Neumann entropy. Replacing the forward marginal by D^2 generally destroys zero-error orthogonality; Appendix A includes an explicit check on the flagged magic square.

3 The compiler at a general modular ratio

We use the finite projection and Boolean-row compiler of [6, Section 4], itself based on [5]. Its graph identities, rational-row tests, and normal support proofs remain unchanged. We specify the parameter change and prove its consequences. Each flagged-magic-square (FMS) bridge forces a density ratio of two between consecutive selected corners (Figure 1). A chain of n bridges therefore gives the ratio 2^n from source to far corner. Transporting a source-to-far creation operator back to the source gives $\lambda = 2^{-n}$. Four orthogonal creations then generate a rooted tree with four branches per vertex. The density of each word sector is $\lambda^{|w|}$ times the root density. Summing over the 4^ℓ words at depth ℓ gives a geometric depth distribution. The exact spectrum identity below accounts for the finite scaffold and auxiliary factors. Its square-root trace determines whether perfect play gives a normal channel code.

$$\underset{1}{e_0} \xrightarrow{w_1} \underset{2}{e_1} \xrightarrow{w_2} \underset{2^2}{e_2} \longrightarrow \cdots \xrightarrow{w_n} \underset{2^n}{e_n}$$

Figure 1: Selected corners and their masses relative to the source. Each bridge also has three further magic corners of the same mass as its selected endpoint. Returning along the chain gives the creation ratio $\lambda = 2^{-n}$.

3.1 Admissible input and the n-bridge assembly

Definition 3.1. Fix an integer $n \geq 3$ and put $\lambda = 2^{-n}$. An admissible presentation consists of four contraction symbols A_α, $\alpha \in \{0,1\}^2$, a centralizer root projection $P_\circ$, and finitely many centralizer labels typed as unitaries, projections, positive contractions, or general contractions. Besides rational $*$-polynomials in the centralizer labels, its equations are

$$A_\alpha^* A_\beta = \delta_{\alpha\beta} I, \qquad \sum_\alpha A_\alpha A_\alpha^* = I - P_\circ,$$

endpoint equations $LA_\alpha = A_\beta B$ or $LA_\alpha = 0$, and finite sums $\sum_\nu B_\nu A_{\alpha_\nu} C_\nu = 0$. The endpoint coefficients are centralizer polynomials; every label appearing at a left endpoint is transported. No other monomial contains more than one creation symbol.

A normal realization is a nonzero Hilbert space with a faithful density h_0, bounded operators of the declared types, all the application identities, and

$$[h_0, L] = 0 \quad \text{for centralizer labels,} \qquad h_0 A_\alpha = \lambda A_\alpha h_0. \tag{4}$$

The change from [6, Definition in Section 4.4] is the number of bridges and the covariance ratio. In particular the ratio is imposed by the assembly, rather than by a real coefficient in the Boolean game.

Set

$$\Omega_n = 1 + 4\sum_{g=1}^{n} 2^g = 2^{n+3} - 7, \qquad \mu_n = \Omega_n^{-1}(1; 2^{\oplus 4}; 4^{\oplus 4}; \ldots; (2^n)^{\oplus 4}). \tag{5}$$

The notation $t^{\oplus 4}$ denotes four entries equal to t.

Lemma 3.2 (n-bridge compiler)**.** *Every admissible presentation effectively gives a finite rational Boolean game $G^{(n)}$. Perfect normal spatial strategies exist exactly when normal realizations exist. The completeness map appends finite scaffold, qutrit, and row factors. The assembly parameters are*

quantity	*value*
orthogonal scaffold corners	$4n+1$
frame-bound parameters	$6n+1$
cover constant	$\Omega_n = 2^{n+3}-7$
chain covariance	$hv = 2^n vh$
normalized source density	$\Omega_n e_0 h e_0$
far-corner weight relative to source	2^n

Here h is the normalized scaffold density after removing its finite scalar factors. The completeness construction preserves finite marginal entropy.

Proof. Use mutually orthogonal synchronized corners

$$e_0; \quad m_{g,00}, m_{g,01}, m_{g,10}, m_{g,11} \quad (1 \le g \le n),$$

put $e_g = m_{g,00}$ and $M_g = \sum_{a,b} m_{g,ab}$, and impose $e_0 + \sum_g M_g = I$. Conditional flagged-magic-square bridge g has flag e_{g-1}, magic support M_g, and activity $e_{g-1} + M_g$. Retain the original questions, all activity/event bindings in both orientations, and the label refinements of [6, Lemmas 4.1–4.2]. The normal bridge gives partial isometries

$$w_g^* w_g = e_{g-1}, \qquad w_g w_g^* = e_g, \qquad h w_g = 2 w_g h.$$

The other three magic corners are identified by synchronized Pauli matrix units and have the same density block as e_g. Thus $v = w_n \cdots w_1$ satisfies

$$v^* v = e_0, \qquad v v^* = e_n, \qquad hv = 2^n vh.$$

Cyclicity of trace makes each selected corner have mass 2^g times the source mass. The cover gives source mass $1/\Omega_n$, hence a nonzero source. There are $4n+1$ corners; adding the two positive Pauli-axis events for each bridge gives $6n+1$ parameters bound to the scalar qutrit frame.

Here are the remaining equations, to fix the assembly unambiguously. A unitary label uses itself, a projection uses $2P - I$, a positive contraction T uses $(2I + u + u^*)/4$, and a general contraction uses $(u_+ + u_-)/2$. Let Λ contain the encoding of $P_\circ$ and all unitary encodings required at left endpoints. The root projection is transported even when it occurs in no endpoint equation. For $0 \le r \le n$ impose

$$(u_{r,t} - I)(I - e_r) = 0 \quad (t \in \Lambda).$$

Decode the corner copies L_r using the corner identity e_r. For each bridge impose the block-unitary equation

$$v_{g,t} = I - e_{g-1} - M_g + e_{g-1} u_{g-1,t} + \sum_{a,b} \mathfrak{x}_g^a \mathfrak{z}_g^b e_g u_{g,t} \mathfrak{z}_g^{-b} \mathfrak{x}_g^{-a}. \tag{6}$$

Here $\mathfrak{x}_g, \mathfrak{z}_g$ are the bound Pauli reflections extended by identity outside M_g. Refine each retained sign question, on each party, for the six frame projections

$$F_1, F_2, F_3, G_{12}(1), G_{13}(1), G_{23}(1)$$

and for $G_{12}(v_{g,t})$, $t \in \Lambda$. The shared opposite questions force every refined coarse sign to equal the original one. The six frame commutators give $w_g = I_3 \otimes \widetilde{w}_g$. The graph-label commutator then gives the actual transport equation

$$w_g u_{g-1,t} = u_{g,t} w_g, \qquad L_n v = v L_0.$$

Distinct transported labels need not commute with each other.

Introduce centralizer contractions $a_\alpha = e_n a_\alpha e_0$, encoded by two unitaries. Translate the application equations as follows:

root equation	centralizer equation
$A_\alpha^* A_\beta = \delta_{\alpha\beta} I$	$a_\alpha^* a_\beta = \delta_{\alpha\beta} e_0$
$\sum_\alpha A_\alpha A_\alpha^* = I - P_\circ$	$\sum_\alpha a_\alpha a_\alpha^* = e_n - (P_\circ)_n$
$LA_\alpha = A_\beta B$	$L_n a_\alpha = a_\beta B_0$
$LA_\alpha = 0$	$L_n a_\alpha = 0$
$\sum_\nu B_\nu A_{\alpha_\nu} C_\nu = 0$	$\sum_\nu (B_\nu)_n a_{\alpha_\nu} (C_\nu)_0 = 0$
centralizer polynomial	same polynomial in source copies

Extraction $A_\alpha = v^* a_\alpha$ reverses every row, since v is unitary between the two corners and intertwines all left endpoint labels. It gives (4), because $hv^* = 2^{-n} v^* h$. Only the displayed centralizer polynomials are submitted to the graph compiler; products of modular transports are never treated as centralizer words.

For completeness, from a realization take the scaffold density

$$h_{\mathrm{sc}} = \mathrm{diag}(\mu_n) \otimes h_0. \tag{7}$$

Use identity corner identifications and put $a_\alpha = |e_n\rangle\langle e_0| \otimes A_\alpha$. The covariance makes a_α commute with h_{sc}. Its positive eigenspaces are finite dimensional. Within each such block the polar partial isometry of any contraction T extends to a unitary V, and

$$U_\pm = V\big(|T| \pm i\sqrt{I - |T|^2}\big), \qquad T = (U_+ + U_-)/2.$$

Their direct sums are bounded centralizer unitaries. Positive contractions use $u = 2T - I + 2i\sqrt{T(I-T)}$; projections use their reflections. Copy the same source encodings through the bridge identifications. This realizes all block and support equations simultaneously.

Append a maximally entangled qutrit and apply the actual graph and triangle formulas of [6, Section 4.2]. They certify products of the common labels. Finally apply the Boolean rational-row construction [6, Lemma 4.3], adjoining a finite maximally entangled row ancilla of dimension $d_{\mathrm{row}} = d_{\max}$. Its old-projection/scalar-axis joint questions put every named synchronized old projection in the commutant of $\mathrm{Mat}_{d_{\mathrm{row}}}$. The FMS and refinement tests remain in the game. These unchanged normal support arguments prove both implications. All new indices are finite; the coefficients and sampling probabilities are rational. Padding questions and accepting unspecified pairs gives a finite game effectively. The appended factors have finite entropy. □

3.2 An exact spectrum identity

One must be careful when removing the row factor in soundness: the FMS signs were not synchronized, so their transports need not commute with $\mathrm{Mat}_{d_{\mathrm{row}}}$. The following elementary cancellation supplies the needed reduction without a partial trace over an infinite space.

Finite-amplification cancellation. If two finite-dimensional tuples of unitaries become unitarily equivalent after d_{row} identical copies, the original tuples are unitarily equivalent. Indeed they are finite-dimensional representations of the universal $*$-algebra on the tuple. The image in their direct sum is a finite-dimensional C^*-algebra. Decompose its two representations into irreducibles; amplification multiplies every multiplicity by d_{row}, so equality of amplified multiplicities cancels d_{row}.

Lemma 3.3 (Exact spectrum identity)**.** *For every pure perfect strategy of $G^{(n)}$, with marginal h and Schmidt operator $D = \sqrt{h}$, there is a normal realization with density h_0 such that, including multiplicities,*

$$\mathrm{spec}(h) = \mathrm{spec}\left(\mathrm{diag}(\mu_n) \otimes h_0 \otimes I_3/3 \otimes I_{d_{\mathrm{row}}}/d_{\mathrm{row}}\right). \tag{8}$$

Conversely every realization has a perfect strategy with this spectrum. In particular, in the extended nonnegative reals,

$$\operatorname{Tr} D = c_n \operatorname{Tr} \sqrt{h_0}, \qquad c_n = \sqrt{3 d_{\mathrm{row}}}\, \frac{1 + 4 \sum_{g=1}^{n} 2^{g/2}}{\sqrt{\Omega_n}}. \tag{9}$$

Proof. Restrict to the original marginal support. Synchronization makes the certified scalar matrix algebras commute with the density. The named old projections commute with $\mathrm{Mat}_{d_{\mathrm{row}}}$; the bound parameters and graph units also provide the scalar Mat_3 frame. These two scalar algebras commute, so their exact matrix-unit decompositions give a density $I_3/3 \otimes I_{d_{\mathrm{row}}}/d_{\mathrm{row}} \otimes h_{\mathrm{base}}$ and factorizations of every named old synchronized projection. This is a finite matrix-unit decomposition, not an uncontrolled partial trace.

The magic matrix units identify all four density blocks at each bridge. The w_g identify consecutive normalized selected density blocks, with ratio two. The cover therefore implies that the full density has the spectrum in (8), where h_0 is its normalized source block after removal of both scalar factors. It remains to verify that this h_0 is the density of a realization; simply compressing $v^* a_\alpha$ across $\mathrm{Mat}_{d_{\mathrm{row}}}$ would not prove that assertion.

The frame refinements make v act trivially on Mat_3. After removing that frame it is a unitary between the d_{row}-amplifications of the source and far spaces, intertwining their normalized densities and every transported unitary label. In each positive density eigenspace these spaces are finite dimensional. Apply finite-amplification cancellation to their transported unitary tuples. This gives an unamplified unitary for each positive eigenvalue. Their Hilbert direct sum $\widetilde{v}$ intertwines h_0 with the normalized far density, and intertwines all transported labels simultaneously. The densities are faithful, so these eigenspaces exhaust both spaces.

All centralizer contractions a_α are polynomials in old labels and factor across both scalar algebras; write their reduced forms $\widetilde{a}_\alpha$. Define $\widetilde{A}_\alpha = \widetilde{v}^* \widetilde{a}_\alpha$. The reduced translated equations in Lemma 3.2 give every application identity. Intertwining of the normalized densities, together with the source/far weight 2^n, gives $h_0 \widetilde{A}_\alpha = 2^{-n} \widetilde{A}_\alpha h_0$. This proves the realization assertion. The new creations need not be restrictions of the original creations.

Completeness uses (7), followed by the two normalized maximally mixed factors, so has exactly the same spectrum. Taking the square root and the trace of this finite tensor product proves (9); nonnegative spectral sums justify it even when the answer is infinity. □

3.3 The tree at general λ

The tree part of [6, Section 5] works for every $0 < \lambda < 1/4$. For clarity we give the density calculation. For a word w in four letters, write A_w for the product of its creations and

$$Q_\ell = \sum_{|w|=\ell} A_w A_w^*.$$

The isometry relations and covariance imply

$$\operatorname{Tr}(h_0 Q_\ell) = (4\lambda)^\ell, \qquad \operatorname{Tr}(h_0 P_\circ) = 1 - 4\lambda.$$

The decreasing Q_ℓ have a strong limit of zero h_0-weight, hence zero by faithfulness. Their successive differences are $\sum_{|w|=\ell} A_w P_\circ A_w^*$. Thus, with $\mathcal{K} = P_\circ \mathcal{H} \neq 0$,

$$\mathcal{H} \simeq \ell^2(\mathcal{W}) \otimes \mathcal{K}, \quad h_0 = h_{F,\lambda} \otimes \sigma, \quad h_{F,\lambda} = (1 - 4\lambda) \sum_w \lambda^{|w|} |w\rangle\langle w|, \tag{10}$$

where σ is a faithful density. The covariance makes every word-sector projection commute with h_0, so transporting the root block proves the equality, not only its diagonal entries. Depth ℓ has probability $(1 - 4\lambda)(4\lambda)^\ell$; $(4\lambda)^\ell$ is the mass at depth at least ℓ.

The coordinate, Weyl-clock, bounded period-reader and deterministic tableau equations of [6, Sections 5–7] contain no Fock density ratio. Their proofs use only faithful normality, positive finite-dimensional density eigenspaces, the exact tree decomposition, and a positive weight at every finite depth. All hold in (10). Consequently their soundness and canonical completeness hold unchanged for every $\lambda = 2^{-n} < 1/4$. In particular the Weyl determinant still forces a root phase of finite order: on a positive density eigenspace of dimension q_{clk}, $J_{\mathrm{clk}}K_{\mathrm{clk}} = \zeta K_{\mathrm{clk}}J_{\mathrm{clk}}$ gives $\zeta^{q_{\mathrm{clk}}} = 1$. No clock relation is squared or otherwise changed.

3.4 Summability and its critical point

Equation (10) gives

$$\operatorname{Tr}\sqrt{h_{F,\lambda}} = \sqrt{1-4\lambda}\sum_{\ell=0}^{\infty}(4\sqrt{\lambda})^{\ell} = \frac{\sqrt{1-4\lambda}}{1-4\sqrt{\lambda}} \tag{11}$$

when $\lambda < 1/16$, and infinity when $1/16 \le \lambda < 1/4$. For the canonical YES realization $\sigma = I_{q_{\mathrm{clk}}}/q_{\mathrm{clk}}$, Lemma 3.3 therefore gives

$$\operatorname{Tr} D = \sqrt{3q_{\mathrm{clk}}d_{\mathrm{row}}}\,\frac{1+4\sum_{g=1}^{n}2^{g/2}}{\sqrt{\Omega_n}}\,\frac{\sqrt{1-2^{2-n}}}{1-2^{2-n/2}} \quad (n>4). \tag{12}$$

For five bridges and the optional eight-bridge construction, the exact coefficients are

$$n=5: \quad \operatorname{Tr} D = \frac{(81+53\sqrt{2})\sqrt{7}}{2\sqrt{83}}\sqrt{q_{\mathrm{clk}}d_{\mathrm{row}}}, \tag{13}$$

$$n=8: \quad \operatorname{Tr} D = \frac{(121+60\sqrt{2})\sqrt{21}}{2\sqrt{2041}}\sqrt{q_{\mathrm{clk}}d_{\mathrm{row}}}. \tag{14}$$

They are approximately $22.65\sqrt{q_{\mathrm{clk}}d_{\mathrm{row}}}$ and $10.44\sqrt{q_{\mathrm{clk}}d_{\mathrm{row}}}$.

Remark (The original four bridges are critical). At $n = 4$, every realization has $\operatorname{Tr}\sqrt{h_0} = \infty$, regardless of σ, because the per-depth amplitude sum in (11) is constant and nonzero. Lemma 3.3 then makes every perfect pure game strategy have infinite amplitude sum. Theorem 2.3 implies that the channel of every game G_e of [6, Theorem 7.1] has no k-message normal code, for every input index. Applying its original games directly would therefore give an empty YES set, rather than the desired hardness reduction.

4 The normal and finite-dimensional complexity rows

4.1 Upper membership

Convert a channel instance (N, m) to a finite game $\mathcal{G}_{\mathrm{code}}(N, m)$. Alice receives $i \in [m]$ and answers a channel input x; Bob receives a channel output y and answers $j \in [m]$. All question pairs have positive rational probability. A tuple loses exactly when $N(y \mid x) > 0$ and $j \ne i$. With the same resource and measurements,

$$\mathrm{ZE}_t(N, m) \iff \mathcal{G}_{\mathrm{code}}(N, m) \text{ has a perfect } t\text{-strategy}. \tag{15}$$

Indeed the error probability of the code is the finite sum of these nonnegative expectations weighted by $N(y \mid x)$. It vanishes exactly when every forbidden expectation vanishes. The equivalence preserves marginal entropy as well.

For completeness we restate the effective compact-stratum argument of [6, Section 3]. A pure normal vector has a largest positive Schmidt amplitude s_1. Divide its Schmidt operator by s_1, and choose an integer K with

$$\delta_1 = 1, \quad 1 \ge \delta_2 \ge \cdots \ge 0, \quad \sum_i \delta_i^2 \le K.$$

Then $\delta_{r+1} \le \sqrt{K/(r+1)}$. These diagonal operators form an operator-norm compact family. Finite families of effects are weak-operator compact. If E_j, F_j converge weakly and Δ_j converges in norm to a compact Δ, then $E_j \Delta_j F_j$ converges weakly to $E\Delta F$: $\Delta F_j v$ converges in norm by compactness, and one pairs it with the weakly convergent $E_j^* u$.

Define $R(G, K, r)$ to require r-dimensional POVMs and a sorted diagonal Δ_r with first entry one, squared sum at most K, and

$$|(E_{a|u} \Delta_r F_{b|v}^{\mathsf{T}})_{ij}|^2 \le K/(r+1)$$

for every losing tuple and $i, j \le r$. This is a rational semialgebraic feasibility problem and is decidable by real quantifier elimination. Compression of an exact strategy supplies all stages: the omitted diagonal tail has norm at most $\sqrt{K/(r+1)}$. Conversely extend stage effects by a deterministic POVM on the complement, take a compact subsequential limit, and use the compact-sandwich observation. Every losing sandwich is zero. The limit has first diagonal entry one and squared sum between one and K, so its normalized vectorization is a nonzero normal perfect strategy. Consequently

$$G \in \mathrm{PERF}_{\mathrm{qs}} \iff \exists K \; \forall r \; R(G, K, r).$$

Mixed strategies reduce to pure perfect components, as in Lemma 2.1. Equation (15) proves $\mathrm{ZE}_{\mathrm{qs}} \in \Sigma_2^0$.

For finite entropy one uses entropy budgets rather than the largest Schmidt amplitude [6, Section 3.4]. Here are the effective details. Sorted probabilities satisfy $p_i \le 1/i$, so $H(p) \le B$ implies

$$\sum_{i>r} p_i \le B / \log_2(r+1).$$

The normalized first-r truncation majorizes p and has entropy at most B. Its vector trace distance is the square root of the omitted mass. Thus a perfect strategy supplies finite strategies whose loss is at most a computable rational $\eta_{B,r} > \sqrt{B/\log_2(r+1)}$, tending to zero, and whose entropy is at most B.

To decide finite relaxations approximate H uniformly on the finite probability simplex by computable rational polynomials $H_{r,a}$ within $1/a$. This is effective since $-t \log_2 t$, extended by zero at zero, is computably continuous; rational Bernstein approximations suffice. Require sorted p, $s_i \ge 0$, $s_i^2 = p_i$, finite POVMs, loss at most $\eta_{B,r}$, and

$$H_{r,a}(p) \le B + 1/r + 1/a.$$

Each relaxation is decidable. If all a stages at fixed r are feasible, finite-dimensional compactness gives a witness of entropy at most $B + 1/r$. Uniform entropy bounds give tightness of the sorted distributions in ℓ^1, hence compactness of their square roots in ℓ^2. Passing over r to a limit, with weak limits of the effects, gives zero loss and entropy at most B by lower semicontinuity. Conversely a perfect strategy with entropy at most B satisfies every relaxation. Flattening the two universal indices gives a Π_1^0 budget stratum, and taking the union over integer B gives Σ_2^0.

If a mixed perfect resource has finite entropy on one marginal, entropy concavity gives a positive-weight pure perfect component of finite entropy. For a pure bipartite component the two marginal entropies agree. Hence using one or both finite marginal entropies leads to the same feasibility set, and (15) proves $\mathrm{ZE}_{\mathrm{finS}} \in \Sigma_2^0$.

4.2 Five bridges for normal hardness

Let $S = \{e : \exists j \; \forall t \; R(e, j, t)\}$ for any decidable predicate R. Use the presentation of [6, Sections 5–7], with its computation program hardwiring e, and change only its bridge count to

five. Denote the resulting game by $G_e^{(5)}$. For reference its unchanged clock and computation mechanism is as follows.

The four-letter tree carries, at depth r, a 2^r-by-2^r coordinate square. Centralizer unitaries and projections impose cyclic coordinate increments and boundary indicators. A root Weyl pair $J_{\mathrm{clk}}K_{\mathrm{clk}} = ZK_{\mathrm{clk}}J_{\mathrm{clk}}$ commuting with the root density has a nonzero finite-order phase sector; its determinant on a positive density eigenspace forces this. The clock recurrence gives $Z_{\mathrm{clk}} = Z^y$ at coordinate (x, y), while a positive injective counter has value 2^{-y}. The two reader equations

$$(Z_{\mathrm{clk}} - I)P_{\mathrm{ret}} = 0, \qquad T_{\mathrm{inv}}(Z_{\mathrm{clk}} - I) = \tfrac{1}{2}C_{\mathrm{ctr}}(I - P_{\mathrm{ret}})$$

force P_{ret} to be exactly the kernel projection of $Z_{\mathrm{clk}} - I$. On a phase of order q_{clk}, the first positive return is precisely time q_{clk}.

The deterministic tableau freezes a waiting front at that time and reads its coordinate as unary input. It halts if q_{clk} is one or composite. If $q_{\mathrm{clk}} = p_{j+1}$ is prime it checks $R(e, j, 0), R(e, j, 1), \ldots$ and halts at the first failure. Later pulses have no effect. A halt is forbidden by an operator equation. Every depth has positive normal weight, so a finite halting event eventually appears in a sufficiently large square and contradicts perfection. These are exact identities, not assertions about high-probability truncations.

If the game is perfect, Lemma 3.2 extracts this mechanism and therefore a witness j. Conversely a witness j uses the canonical tree, a q_{clk}-dimensional Weyl pair for $q_{\mathrm{clk}} = p_{j+1}$, the good primitive phase and bounded reader from [6, Section 6], and its nonhalting classical tableau. The reader norm estimate depends on the phase and the counter, not on λ. Its density is $h_{F,1/32} \otimes I_{q_{\mathrm{clk}}}/q_{\mathrm{clk}}$. This is the simultaneous canonical realization, so Lemma 3.2 supplies a perfect game strategy. Formula (13) makes its Schmidt amplitudes summable.

Now put

$$N_e = N_{\Gamma(G_e^{(5)})}, \qquad k_e = |U_e| + |V_e|.$$

These data are finite and rational and are computed from e without knowing its witness. Theorem 2.3 and the preceding two implications give

$$\mathrm{ZE}_{\mathrm{qs}}(N_e, k_e) \iff e \in S.$$

This proves Σ_2^0-hardness and the first part of Theorem 1.1. No channel in this reduction has a finite-dimensional saturating code: otherwise the finite converse would extract four orthogonal isometries on a nonzero finite-dimensional root space, an impossibility. We make no claim here that a value-one promise is inherited from the source construction.

4.3 Finite entropy already at five bridges

Use exactly the same five-bridge reduction. The forward code marginal is the normalized square root of the canonical game marginal. Its tree factor is

$$\widehat{h}_5 = \frac{\sqrt{h_{F,1/32}}}{\mathrm{Tr}\,\sqrt{h_{F,1/32}}} = (1 - 2^{-1/2}) \sum_w 2^{-5|w|/2} |w\rangle\langle w|.$$

At depth ℓ there are 4^ℓ words, so the depth distribution is geometric with ratio $r_5 = 2^{-1/2}$. Its mean is $r_5/(1 - r_5) = 1 + \sqrt{2}$. Each eigenvalue at depth ℓ has negative base-two logarithm $-\log_2(1 - r_5) + (5/2)\ell$. Taking its expectation proves

$$S(\widehat{h}_5) = -\log_2(1 - 2^{-1/2}) + \frac{5}{2}(1 + \sqrt{2}) \approx 7.807 \text{ bits}. \tag{16}$$

The full canonical code marginal is

$$\mathrm{diag}(\nu_5) \otimes \widehat{h}_5 \otimes I_{q_{\mathrm{clk}}}/q_{\mathrm{clk}} \otimes I_3/3 \otimes I_{d_{\mathrm{row}}}/d_{\mathrm{row}}, \qquad \nu_{5,i} = \frac{\sqrt{\mu_{5,i}}}{\sum_j \sqrt{\mu_{5,j}}}.$$

Thus its entropy is $H(\nu_5) + S(\widehat{h}_5) + \log_2(3q_{\mathrm{clk}}d_{\mathrm{row}})$, which is finite. The pure code has the same entropy on both sides. A NO input has no normal code at all, hence no finite-entropy code. The same map therefore proves the second half of Theorem 1.1.

Remark (Eight bridges). With $n = 8$, the normalized square root of $h_{F,1/256}$ is exactly $h_{F,1/16}$, with entropy $\log_2(4/3) + 4/3$. Its amplitude sum is (14). This is another finite-entropy construction; five bridges already suffice for hardness.

4.4 The finite-dimensional row, as an attributed corollary

For each fixed dimension, the channel code game in (15) has a rational semialgebraic perfect-strategy predicate: split matrix entries into real and imaginary parts, impose positivity, POVM sums, normalization and zero losing probabilities. Quantifier elimination decides it. Enumerating dimensions gives $\mathrm{ZE}_q \in \Sigma_1^0$.

For hardness, the perfect-completeness reduction of $\mathsf{MIP}^* = \mathsf{RE}$ gives, from a halting instance, a finite game with a perfect finite-dimensional strategy, and from a nonhalting instance a game of entangled value at most $1/2$ [8, Theorem 12.6]. Mančinska's Corollary 7 maps perfect finite-dimensional game feasibility to the saturating one-shot channel threshold [10]. Thus ZE_q is Σ_1^0-complete. This conclusion uses her finite theorem, not the new normal converse. We have not found this corollary stated. Since classical channels are quantum channels, it shows the one-shot threshold for $C_{0,E}^{(1)}$ is undecidable, and Σ_1^0-complete for classical channels [2, Section 1.4]. Here E denotes unrestricted finite-dimensional entanglement; this is a one-shot statement.

5 Commuting operators and arbitrary-code saturation

The code game (15) identifies $\mathrm{ZE}_{\mathrm{qc}}$ with perfect commuting play for a particular finite game. Commuting correlations form a compact set, and the NPA hierarchy converges from above to their value [14]. Its rational moment relaxations can be decided by real quantifier elimination. A NO instance eventually has an infeasible perfect-value relaxation; conversely a feasible relaxation at every order gives a commuting strategy by compactness of the full moment system. Hence $\mathrm{ZE}_{\mathrm{qc}} \in \Pi_1^0$.

We prove the reduction for the lower bound, including codes with arbitrary commuting resources. Slofstra proved Π_1^0-hardness for linear system games [16]; with the NPA bound this is Π_1^0-completeness, which Mousavi, Nezhadi and Yuen reprove for synchronous games [13, Theorem 6.4]. The source is perfect commuting feasibility for finite synchronous games with positive diagonal question probabilities; see [13, Theorem 6.4 and Section 1.3]. Their synchronous framework and tracial representation are stated in [13, Theorem 1.6]. Such a game has the same question and answer sets on both sides, and distinct answers on an equal-question pair are losing.

5.1 Normal functionals from a commuting code

Represent a code on a Hilbert space with vector ξ, encoding effects $E_{i,x}$ in an algebra commuting with Bob's von Neumann algebra $\mathcal{N}$, and decoder effects in $\mathcal{N}$. Define normal positive functionals

$$\phi(T) = \langle \xi, T\xi \rangle, \qquad \phi_{i,x}(T) = \langle \xi, E_{i,x}T\xi \rangle \quad (T \in \mathcal{N}).$$

The two operators commute, so these functionals are positive and $\sum_x \phi_{i,x} = \phi$. Compress $\mathcal{N}$ to the support of ϕ. This makes ϕ faithful. It preserves all the functionals, their sums, and decoder tests, since each $\phi_{i,x} \le \phi$ is supported on the same support. The compressed encoders still commute with the compressed Bob algebra.

When $i \neq j$ and $x \sim x'$, the functionals $\phi_{i,x}, \phi_{j,x'}$ have orthogonal supports in $\mathcal{N}$. To see this, for any common output y their positively weighted functionals are summands of the two distinct-message output functionals. Perfect decoding makes the support of one lie in the eigenvalue-one space of a decoder effect, and that of the other in its kernel. These spectral spaces are orthogonal. Support inclusion for positive summands proves the assertion.

Lemma 5.1 (Functional saturation)**.** *Suppose the input graph is partitioned into k cliques $\mathcal{C}_c$ and a commuting code sends k messages. For*

$$\psi_{i,c} = \sum_{x \in \mathcal{C}_c} \phi_{i,x}, \qquad s_{i,c} = \operatorname{supp} \psi_{i,c},$$

the $s_{i,c}$ lie in the centralizer $\mathcal{N}_\phi = \{s : \phi(sT) = \phi(Ts) \text{ for all } T \in \mathcal{N}\}$ and satisfy

$$\psi_{i,c}(T) = \phi(s_{i,c}T) = \phi(Ts_{i,c}).$$

Their rows and columns are orthogonal projection partitions.

Proof. For each c the supports are orthogonal in i. Domination by ϕ gives

$$\sum_i \psi_{i,c}(1) \le \sum_i \phi(s_{i,c}) \le 1.$$

The sum over c is k, so every inequality is an equality. Faithfulness gives $\sum_i s_{i,c} = 1$. For each i, c, $\eta = \phi - \psi_{i,c}$ is positive and $\eta(s_{i,c}) = 0$. Cauchy–Schwarz for a positive functional gives $\eta(s_{i,c}T) = \eta(Ts_{i,c}) = 0$. The support property of $\psi_{i,c}$ gives $\psi_{i,c}(s_{i,c}T) = \psi_{i,c}(Ts_{i,c}) = \psi_{i,c}(T)$. Combining them proves the formula and centralizer membership.

Summing over c for fixed i gives $\phi((\sum_c s_{i,c} - 1)T) = 0$ for every T. Take $T = (\sum_c s_{i,c} - 1)^*$ and use faithfulness to obtain the row sum. A finite sum of projections equal to identity is orthogonal, as in Lemma 2.2. □

5.2 Synchrony makes the fine supports tracial

Apply the lemma to $\Gamma(G)$, with $k = 2|U|$, for a synchronous game G. Aggregate the fine functionals

$$\psi_{c,a} = \sum_i \phi_{i,(c,a)}.$$

They sum to ϕ for each tagged question c. Across a cross losing edge their supports are orthogonal: unequal message indices use the code condition, and equal message indices use distinct clique supports in one row of $(s_{i,c})$. Orthogonality passes to finite sums.

Fix an original question u. Write $\alpha_a = \psi_{(L,u),a}$, $\beta_a = \psi_{(R,u),a}$, and $p_a = \operatorname{supp} \alpha_a$. For $a \neq b$, synchrony gives $p_a \operatorname{supp} \beta_b = 0$. Hence

$$\alpha_a(1) \le \phi(p_a) = \beta_a(p_a) \le \beta_a(1).$$

Both outside families sum to one, so these inequalities are equalities for every a. The same Cauchy–Schwarz argument used in Lemma 5.1, now for $\phi - \alpha_a$ and $\phi - \beta_a$, gives

$$\alpha_a(T) = \beta_a(T) = \phi(p_aT) = \phi(Tp_a).$$

Faithfulness implies that their supports are the same p_a, and that $\sum_a p_a = 1$. Thus the fine outcomes are projections in $\mathcal{N}_\phi$.

The restriction $\tau = \phi|_{\mathcal{N}_\phi}$ is a faithful normal tracial state. For any original losing tuple (u, a, v, b) the preceding support orthogonality gives $p_{u,a}p_{v,b} = 0$. The left/right multiplication construction on $L^2(\mathcal{N}_\phi, \tau)$ therefore gives a perfect commuting strategy for G. This argument starts from an arbitrary saturating code; it assumes neither projective encoders nor a tracial original resource.

Conversely a perfect commuting strategy for synchronous G has its usual tracial projection representation [15, Theorem 5.5]; see also [9, 13]. For each tagged question use its projection partition to encode its designated message in the left algebra of the tracial GNS space; use the right algebra to decode. The conditional functionals are $T \mapsto \tau(p_{c,a}T)$. For distinct designated messages and confusable vertices, their supports are orthogonal by the losing relation. Each output's distinct-message supports therefore give a decoding measurement. This is a commuting k-message code for $N_{\Gamma(G)}$.

Theorem 5.2. *The one-shot feasibility problem* $\mathrm{ZE}_{\mathrm{qc}}$ *for finite rational classical channels is* Π_1^0*-complete.*

Proof. For each finite synchronous game G with positive diagonal question probabilities, the two constructions give the effective equivalence

$$G \text{ is commuting-perfect} \iff \mathrm{ZE}_{\mathrm{qc}}(N_{\Gamma(G)}, 2|U|).$$

The source problem is Π_1^0-complete, so this proves hardness. The NPA upper predicate gives membership. □

6 A channel whose codes require infinite entropy

6.1 The squared Catalan density

Let $\mathcal{W}_{\mathrm{Cat}}$ consist of the empty word and the legal chronological words in an up-step A and a down-step B. An up-step is always allowed; a down-step is allowed only at height at least two. Let $u(w)$ count up-steps. The number of words with j up-steps is the Catalan number Cat_j. On $\ell^2(\mathcal{W}_{\mathrm{Cat}})$ let A, B append the corresponding legal letter, with B zero when forbidden. Let C_{Cat} have eigenvalue $2^{-\mathrm{ht}(w)}$, let P project onto the empty word, and let Q project onto height one.

We require the local relations of [5, Section 3]:

$$A^*A = I, \qquad AA^* + BB^* + P = I, \qquad B^*B = I - P - Q, \tag{17}$$

$$C_{\mathrm{Cat}}P = P, \qquad C_{\mathrm{Cat}}Q = Q/2, \qquad 0 \le C_{\mathrm{Cat}} \le I, \tag{18}$$

$$C_{\mathrm{Cat}}A = AC_{\mathrm{Cat}}/2, \qquad C_{\mathrm{Cat}}B = 2BC_{\mathrm{Cat}}. \tag{19}$$

In this paper the modular ratio is squared:

$$\kappa A = A\kappa/16, \qquad [\kappa, B] = [\kappa, C_{\mathrm{Cat}}] = [\kappa, P] = [\kappa, Q] = 0. \tag{20}$$

Here κ is a faithful normal density. The counter relation $C_{\mathrm{Cat}}A = AC_{\mathrm{Cat}}/2$ is unchanged.

Lemma 6.1 (Normal classification at ratio 1/16)**.** *Every normal realization of* (17)–(20) *is, up to a unitary,*

$$\mathcal{H} = \ell^2(\mathcal{W}_{\mathrm{Cat}}) \otimes \mathcal{J}, \qquad \kappa = k_{\mathrm{Cat}} \otimes \sigma, \qquad k_{\mathrm{Cat}} = \frac{1}{8 - 4\sqrt{3}} \sum_w 16^{-u(w)} |w\rangle\langle w|,$$

where σ is a faithful normal density. All five local operators act canonically on the first factor and as identity on the second.

Proof. We give the normality argument to check the ratio change in [5, Lemma 3.1]. The boundary equations put P, Q in the counter eigenspaces one and one half and give $PQ = 0$. The range projections AA^*, BB^*, P are orthogonal and sum to identity; B is a partial isometry with initial projection $I - P - Q$. All these projections commute with C_{Cat}.

There is no spectrum of C_{Cat} in $(1/2, 1)$, since there B is isometric and $C_{\mathrm{Cat}}B = 2BC_{\mathrm{Cat}}$ would transport it above one. The same reasoning at eigenvalue one gives $1_{\{1\}}(C_{\mathrm{Cat}}) = P$. At eigenvalue one half, B would transport the complement of Q into P, orthogonally to its own range, so $1_{\{1/2\}}(C_{\mathrm{Cat}}) = Q$. Iterating B excludes every interval $(2^{-j-1}, 2^{-j})$. Functional calculus for the bounded intertwining relation justifies these spectral transports.

The zero-counter space reduces the operators and κ. On it $P = Q = 0$, so A, B are isometries with orthogonal ranges summing to identity. If its κ-mass is t, cyclicity and (20) make the two range masses $t/16$ and t. Their sum is t, forcing $t = 0$. Faithfulness removes this space.

Apply legal words to $P\mathcal{H}$. Their sectors are orthogonal: different last letters have orthogonal ranges; equal last letters cancel with $A^*A = I$ or with $B^*B = I - P - Q$ on a legal predecessor; and a nonempty sector is orthogonal to P. Denote the word isometry by V_w^{Cat}. Covariance gives

$$\kappa V_w^{\mathrm{Cat}} = 16^{-u(w)} V_w^{\mathrm{Cat}} \kappa.$$

The word-sector projections commute with κ, and their closed span reduces all the operators. If its orthogonal complement were nonzero, the compact faithful density there would have a largest positive eigenvalue, with nonzero finite-dimensional eigenspace L. The adjoint covariance gives $A^*L = 0$. Since $P = 0$ on this complement, $BB^* = I$ on L. Commutation with κ makes L reduce B, so $B|_L$ is unitary. It also reduces C_{Cat}. Tracing $C_{\mathrm{Cat}}B = 2BC_{\mathrm{Cat}}$ on L forces $\mathrm{Tr}_L\, C_{\mathrm{Cat}} = 0$, hence $C_{\mathrm{Cat}}|_L = 0$, contrary to the absence of zero-counter mass. Thus the word sectors exhaust the support.

Their density blocks are $16^{-u(w)} V_w^{\mathrm{Cat}} (P\kappa P)(V_w^{\mathrm{Cat}})^*$. The word count is Cat_j: for $j \geq 1$, words with j up-steps and b down-steps, $0 \leq b \leq j - 1$, number

$$\binom{j+b-1}{b} - \binom{j+b-1}{b-1};$$

reflection of the first bad prefix gives this formula, and summing over b gives Cat_j. The empty word supplies $\mathrm{Cat}_0 = 1$. The generating function

$$F(z) = \sum_{j \geq 0} \mathrm{Cat}_j\, z^j = \frac{1 - \sqrt{1 - 4z}}{2z}$$

gives $F(1/16) = 8 - 4\sqrt{3}$. Normalization then gives the asserted tensor product. The same sector identifications fix the local operators. □

Define the critical Catalan density of [5] by

$$h_{\mathrm{Cat}} = \frac{1}{2} \sum_w 4^{-u(w)} |w\rangle\langle w|.$$

The generating function yields the exact identities

$$\mathrm{Tr}\, h_{\mathrm{Cat}}^2 = 2 - \sqrt{3}, \qquad k_{\mathrm{Cat}} = \frac{h_{\mathrm{Cat}}^2}{2 - \sqrt{3}}, \qquad \mathrm{Tr}\, \sqrt{k_{\mathrm{Cat}}} = \frac{\sqrt{6} + \sqrt{2}}{2}, \qquad \frac{\sqrt{k_{\mathrm{Cat}}}}{\mathrm{Tr}\, \sqrt{k_{\mathrm{Cat}}}} = h_{\mathrm{Cat}}. \tag{21}$$

Moreover

$$S(h_{\mathrm{Cat}}) = \sum_{j \geq 0} \frac{\mathrm{Cat}_j}{2\, 4^j} (1 + 2j) = \infty, \tag{22}$$

since $\mathrm{Cat}_j\, 4^{-j} \sim (\sqrt{\pi} j^{3/2})^{-1}$. By contrast the squared Catalan factor has

$$S(k_{\mathrm{Cat}}) = \log_2(8 - 4\sqrt{3}) + \frac{4}{\sqrt{3}} - 2 \approx 0.4094 \text{ bits.}$$

Indeed, with F as above and $z = 1/16$, its mean up-step count is $zF'(z)/F(z) = 1/\sqrt{3} - 1/2$; averaging $\log_2 F(z) + 4u(w)$ gives the formula. Its multiplicity-weighted probabilities decay geometrically. The canonical perfect strategy of the game H_{Cat} below therefore has finite entropy: $S(k_{\mathrm{Cat}})$ is its infinite Catalan factor's contribution, and all its other canonical factors are finite dimensional. The channel of H_{Cat} instead forces the factor h_{Cat} and hence infinite entropy. The communication task itself creates the entropy demand through the square-root map.

6.2 A four-bridge Catalan assembly

The Catalan relations are not the four-creation syntax of Definition 3.1. We therefore specify their own assembly, extending [5, Section 6] from two bridges to four. Use the seventeen corners, twenty-five frame-bound parameters, cover $\Omega_4 = 121$, axis equations, support-copy equations and block-unitary transport equations of Section 3. The transported unitary encodings are those of $B, C_{\mathrm{Cat}}, P, Q$: two unitaries b_+, b_- for B, one for the positive contraction C_{Cat}, and the reflections for P, Q. Thus $|\Lambda| = 5$. For each bridge use the six frame refinements and these five graph-label refinements, on both parties, retaining the original opposite FMS questions.

For $r = 0, \dots, 4$, denote the decoded corner copies by b_r, c_r, p_r, q_r. Impose the root reflection equations for p_0, q_0. Introduce one centralizer contraction $a = (a_+ + a_-)/2$. The remaining equations are

$$a = e_4 a e_0, \qquad a^* a = e_0, \qquad aa^* + b_4 b_4^* + p_4 = e_4, \tag{23}$$

$$b_0^* b_0 = e_0 - p_0 - q_0, \qquad p_0 q_0 = 0, \qquad c_0 p_0 = p_0, \tag{24}$$

$$c_0 q_0 = q_0/2, \qquad c_4 a = a c_0/2, \qquad c_0 b_0 = 2 b_0 c_0. \tag{25}$$

In particular the range equation is at the far corner. It is not $aa^* + b_0 b_0^* + p_0 = e_0$.

Let $v = w_4 w_3 w_2 w_1$. Extraction on the source gives

$$A = v^* a, \qquad B = b_0, \qquad C_{\mathrm{Cat}} = c_0, \qquad P = p_0, \qquad Q = q_0.$$

Conjugate the far range equation by v, using the actual transported copies. This gives $AA^* + BB^* + P = e_0$. The other local equations follow directly; for example

$$C_{\mathrm{Cat}} A = c_0 v^* a = v^* c_4 a = v^* a c_0/2 = A C_{\mathrm{Cat}}/2.$$

Since a is a centralizer contraction and $hv = 16vh$, its source density has $\kappa A = A\kappa/16$. The other decoded labels commute with it. After removal of the finite scalar factors, use exactly the finite-amplification cancellation proof of Lemma 3.3, now for these five transported unitary labels. It gives a realization of (17)–(20) and the same spectrum identity with μ_4.

Conversely use the canonical Catalan operators and density k_{Cat} in $\mathrm{diag}(\mu_4) \otimes k_{\mathrm{Cat}}$, put $a = |e_4\rangle\langle e_0| \otimes A$, and use identity transports. The ratio makes a commute with the scaffold density. The blockwise polar encodings, actual graphs, label refinements and finite Boolean-row extension used in Lemma 3.2 realize all the displayed equations simultaneously. Thus they compute a finite rational game H_{Cat} with a perfect normal strategy. This construction has not assumed that the Catalan relations were admissible Fock input.

6.3 Every normal code has infinite entropy

Set $N = N_{\Gamma(H_{\mathrm{Cat}})}$ and $m = |U_{H_{\mathrm{Cat}}}| + |V_{H_{\mathrm{Cat}}}|$. The canonical strategy just constructed has finite amplitude sum, by (21) and its finite scaffold and scalar factors. Theorem 2.3 therefore gives a normal m-message code.

Consider any normal code, including a mixed resource, and let ρ_B be its Bob marginal. The assemblage converse and Theorem 2.3 construct a pure perfect game strategy with

$$h_{\mathrm{game}} = \frac{\rho_B^2}{\operatorname{Tr}\rho_B^2}, \qquad \rho_B = \frac{\sqrt{h_{\mathrm{game}}}}{\operatorname{Tr}\sqrt{h_{\mathrm{game}}}}.$$

The assembly spectrum identity and Lemma 6.1 give

$$h_{\mathrm{game}} \simeq \operatorname{diag}(\mu_4) \otimes k_{\mathrm{Cat}} \otimes \sigma \otimes I_3/3 \otimes I_{d_{\mathrm{row}}}/d_{\mathrm{row}}.$$

Here $\simeq$ means unitary equivalence of positive operators. Trace summability forces $\operatorname{Tr}\sqrt{\sigma} < \infty$. Taking the normalized square root gives

$$\rho_B \simeq \operatorname{diag}(\nu_4) \otimes h_{\mathrm{Cat}} \otimes \sigma_{\mathrm{code}} \otimes I_3/3 \otimes I_{d_{\mathrm{row}}}/d_{\mathrm{row}}, \quad \nu_{4,i} = \frac{\sqrt{\mu_{4,i}}}{\sum_j \sqrt{\mu_{4,j}}}, \quad \sigma_{\mathrm{code}} = \frac{\sqrt{\sigma}}{\operatorname{Tr}\sqrt{\sigma}}. \tag{26}$$

The Catalan factor is exactly h_{Cat}; the full marginal also includes the finite factors and the possible spectator σ_{code}. Equation (22) gives $S(\rho_B) = \infty$ for every code.

For a pure resource its two marginal spectra coincide, so Alice's entropy is infinite as well. For a mixed resource $\omega = \sum_j t_j |\psi_j\rangle\langle\psi_j|$, each positive-weight component is a perfect code under the same encoding and decoding measurements. This follows by applying positivity to each forbidden error expectation. Each such pure component has infinite entropy on both marginals. Extended entropy concavity gives

$$S(\omega_A) \geq t_j S((\psi_j)_A) = \infty,$$

and similarly for Bob. This proves Theorem 1.2 for all normal codes, rather than only canonical ones.

7 Capacity bounds and open questions

Let $a_t(r)$ be the largest message count achievable with model t over $N^{\otimes r}$. Product codes give $a_t(r+s) \geq a_t(r)a_t(s)$, including for normal and commuting resources. The singleton-clique bound gives $1 \leq a_t(r) \leq |X|^r$. Fekete's lemma therefore gives

$$C_{0,t}(N) = \lim_{r\to\infty} \frac{\log_2 a_t(r)}{r} = \sup_{r\geq 1} \frac{\log_2 a_t(r)}{r}.$$

For a rational b, the strict lower-cut condition is

$$b < C_{0,t}(N) \iff \exists r, m \ [\mathrm{ZE}_t(N^{\otimes r}, m) \text{ and } b < \log_2(m)/r].$$

The logarithmic comparison is decidable for rational b: after clearing denominators it compares integer powers (with trivial cases for $b < 0$).

The q predicate is computably enumerable, so $C_{0,q}$ is uniformly left-c.e. For qs the one-shot predicate is Σ_2^0, hence c.e. relative to the halting oracle $0'$. For qc it is Π_1^0, hence decidable relative to $0'$. Taking the displayed existential union proves that both $C_{0,\mathrm{qs}}$ and $C_{0,\mathrm{qc}}$ are uniformly left-c.e. relative to $0'$. These are upper bounds only.

One-shot hardness does not imply capacity hardness: other message counts or tensor powers can hide the one-use threshold. Our reductions do not control these possibilities. The computability and precise arithmetical complexity of normal-entanglement zero-error capacity remain open. So does a strict separation of $C_{0,q}$ from $C_{0,\mathrm{qs}}$ for a finite classical channel. The channels constructed here separate one-shot attainment or entropy requirements; no asymptotic separation follows from them.

How this work was done

The authors selected the questions and proof standards, directed the research, and take responsibility for every claim. AI systems from more than one developer assisted with exploration, derivation, programming and drafting.

The mathematical evidence consists of the proofs in this paper and the explicitly cited external theorems. The ancillary programs provide reproducible finite checks. Independent reviews examined the operator arguments, attribution, complexity reductions and entropy calculations. The finite programs test strategy and code matrices, exact bridge constants, spectrum identities and Catalan generating-function identities. Their arithmetic and scope are stated in Appendix A; numerical checks are distinguished from exact checks.

These finite checks do not prove infinite-dimensional existence, compactness or arithmetical hardness by sampling. Those conclusions rest on the written proofs and the cited source reductions. This preprint has not been peer reviewed.

A Reproducibility and claim attribution

The accompanying portable verification directory contains finite matrix checks, explicit pass/fail assertions, and exact general-bridge checks. Run `python run_all.py` after installing the packages in `requirements.txt`. Each invocation writes to a fresh output directory. Floating-point matrix checks use explicit tolerances; the bridge and Catalan constants are checked symbolically. The runner reports a failure if any assertion fails. The recorded results, dependency versions and source hashes accompany the bundle.

The magic-square graph has classical independence number five and a six-message entangled code. The flagged-magic-square graph has fifty vertices, classical independence number nine and a ten-message entangled code. The reverse map is also tested after a noncommutative row-and-column scrambling. These examples verify the implementation of the finite correspondence already proved in [10]; they are not novelty claims. The naive D^2 forward marginal is tested as a negative control.

The exact script checks $4n + 1$, $6n + 1$, Ω_n, normalized scaffold weights, source/far mass ratios, convergence and criticality of the amplitude series, the five/eight constants, the five-bridge code entropy, the squared Catalan entropy, and all identities in (21). The infinite entropy conclusion uses Catalan asymptotics in the proof, not a finite partial entropy sum.

claim	attribution	proof/evidence
finite game–channel correspondence	known: Mančinska	Lemma 5, Theorem 6 and Corollaries 7–8 of [10]
normal trace-class correspondence	new extension	Section 2; bounded Douglas factorization
n-bridge spectrum control	new extension of authors' compiler	Section 3; finite-amplification cancellation
normal and finite-entropy complexity	new	Section 4; source reduction [6]
finite-dimensional complexity	attributed corollary	Mančinska plus [8]
commuting complexity	new arbitrary-code reduction	Section 5; source hardness [16, 13]
infinite-entropy-only channel	new	Section 6; source Catalan tools [5]
capacity computability	upper bounds only	Section 7; no capacity hardness claim

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